Learning with Misattribution of Reference Dependence

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Abstract

This paper explores how an individual who assesses outcomes relative to a reference point may develop biased beliefs when learning from experience. We consider an agent whose utility depends on both the intrinsic value of an outcome and how that value compares to expectations. A misattributing agent neglects how the utility from a positive or negative surprise contributes to her overall experience, and wrongly attributes this sensation to the intrinsic value of an outcome. Our model provides an intuition for why first-hand experience influences beliefs differently than second-hand observations containing identical information and why losses can impact beliefs more than gains. Our main results study implications of misattribution in dynamic environments. First, a misattributor’s expectations are over-influenced by recent experiences and under-influenced by earlier ones. Second, long-run beliefs grow pessimistic and undervalue prospects in proportion to their variability, leading a decision maker to abandon some risky options that are optimal. Third, when outcomes are autocorrelated, a misattributor persistently forms extrapolative and volatile forecasts about future payoffs. Applying the model, we show that (i) uncertain availability of a good can increase its perceived value, (ii) a misattributor may over-invest to insure against undesirable outcomes, and (iii) a misattributing principal may overestimate the ability of a manipulative agent who initially suppresses expectations and beats them thereafter.

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1 Introduction

Learning from personal experience guides a wide range of decisions, but recent studies indicate some shortcomings in how we learn from our past.¹ This paper explores a potentially important way that learning from experience can lead to biased beliefs and mistaken choices. We build from the well-known idea that people evaluate experiences relative to their expectations: the utility from an outcome depends on both its “intrinsic value” and how that value compares to expectations (e.g., Kahneman and Tversky 1979; Bell 1985; Kőszegi and Rabin 2006). This reference dependence can complicate learning, since today’s utility—which depends on current expectations—may not match what a person will earn after her expectations change. Thus, when using past experiences to forecast future utility, rational learning requires a person to disentangle the intrinsic value of an outcome from the sensations of positive or negative surprise it generated. Intuition and a growing body of research suggests that people may fail to fully separate these two sources of utility, and may incorrectly attribute the sensation of surprise to their intrinsic taste for an outcome.²

Imagine, for example, a traveler flying on a respected airline. If the service falls short of expectations, she will feel unhappy both because of the subpar service and the negative surprise. A rational traveler will understand that her bad experience derived in part from her high expectations. A less introspective traveler, however, might misattribute her disappointment to the airline’s quality, and consequently underestimate the service she experienced. Alternatively, consider a researcher collaborating with a new colleague. If the colleague finishes an unexpectedly large share of the project on his own, the researcher will feel happy both because her workload has diminished and because this came as a pleasant surprise. If she misattributes this latter feeling to her partner’s performance, she may recall an exaggerated perception of his contribution. As these examples suggest, surprises may distort perceived outcomes: exceeding expectations inflates perceptions, and falling short deflates them.

This paper explores the learning process and behavior of an individual who neglects how sen-

¹For instance, recent evidence shows that people rely too heavily on their own past experience when making decisions. Personal successes and failures play an important role in IPO subscription (Kaustia and Knüpfer 2008; Chiang et al. 2011), risk taking and stock-market participation (Malmendier and Nagel 2011), insurance take-up (Gallagher 2014), college-major choice (Xia 2016), and compliance with deadlines (Haselhuhn et al. 2012).

²Our companion paper, Bushong and Gagnon-Bartsch (2016), provides direct evidence that people fail to account for their former expectations when inferring from past experience. In an experiment, we manipulate participants’ expectations prior to learning about unfamiliar real-effort tasks. Consistent with our model, we observe systematic and persistent changes in subsequent willingness to work at those tasks. A related literature on other forms of misattribution demonstrates how the effects of extraneous situational factors may be attributed to the inherent characteristics of a good or person. For instance, Haggag and Pope (2016) demonstrate that, when assessing the value of a good, people have difficulty separating state-dependent utility caused by temporary circumstances such as thirst from the quality of the good. Dutton and Aron (1974) find that subjects who form opinions about people they meet for the first time exhibit judgments dependent on unrelated factors (e.g., their current state of fear or excitement). We discuss this literature at greater length in Section 2.
sations of positive and negative surprise influenced her experiences, and who misattributes this aspect of her overall utility to the quality of the outcomes she faced. We examine the implications of this mistake across a range of settings in which a decision maker uses past experience to form expectations about future payoffs. The model has several immediate implications that organize a variety of known errors in belief formation. First, a misattributor tends to over-infer from first-hand experiences—outcomes with direct utility consequences—relative to second-hand observations that contain identical information. Outcomes that incite sensations of elation or disappointment are misencoded, while signals that do not directly influence utility are not. Second, when a misattributor is loss averse, negative surprises influence experienced utility more than positive surprises, and hence outcomes that come as a loss may influence beliefs more than those that come as a gain (e.g., Kuhnen 2015; Ben-David, Graham, and Harvey 2013; Skinner and Sloan 2002). Additionally, misattribution can generate sequential “contrast effects”: a misattributor perceives today’s outcome as better the worse was yesterday’s. Insofar as yesterday’s experience lowers expectations, today’s outcome comes as a greater gain (or a lesser loss).

Our general results explore the dynamics of beliefs and behavior in both the short and long run. Misattribution can lead to persistent mistakes about an action’s benefit, and we characterize how features of the environment shape these long-run beliefs. In particular, variability in the outcomes an action yields causes a misattributor to become pessimistic about that action, meaning she may turn down risky-but-optimal prospects. Our model additionally provides testable implications for short-run patterns in beliefs. Namely, misattribution induces a “recency bias”—expectations weight recent outcomes more than earlier ones—and “order effects”—a misattributor is most optimistic about an action when, ceteris paribus, its outcomes are experienced in an increasing order from worst to best. We also present two applications that reveal how misattribution dynamically distorts incentives. First, we show that paying to insure against undesirable outcomes can endogenously lead a misattributor to pay more and more over time. Second, we examine how a sophisticated agent might exploit a misattributing principal in signaling environments. Matching intuitions about “expectations management” from marketing and politics, an agent attempting to signal high value will restrain performance early but consistently beat expectations thereafter.

Section 2 introduces the model. Each period, the decision maker selects an action and then experiences reference-dependent utility composed of two parts: consumption utility—the classi-

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3Beyond misattribution, there are several plausible factors that may lead people to overweight personal experiences when making decisions. For instance, both the availability heuristic (Tversky and Kahneman 1973) or under-inference from large samples (Benjamin, Rabin, Raymond 2016) likely play a role. Misattribution predicts that such overweighting of personal experience is more pronounced for surprising episodes that carry utility consequences.

4Sequential contrast effects have been documented in numerous settings, including sequential decisions made by teachers (Bhargava 2007) and speed daters (Bhargava and Fisman 2014). In a financial setting, Hartzmark and Shue (2016) demonstrate that prior-day earnings announcements of other firms negatively correlate with stock-price reactions to contemporaneous announcements.
cal notion of payoffs—and gain-loss utility, which is proportional to the difference between her realized consumption utility and what she expected. Similar to a bandit problem, we assume the outcome from each action may be stochastic and that the person is initially uncertain about the distribution of outcomes associated with each action. She updates her beliefs about these distributions based on past experience.

Within this setting, we consider a decision maker who neglects the extent to which her past experiences were influenced by her reference point. Her utility from outcome $x \in \mathbb{R}$ when expecting $r \in \mathbb{R}$ is $u(x|r) = x + \eta n(x|r)$, where her gain-loss utility $n(x|r)$ is proportional to the difference between $x$ and $r$ and parameter $\eta > 0$ measures the impact of gains and losses on utility. We assume that a misattributor wrongly infers from past utility as if she weighted gains and losses by a diminished factor $\hat{\eta} < \eta$. While she correctly recalls how she felt after each outcome $x$, she neglects how gains and losses influenced her assessment of $x$ and thus encodes a distorted perception of that value. This implies that when $x$ beats expectations, she encodes a value $\hat{x} > x$, and when $x$ falls short of expectations, she encodes $\hat{x} < x$. The decision maker is otherwise rational and updates according to Bayes’ Rule as if outcome $\hat{x}$ truly occurred. We conclude Section 2 by discussing evidence of attribution errors more broadly. In particular, we highlight our experimental companion paper, Bushong and Gagnon-Bartsch (2016), which finds evidence consistent with the model above.

In Section 3, we analyze a few simple examples that demonstrate how our model leads to mistaken beliefs in the most basic environments. We first consider a deterministic setting in which a consumer repeatedly experiments with a new product that has the same quality each round. A misattributor will perceive false variation in quality over the short run, but will eventually learn correctly if she continues to use the good. However, mistakes will persist in stochastic environments. For instance, suppose that sometimes the person’s desired service (e.g., Uber) is unavailable and she is stuck with a lesser alternative (e.g., a taxi). When a misattributor finds out which option she will receive just prior to consuming it, she will persistently overvalue her preferred option and undervalue the fall-back. Furthermore, the less likely she is to receive the good outcome, the more she overvalues it.

Section 4 explores more generally how misattribution influences long-run perceptions and choices over actions that have stochastic outcomes—e.g., services with fluctuating quality or risky investments. The interplay between beliefs and realized utility prevents a misattributor from reaching

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5 When a person learns about her upcoming consumption is important in our model. If she learns which outcome she will face sufficiently ahead of time, then her reference point adapts to that particular outcome. We could extend our model, however, to incorporate misattribution of news utility (Kőszegi and Rabin 2009). In that case, a person misencodes signals about future consumption in the same way that she misencodes current consumption. While this extension preserves many of our basic insights, it introduces both new complications and comparative statics. We discuss this extension in the conclusion.
correct expectations despite ample experience. As such, we characterize her steady-state beliefs about the distribution of outcomes associated with each of her actions in terms of their true mean and variance. A misattributor will perceive a distribution that is excessively variable and, due to loss aversion, negatively skewed relative to the true distribution. Hence, she underestimates a prospect’s mean outcome. Increasing the true variability of a prospect causes a misattributor to underestimate its mean by a greater extent, as increased variability amplifies sensations of loss on average. Importantly, because riskier actions induce more biased, pessimistic beliefs, a misattributor excessively avoids risk and potentially incurs large welfare losses.6

Section 5 analyzes how the order in which a misattributor experiences outcomes can distort her perceived benefit of an action. For instance, consider a worker who updates her beliefs about a collaborator’s productivity each day. Even when the colleague’s performance is i.i.d., misattribution can generate a recency bias—recent outcomes influence beliefs more than older ones.7 This stems from the contrast effect inherent in our form of misattribution: low initial outcomes that reduce expectations cause later outcomes to be seen in better light, while high initial outcomes that increase expectations cause later outcomes to be judged more harshly. When misattribution is not too extreme, this result implies that a misattributor forms the highest estimate of an action’s benefit following an increasing sequence of outcomes. That is, fixing the amount of work the colleague actually completes, the misattributor is most optimistic about her partner’s productivity when each of his contributions is greater than the last.8

Section 5 also explores how misattribution generates persistent overly-extrapolative forecasts in environments with autocorrelated outcomes. When today’s outcome beats expectations, a misattributor exaggerates that outcome and expects higher outcomes in the future. However, since she

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6Malmendier and Nagel (2011) document increased apparent risk aversion as investors exit the stock market in response to adverse personal experiences. They argue this stems from biased beliefs rather than altered preferences, and later work (Malmendier and Nagel 2016) provides more direct support for the belief channel. A related literature attempts to explain such phenomena by positing that risk preferences depend directly on a decision maker’s history of elations and disappointments. While models of history-dependent risk attitudes such as Dillenberger and Rozen (2015) help explain behavior in situations where there is nothing to learn and hence biased beliefs have little bite (e.g., Thaler and Johnson 1990), these models additionally imply a “primacy effect”: early experiences have a greater impact on behavior than later experiences. This implication stands at odds with the recency effect documented by Malmendier and Nagel (2011). Our model additionally captures a recency effect.

7Such a “recency effect” is a well-known sequencing bias that has been documented in a range of economic decisions, such stock-market participation and hiring decisions (Highhouse and Gallo 1997). Recent models (e.g., Malmendier, Pouzo, and Vanasco 2017) incorporate a recency effect into learning models, demonstrating how it helps explain phenomenon such as excessive volatility in asset prices.

8A literature studying the role of improving sequences on perceptions demonstrates that, fixing the set of outcomes, people tend to form the most optimistic impressions when they experience outcomes in increasing order (e.g., Ross and Simonson 1991; Haisley and Loewenstein 2011). However, such a recency bias stands at odds with confirmation bias (Rabin and Schrag 1999; Fryer, Harms, and Jackson 2015). The mechanisms behind these two errors are not mutually exclusive. Indeed, Hogarth and Einhorn’s (1992) meta-study on order effects finds strong support for both recency and confirmatory effects. Which effect dominates depends on the type of feedback received: confirmatory effects dominate as evidence becomes more ambiguous and difficult to interpret.
holds unrealistically optimistic expectations, the next outcome typically comes as a disappointment. Misattributing this negative surprise swings her beliefs toward unrealistically pessimistic expectations. This pattern will continue over time: the person forms exaggerated forecasts in the direction of the most recent outcome, which leads to subsequent “surprises” in the opposite direction. Our basic prediction of extrapolative and volatile forecasts accords with a range of evidence, including Greenwood and Shleifer (2014) and Gennaioli, Ma, and Shleifer (2015) who find that investors’ and managers’ predictions of their future earnings exhibit forecast errors that negatively relate with past performance.9

In our first application, presented in Section 6, we consider scenarios where the decision maker can pay (through costly effort, money, etc.) to insure herself against bad states (e.g., the loss of crops due to adverse weather or disease). With misattribution, taking measures to reduce the chance of a bad outcome endogenously increases the person’s demand for such insurance, leading her to take more extreme measures over time. Roughly put, the more effort she exerts to reduce the chance of bad states, the more optimistic are her expectations. However, compared to these new expectations, a bad outcome (when it does occur) will seem worse than it previously did. As such, a misattributor overestimates the benefit from further reducing the chance of bad outcomes.

Finally, the order effects implied by our model suggest that a sophisticated agent may be able to strategically manipulate outcomes to influence the evaluations of a misattributor. Section 7 explores this idea in a career-concern setting where a misattributing (but otherwise rational) principal sequentially updates her beliefs about an agent’s ability and compensates him based on those inferences. For instance, imagine a client learning the ability of a contractor (the agent) over the course of several projects. While classical models like Holmström (1999) predict that the agent’s effort inefficiently declines over time, biased evaluations introduce new incentives that counter this result. Although the agent could surprise the principal with high effort today to increase tomorrow’s compensation, doing so generates a negative “internality” on subsequent compensation: it raises the principal’s expectations and causes her to judge later output more harshly. In response, a manipulative agent may initially under-perform relative to the principal’s expectations, but consistently beat them thereafter. This result matches common strategies of “expectations management” used in marketing, accounting, and politics.10

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9Our model applies to such settings involving money so long as earnings carry immediate hedonic consequences—a notion discussed in Kőszegi and Rabin (2009). While other models give rise to this general pattern of extrapolative beliefs and systematic reversals (e.g., Barberis, Shleifer, and Vishny 1998; Bordalo, Gennaioli, and Shleifer 2016), our model provides additional predictions which may help empirically disentangle the mechanism at play. For instance, we predict that these patterns are more pronounced when forecasting about one’s own earnings (i.e., outcomes carry utility consequences for the forecaster) and that beliefs respond more to bad news than good.

10For instance, research from accounting and finance finds that firms commonly use a variety of mechanisms to “walk down” investors’ expectations prior to earnings announcements—strategic accounting of working capital and cash flow (Burgstahler and Dichev 1997), sales (Roychowdhury 2006), or distorting analyst forecasts (e.g., Richardson, Teoh, and Wycoki 2004). Furthermore, Bartov, Givoly and Hayn (2002) argue that such efforts to meet or beat
We conclude in Section 8 by discussing how our model contributes to a growing literature on learning with misspecified models and by reviewing our model’s empirical predictions. In particular, we note ways that researchers can distinguish misattribution from statistical biases, such as base-rate neglect (e.g., Benjamin, Bodoh-Creed, and Rabin 2016), that share similar qualitative predictions. Finally, we present some natural extensions of our model. Although we study an agent who retrospectively misunderstands the source of her own utility, our model can be reframed as an interpersonal bias where an observer neglects how expectations shape the experiences of others. For instance, a person reading reviews for a class may fail to appreciate that a bad rating could reflect the reviewer’s high expectations rather than a low-quality professor. In addition to hindering social learning, failing to account for others’ expectations may have important implications for how policy makers interpret surveys measuring satisfaction. There is suggestive evidence that reported satisfaction with public services declines with expectations (e.g., James 2009; Van Ryzin 2004). If policy makers neglect the role of expectations in these reports, they may wrongly attribute such a decline to poor quality or changing tastes and consequently suggest ill-suited reforms.

2 A Model of Misattribution of Reference-Dependent Utility

This section presents our model of a decision maker who errs when learning from experience. We first introduce the decision maker’s reference-dependent utility function and then describe our general consumption and learning environment. Section 2.3 formalizes our novel assumption: the decision maker misattributes sensations of positive and negative surprise to the intrinsic value of an outcome, which leads to biased beliefs. We also discuss motivating evidence for the main assumptions of our model (see our experimental companion paper, Bushong and Gagnon-Bartsch 2016, for additional discussion).

2.1 Reference-Dependent Utility

We begin by specifying the agent’s reference-dependent utility from consumption bundle \( c = (c^1, \ldots, c^K) \in \mathbb{R}^K \). As in Kőszegi and Rabin (2006) (henceforth KR), we assume that overall utility has two components. The first component, “consumption utility”, corresponds to the

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11 Psychologists have discussed several forms of a misattribution spanning a broad range of concepts. However, to ease exposition, we slightly abuse this terminology and use “misattribution” (misattributor, etc.) to refer specifically to misattribution of reference dependence.

12 While Kahneman and Tversky’s (1979) model of reference-dependent preferences could be interpreted purely as a descriptor of choice, we and others interpret it as reflective of reference-dependent hedonics. For instance, Bell
outcome-based payoff traditionally studied in economics. We assume that consumption utility is additively separable across dimensions, and we let \( x^k \in \mathbb{R} \) denote the consumption utility from \( c^k \).\(^{13}\) The second component, “gain-loss utility”, derives from comparing \( x^k \) with that dimension’s reference level of utility, denoted \( r^k \in \mathbb{R} \). When the reference point is deterministic, gain-loss utility—denoted by \( n(x^k | r^k) \)—is given by

\[
n(x^k | r^k) = \begin{cases} 
    x^k - r^k & \text{if } x^k \geq r^k \\
    \lambda (x^k - r^k) & \text{if } x^k < r^k.
\end{cases}
\]

(1)

We assume gain-loss utility is piecewise linear with weight \( \lambda \geq 1 \) on losses.\(^{14}\) Although we often assume loss aversion with \( \lambda > 1 \), we sometimes consider \( \lambda = 1 \) to highlight results independent from loss aversion. The person’s total utility from \( x \) given reference point \( r \) is

\[
u(x | r) = \sum_{k=1}^{K} \left\{ x^k + \eta n(x^k | r^k) \right\},
\]

(2)

where parameter \( \eta \geq 0 \) determines the weight given to sensations of gains and losses relative to absolute outcomes.

Our basic formulation of misattribution, introduced below, can be applied using any definition of the reference point. We assume the person’s reference point is her recent expectation about consumption along each dimension. This assumption seems intuitive in settings where a person learns about consumption values over time and is supported by a range of evidence.\(^{15}\)

We consider two ways to extend the gain-loss utility in Equation 1 to capture expectations-based reference points. Suppose the person believes \( x \) is drawn according to the probability measure \( F \). Our first specification of gain-loss utility assumes a deterministic reference point: following Bell (1985), \( x^k \) is interpreted as a gain whenever \( x^k \geq \mathbb{E}_F [x^k] \) and a loss otherwise. Equation 1 is thus

\(^{13}\)We implicitly assume that consumption utility \( x^k \) derives from a classical Bernoulli utility function \( m : \mathbb{R} \rightarrow \mathbb{R} \) such that \( x^k = m(c^k) \). However, we simply write \( x^k \) to reduce notational clutter.

\(^{14}\)This gain-loss utility function follows the specification of KR under their Assumption A3’. While this specification shares many similarities with Kahneman and Tversky’s (1979) value function, it abstracts from issues of diminishing sensitivity and probability weighting in order to highlight the role of loss aversion. Many of our basic qualitative results continue to hold with diminishing sensitivity.

\(^{15}\)A number of recent experimental studies find evidence of expectations-based reference points (e.g., Abeler et al. 2011; Gill and Prowse 2012; Karle, Kirchsteiger, and Peitz 2015; Sprenger 2015). There is additionally evidence from the field spanning labor supply among taxi drivers (Crawford and Meng 2011), domestic violence resulting from unexpected football losses (Card and Dahl 2011), and decisions in game shows and sports (Post et al. 2008; Pope and Schweitzer 2011; Markle et al. 2015).
given by

\[
n(x^k|F) = \begin{cases} 
  x^k - \mathbb{E}_F[x^k] & \text{if } x \geq \mathbb{E}_F[x^k] \\
  \lambda(x^k - \mathbb{E}_F[x^k]) & \text{if } x < \mathbb{E}_F[x^k]. 
\end{cases}
\] (3)

where \( r^k = \mathbb{E}_F[x^k] \). Alternatively, as in KR, the reference point may be \textit{stochastic}: the sense of gain or loss derives from comparing outcome \( x \) with each counterfactual outcome that was possible given \( F \). As KR note, when a person expects a gamble between $0 or $100, receiving $50 may feel like a gain relative to $0 and a loss relative to $100. Formally, KR assume

\[
n(x^k|F) \equiv \int_{x^k < x} (x^k - x^k) \, dF(x) + \lambda \int_{x^k \geq x} (x^k - x^k) \, dF(x),
\] (4)

meaning outcome \( x^k \) is compared against each hypothetical outcome along that dimension and this comparison is weighted by the probability of the hypothetical outcome.\(^{16}\)

For tractability, we assume deterministic reference points (Equation 3) throughout the paper unless otherwise noted. At several points, however, we discuss the relationship between these two assumptions and emphasize that our main qualitative predictions hold under either approach.\(^{17}\)

### 2.2 Consumption and Learning Environment

We analyze a decision maker who faces a series of choices over time. The consumption utility associated with each of her available actions is stochastic, and she updates her beliefs about these payoffs from experience. These beliefs form the expectations from which she derives reference-dependent utility.

Each period \( t \in \{1,2,\ldots,T\} \) is structured as follows. First, the person chooses an action \( a_t \) from set \( \mathcal{A} \equiv \{1,\ldots,N\} \). Second, an outcome \( x_t \equiv (x_t(1),\ldots,x_t(N)) \in \mathbb{R}^{K \times N} \) is realized, where each element \( x_t(a) = (x^1_t(a),\ldots,x^K_t(a)) \) is the \( K \)-dimensional vector of consumption utility achieved from taking choice \( a \) that period. Third, the person earns total utility \( u_t \in \mathbb{R} \) that depends on both the outcome \( x_t(a) \) and her expectations of \( x_t(a) \). In terms of the introductory examples, \( a_t \) might be a traveler’s chosen airline for trip \( t \) and \( x_t(a) \in \mathbb{R} \) is the quality of service, or \( a_t \) might be a researcher’s chosen collaborator and \( x_t(a) \) is that colleague’s contribution.

For each action \( a \in \mathcal{A} \), the outcome \( x_t(a) \) depends on an unknown payoff-relevant parameter \( \theta(a) \) drawn from set \( \Theta(a) \subseteq \mathbb{R} \).\(^{18}\) Aside from our treatment of autocorrelated outcomes in Section 5.2,

\(^{16}\)Gul (1991) suggests a third specification of an expectations-based reference points that is recursively defined as the certainty equivalent of a lottery which yields disappointment relative to that certainty equivalent. In our settings, this specification is qualitatively similar to the deterministic reference points described above.

\(^{17}\)Note that if either \( \lambda = 1 \) or if the reference distribution \( F \) is degenerate, then the stochastic reference-point model reduces to the deterministic model.

\(^{18}\)For convergence results in Section 4, we assume \( \Theta \) is compact.
we assume that, conditional on $\theta(a)$, $x_t(a)$ is i.i.d. with c.d.f. $F^a(\cdot|\theta(a))$. For example, we often consider environments with $x_t(a) = \theta(a) + \varepsilon_t(a)$ where $\varepsilon_t(a)$ are i.i.d. normal shocks. In this case, the collection of parameters $\theta \equiv (\theta(1), \ldots, \theta(N))$ represents, for instance, the average quality of each available product or the mean productivity of each potential colleague.

We assume each parameter $\theta(a)$ is independent of $\theta(b)$ for all $b \in \mathcal{A}$, $b \neq a$, and the person begins with a full-support prior $\pi_1^a \in \Delta(\Theta(a))$ over each $a \in \mathcal{A}$. She updates this belief after each experience and $\pi_t^a \in \Delta(\Theta)$ denotes her beliefs entering period $t$. Critically, we assume these beliefs form her reference point in round $t$: given action $a$, her reference distribution is her subjective belief over $x_t(a)$ given $\pi_t^a$, denoted by $\tilde{F}_t(x|a) \equiv \int_{\theta(a)} F^a(x|\theta(a))d\pi_t^a(\theta(a))$. Given action $a_t = a$, the person earns total utility $u(x_t(a)|\tilde{F}_t(\cdot|a))$ given by Equations 2 and 3.

Unless otherwise noted, we assume that the decision receives information in round $t$ only about the action she chooses that round. However, at times we consider when a person observes the counterfactual outcomes of actions she did not take. Such settings, which we say have full observability, allow us to explore how a misattributor updates from outcomes that carry utility consequences versus those that do not. We analyze full observability primarily in Section 4.2.

### 2.3 Misattribution of Gain-Loss Utility

We now turn to the central assumption of the model: the decision maker neglects how her past experiences were influenced by reference dependence, and misattributes her gain-loss utility to an action’s underlying consumption utility. For instance, consider the workplace example where the worker’s colleague contributes more than expected. While the worker recalls a positive experience with her colleague, she neglects that her utility derived from beating expectations in addition to the colleague’s absolute level of effort. In our framework, this mistake causes the worker to recall an inflated sense of her colleague’s effort and form overly high expectations going forward.

Recall the decision maker seeks to learn the distribution of outcomes associated with an action $a$, $F^a(\cdot|\theta(a))$. A rational updater faced with signal $u_t = x_t(a) + \eta n(x_t(a)|\tilde{F}_t(\cdot|a))$ understands that signal is “contaminated” by a gain-loss term, and properly accounts for this when using $u_t$ to update her beliefs about $\theta(a)$. We assume that a misattributor errs in this step: she infers from $u_t$ as if her utility function weights gains and losses by a diminished factor $\hat{\eta} \in [0, \eta)$. Hence, a misattributor treats signal $u_t$ as if $u_t = x_t(a) + \hat{\eta} n(x_t(a)|\tilde{F}_t(\cdot|a)) \equiv \hat{u}(x_t(a)|\tilde{F}_t(\cdot|a))$, where $\hat{\eta} < \eta$. After each period, the person uses her (correct) memory of $u_t$ along with her misspecified model of utility $\hat{u}$ to infer the outcome $x_t(a)$ she must have received. We denote this *encoded outcome* by

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19 Whether or not the decision maker observes elements of $x_t$ beyond $x_t(a_t)$ depends on the particular application. In some scenarios, observing the outcomes of counter-factual decisions seems realistic (e.g., betting markets), while in others it does not (e.g., experimental consumption).
\[ \hat{x}_t(a) = (\hat{x}_t^1(a), \ldots, \hat{x}_t^K(a)) \] and the misinference described above implies that \( \hat{x}_t(a) \) solves

\[ u(x_t(a)|\hat{F}_t(\cdot|a)) = u_t = \hat{u}_t(\hat{x}_t(a)|\hat{F}_t(\cdot|a)). \]  

We assume that along each dimension \( k \), this misencoded value of \( x_t(a) \) depends solely on gains and losses on that dimension. Thus, \( \hat{x}_t^k(a) = \hat{u}_t^{-1} \left( u_t^{k} | \hat{F}_t^k(\cdot|a) \right) \), where \( \hat{F}_t^k(\cdot|a) \) is the marginal belief about \( x_t^k \).\(^{20}\)

Roughly put, the person’s incorrect model of her past utility understates the degree to which \( u_t \) derives from gain-loss utility. Any gain-loss utility the decision maker fails to account for is attributed to the intrinsic consumption value associated with her choice.

Finally, we assume the person is unaware of her misencoding and uses \( \hat{x}_t(a) \) along with Bayes’ rule to update her beliefs \( \pi_t^a \) over \( \theta(a) \). To ensure the learning problem is well-defined, we assume that for all \( a \in \mathcal{A} \) and \( \theta(a) \in \Theta(a) \), \( x_t(a) \) has full support over \( \mathbb{R}^K \). This assumption implies both that a single observation cannot perfectly rule out a parameter and that a misattributor never observes outcomes she believed impossible.\(^{21}\)

In order to explore implications of misattribution, we first note that encoded outcomes take a simple form. Letting \( \hat{E}_t[x_t^k(a)] \) denote the person’s expectation of \( x_t^k(a) \) with respect to subjective beliefs \( \hat{F}_t(\cdot|a) \), then

\[ \hat{x}_t^k(a) = \begin{cases} 
    x_t^k(a) + \kappa^G \left( x_t^k(a) - \hat{E}_t[x_t^k(a)] \right) & \text{if } x_t^k(a) \geq \hat{E}_t[x_t^k(a)] \\
    x_t^k(a) + \kappa^L \left( x_t^k(a) - \hat{E}_t[x_t^k(a)] \right) & \text{if } x_t^k(a) < \hat{E}_t[x_t^k(a)],
\end{cases} \]  

where

\[ \kappa^G \equiv \left( \eta - \hat{\eta} \right) \quad \text{and} \quad \kappa^L \equiv \lambda \left( \eta - \hat{\eta} \right). \]  

The parameters \( \kappa^G \) and \( \kappa^L \) represent the extent that elations and disappointments, respectively, distort encoded outcomes. Intuitively, \( \kappa^G \) and \( \kappa^L \) increase in the extent of misattribution—i.e., as \( \hat{\eta} \) decreases.

The simple form above (Equation 6) offers several immediate implications. First, the misattributor overreacts to surprising outcomes:

**Observation 1. Outcomes that beat expectations are distorted upward, while those that fall short are distorted downward.** If \( x_t^k(a) > \hat{E}_t[x_t^k(a)] \), then \( \hat{x}_t^k(a) > x_t^k(a) \), and if \( x_t^k(a) < \hat{E}_t[x_t^k(a)] \), then \( \hat{x}_t^k(a) < x_t^k(a) \).

\(^{20}\)The encoded value \( \hat{x}_t^k(a) \) is well defined and unique: fixing any belief \( \hat{F}_t^k(\cdot|a) \), the utility function \( \hat{u}_t \) is strictly increasing in \( \hat{x}_t^k(a) \).

\(^{21}\)Full support is stronger than necessary to ensure that seemingly-impossible outcomes never occur. We maintain this stronger assumption for simplicity: the necessary conditions depend on properties of both the set of possible distributions \( \{ F^\theta(\cdot|a) \}_{\Theta(a) \in \Theta(a)} \) and the parameters, \((\eta, \hat{\eta}, \lambda)\).
Second, when \( \lambda > 1 \), disappointments and elations distort encoded outcomes (and hence beliefs) asymmetrically:

**Observation 2.** *Losses are misencoded by more than equivalently sized gains.* Suppose \( \lambda > 1 \). Consider outcomes \( g^k(a) \equiv \hat{E}_t[x^k_t(a)] + \varepsilon \) and \( l^k(a) \equiv \hat{E}_t[x^k_t(a)] - \varepsilon \). For any \( \varepsilon > 0 \), \( |\hat{g}^k(a) - \hat{l}^k(a)| > |g^k(a) - g^k(a)| \).

Additionally, our model implies a difference in learning from outcomes with direct utility consequences versus observations that contain identical information but do not directly influence payoffs. In particular, given that misattribution stems from a misunderstanding of the source of utility, only those outcomes that incite sensations of elation or disappointment are prone to misencoding.\(^{22}\)

To illustrate these observations, consider a person experimenting with a new physical-therapy technique to reduce pain. Let \( x \) measure the effectiveness of the treatment (in utils), and suppose the patient expects \( x = 50 \). First, imagine \( x = 60 \)—it works better than expected. To decide whether to use this treatment again, she infers the treatment’s efficacy \( x \) from her experienced \( u = 60 + \eta n(60|50) = 60 + \eta 10 \). While she correctly recalls a pleasant experience, she fails to properly disentangle the consumption value of the treatment from the elation due to surprise. From Equation 6, the patient recalls value \( \hat{x} \) such that

\[
\hat{x} = 60 + \left( \frac{\eta - \hat{\eta}}{1 + \hat{\eta}} \right) 10 > x.
\]

If, for instance, \( \eta = 1 \) and \( \hat{\eta} = 1/3 \), then \( \hat{x} = 65 \). Contrastingly, imagine \( x = 40 \) and thus the treatment falls short of expectations. In this case, she encodes a value

\[
\hat{x} = 40 - \lambda \left( \frac{\eta - \hat{\eta}}{1 + \lambda \hat{\eta}} \right) 10 < x.
\]

Again, if \( \eta = 1 \), \( \hat{\eta} = 1/3 \) and \( \lambda = 3 \), then \( \hat{x} = 30 \).

Turning to a description of choice, we assume the decision maker maximizes her true utility function conditional on her erroneous beliefs. We assume the person’s expectations about consumption utility in round \( t \) adjust according to her preceding action. Hence, she maximizes expected utility assuming that, conditional on choice \( a_t \), her reference point will be \( \hat{F}_t(\cdot|a_t) \). This means her reference distribution does not depend on the distributions of foregone alternatives.\(^{23}\)

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\(^{22}\)Charness and Levin (2005) find significantly greater errors in updating about the distribution of balls in an urn when participants observe a sample of draws that have payoff consequences relative to the case in which these signals have no payoff consequences. Their experiment suggests that the affect induced by payments is a critical factor in deviations from Bayesian updating.

\(^{23}\)This solution concept corresponds to KR’s (2007) notion of “choice-acclimating personal equilibrium” aside from the fact that we do not impose rational expectations: equilibrium is with respect to the misattributor’s biased subjective beliefs.
More formally, with full observability—which implies that the optimal dynamic strategy calls for
the myopically-optimal action each round—each action $a_t \in \mathcal{A}$ satisfies

$$
\widehat{E}_t \left[ u(x(a_t)|\widehat{F}_t(\cdot|a_t)) \right] \geq \widehat{E}_t \left[ u(x(b)|\widehat{F}_t(\cdot|b)) \right] \forall b \in \mathcal{A}.
$$

While the above specification of reference points assumes that the chosen action is implemented
deterministically, we can easily extend the model to allow for stochastic choice. This captures
scenarios where the environment adds randomness over the implemented action. For instance,
consider a diner who plans to eat at Restaurant 1 whenever there are open tables, and at Restaurant
2 otherwise. The choice to dine out corresponds to a probability distribution $p_t \in \Delta(\mathcal{A})$ over the
(pure) action she will ultimately realize that round. In such cases, the reference distribution that
enters the person’s utility function is thus $\widehat{F}_t(x|p_t) = \sum^N_{a=1} p_t(a) \widehat{F}_t(x|a)$—the compound lottery
comprised of the reference distribution she would face for each realization $a$, weighted by $p_t(a)$.
For instance, if each night there is 50% chance that Restaurant 1 is full, then her reference point is
a 50-50 mix of what she expects from Restaurant 1 and Restaurant 2.

There are at least two natural interpretations of misencoding outcomes under our model. One
is that the person observes her experienced utility $u_t$ each period, but cannot directly observe the
underlying outcome $x_t$. Hence, $x_t$ must be inferred from $u_t$ and the misattributor errs in doing so.
Another interpretation is that misattribution occurs even when the person can observe $x_t$. Because
the decision maker is unaware that she mistakenly infers $x_t$ from $u_t$, she may treat $u_t$ as a sufficient
statistic for updating her beliefs. Hence, she may “rationally” (from the point of view of her
misspecified model) ignore outcome $x_t$ even when it is observable. Under this interpretation, the
decision maker learns based on how an experience feels without attending to specific outcomes.

While few studies empirically explore misattribution of reference dependence, our companion
paper (Bushong and Gagnon-Bartsch 2016) provides supporting evidence. In that paper’s main
experiment, participants completed one of two unfamiliar tasks: either a neutral task or the same
task with an unpleasant noise played in the background. Hours after they sampled the task and
formed initial impressions, we elicited participants’ willingness to continue working (WTW) on
the same task for additional pay. Identification came from manipulating participants’ expectations
about which task they would face: participants in the control group had no uncertainty over their
task, while each participant in the treatment group determined her task by flipping a coin just
before working. In terms of our model, a treatment participant had a reference point that combined

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This is an application of Gagnon-Bartsch, Rabin, and Schwartzstein (2017), who argue that misspecified models
of the world may persist when a person uses an erroneous theory to guide how she allocates attention. If the decision
maker wrongly believes some variable is redundant given other data she has noticed, then she may rationally (from
the viewpoint of her misspecified model) ignore it. By disregarding those seemingly useless variables, a misattributor
would not be able to discover her mistaken model.
both the neutral and bad tasks, meaning the job she ultimately faced came as either a positive or negative surprise (depending on the outcome). Contrastingly, control participants faced no surprise. Relative to the control group, the treatment group exhibited greater WTW on the neutral task and decreased WTW on the unpleasant task: when the stakes were highest, the WTW of treatment participants was roughly 20% higher (neutral task) and 25% lower (unpleasant task) than the WTW of control participants. These results (along with various robustness checks) are consistent with misattribution of elation and disappointment to the intrinsic enjoyment of the task.

Other forms of misattribution have been discussed in the economics literature. Many resemble the “fundamental attribution error” or “correspondence bias” in psychology (e.g., Ross 1977; Gilbert and Malone 1995), where transient situational factors are incorrectly attributed to underlying, stable characteristics of a person or good. For example, Haggag and Pope (2016) show that experimental participants valued an unfamiliar drink more when they first experienced it while thirsty. Additionally, they find that frequent patrons of an amusement park whose most recent visit was during good weather are more likely to return. In two papers, Simonsohn (2007, 2010) explores the effect of a transient shock (weather) on the subsequent behavior of prospective college students and admissions officers. Simonsohn (2007) demonstrates that applicants with particularly strong academic qualities were evaluated higher by admissions officers when the weather on that evaluation day was poor, and Simonsohn (2010) shows that incoming freshman are more likely to matriculate at an academically rigorous school when the weather on their visit day was cloudy versus sunny. The author interprets both results as a form of attribution bias, though the specific channel is unclear. Relatedly, a series of papers show that CEOs (Bertrand and Mullainathan 2003) and politicians (Wolfers 2007; Cole, Healy, and Werker 2012) are rewarded for luck. Our model shares a common structure with these forms of misattribution, whereby transient sensations—elation and disappointment in our case—are misattributed.

3 Illustrative Examples

This section analyzes two examples that demonstrate how misattribution can bias learning in even the simplest environments. We first consider a decision maker who repeatedly experiments with a single action—for instance, a new product or service—that returns the same outcome each period. While a rational actor learns quickly in this environment, a misattributor will perceive fictitious variation in outcomes. We then consider a stochastic environment in which the person receives at random one of two possible outcomes each period. For example, suppose that a consumer’s preferred product is not always available—some periods she receives her desired good, while others she is stuck with a lesser alternative. A misattributor will overvalue her preferred option and undervalue the fall-back, and the less likely she is to receive her preferred outcome, the more she
Consider a consumer learning how much she enjoys a new service, such as Uber. To make the example stark, suppose there is no variation in the quality of Uber: each ride yields a fixed consumption value \( x \in \mathbb{R} \). Imagine that the person’s first ride surpasses her expectations. When the person misattributes the sensation of positive surprise to Uber’s quality, she perceives a value \( \hat{x}_1 > x \). This impression lifts her expectations prior to her second ride, causing the exact quality she experienced the first day to seem less impressive relative to these new beliefs. She thus perceives a drop in quality, \( \hat{x}_2 < \hat{x}_1 \). Hence, even with constant outcomes, the misattributor makes an error: she perceives variability where there is none. This perceived variability, however, vanishes with ample experience, and a misattributor would correctly learn Uber’s quality if she continued to use the service.

The fact that a misattributor would eventually learn in the example above hinges on the deterministic nature of outcomes. Any randomness in outcomes will cause a misattributor to persistently mislearn, which we emphasize throughout the paper. Following the variant of our model where actions are implemented probabilistically, imagine that Uber (option \( H \)) is available in any given period with probability \( p \in [0, 1] \). When Uber is unavailable, the person settles for a cab (option \( L \)). Suppose that the daily quality of option \( a \in \{H, L\} \) is \( x_i(a) = \theta(a) + \varepsilon_i(a) \), where \( \varepsilon_i(a) \) are i.i.d. normally-distributed shocks with mean zero and variance \( \sigma^2 \). As above, we focus on the noiseless limit where \( \sigma \to 0 \), so the consumption value of option \( a \) is essentially constant at \( \theta(a) \).

Despite correct priors, if the outcome comes as a surprise each period, then her estimates of \( \theta(H) \) and \( \theta(L) \) will grow polarized over time. On the first ride, her expectations incorporate a \( p \) chance of Uber and a \( 1 - p \) chance of a cab—ex ante, she expects consumption utility \( p\theta(H) + (1 - p)\theta(L) \). Since Uber’s quality would come as a pleasant surprise relative to a cab, she encodes an inflated value \( \hat{x}_1 = \theta(H) + \kappa^G[\theta(H) - p\theta(H) - (1 - p)\theta(L)] = \theta(H) + (1 - p)\kappa^G[\theta(H) - \theta(L)] > \theta(H) \). This raises her expectations of Uber. Contrastingly, since a cab’s quality would

\[ \hat{\theta}_t(a) = \frac{\rho^2}{N_t(a)} \sum_{i=1}^{N_t(a)} \hat{x}_i(a) + \frac{\sigma^2}{N_t(a)\rho^2 + \sigma^2} \theta_0(a). \]

Fixing \( \rho \) and taking \( \sigma \to 0 \) implies that the estimate above converges to the sample average of option \( a \)’s encoded performances.
come as a disappointing surprise, she would encode a deflated value \( \hat{x}_1 = \theta(L) - p\kappa^L[\theta(H) - \theta(L)] < \theta(L) \). Hence, no matter her first outcome, she will overestimate the quality difference between the two options.\(^{26}\)

Furthermore, this initial mistake tends to reinforce itself. As the perceived quality difference increases, Uber comes as a greater elation while a cab comes as a greater disappointment. As such, the consumer persistently exaggerates the quality difference and converges to steady-state estimates, denoted \( \hat{\theta}(H) \) and \( \hat{\theta}(L) \), such that \( \hat{\theta}(H) > \theta(H) \) and \( \hat{\theta}(L) < \theta(L) \).\(^{27}\) To see this, note that \( \hat{\theta}(H) \) and \( \hat{\theta}(L) \) must satisfy the following: when expecting a \( p \) chance of \( \hat{\theta}(H) \) and a \( 1 - p \) chance of \( \hat{\theta}(L) \), the outcome \( \theta(H) \) is encoded as \( \hat{\theta}(H) \) and \( \theta(L) \) is encoded as \( \hat{\theta}(L) \). Solving the system of equations implied by these conditions yields long-run beliefs

\[
\begin{align*}
\hat{\theta}(H) & = \theta(H) + G(p, \eta, \lambda, \hat{\eta})[\theta(H) - \theta(L)] \quad (9) \\
\hat{\theta}(L) & = \theta(L) - L(p, \eta, \lambda, \hat{\eta})[\theta(H) - \theta(L)],
\end{align*}
\]

where

\[
G(p, \eta, \lambda, \hat{\eta}) \equiv \frac{(1 - p)\kappa^G(1 + \kappa^L)}{1 + p\kappa^G + (1 - p)\kappa^L} \quad \text{and} \quad L(p, \eta, \lambda, \hat{\eta}) \equiv \frac{p\kappa^L(1 + \kappa^G)}{1 + p\kappa^G + (1 - p)\kappa^L}. \quad (10)
\]

Additionally, because of loss aversion, the consumer’s expected consumption utility from hailing a ride, \( p\hat{\theta}(H) + (1 - p)\hat{\theta}(L) \), is strictly less than the true expected value. These findings foreshadow our more general long-run results in the next section: a misattributor perceives a distribution of outcomes that is more disperse and negatively-skewed relative to the true distribution, causing her to undervalue such prospects.\(^{28}\)

Interestingly, the misattributor overestimates Uber’s quality by more when its availability \( p \) decreases. As taking Uber becomes less likely, the rider experiences a greater elation when catch-

\(^{26}\)Although we share some motivations with Haggag and Pope (2016), misattributors in their model tend to underestimate the payoff difference between two outcomes. Furthermore, unlike mistakes driven by misattribution of reference dependence, biased forecasts in Haggag and Pope’s formulation vanish with experience. This distinction stems from the fact that Haggag and Pope rule out complementaries where past experiences influence today’s consumption utility. Reference dependence clearly introduces this complementarity, as past experiences form the reference point against which today’s consumption is evaluated.

\(^{27}\)For sake of clarity, this intuition sidesteps some features of the dynamics. For instance, if the person takes Uber multiple times in a row, she perceives variation in Uber’s quality as discussed in the single-outcome example above. Roughly put, estimates of Uber’s quality fluctuate about an overestimated mean, while those of a cab fluctuate about an underestimated mean. As these fluctuations diminish over time, the person reaches a stable, yet exaggerated, perception of the quality difference.

\(^{28}\)The consumer forms overly pessimistic forecasts about the utility she will earn from her daily commute. To see this, note that her perceived expected utility of the gamble is \( \mathbb{E}[u] = p\hat{\theta}(H) + (1 - p)\hat{\theta}(L) + p(1 - p)\eta(\lambda - 1)[\hat{\theta}(H) - \hat{\theta}(L)] \). Using the solutions for \( \hat{\theta}(H) \) and \( \hat{\theta}(L) \) above (Equation 9), \( \mathbb{E}[u] = \mathbb{E}[u] - p(1 - p)(\kappa^L - \kappa^G)[\theta(H) - \theta(L)]/(1 + p\kappa^G + (1 - p)\kappa^L) \), where \( \mathbb{E}[u] \) is the expected utility under rational expectations. The bias in expectations vanishes as uncertainty diminishes (\( p \to 0 \) or \( p \to 1 \)).
ing an Uber and consequently overestimates its perceived quality by more. Similarly, when \( p \) increases, she underestimates a cab’s quality by more. This result highlights how randomness in the way a good is allocated can have important consequences on its perceived value and thus implications for firm behavior. For instance, if word of mouth has a strong influence on sales, a firm may choose to limit supply when first introducing a high-quality product with unit demand: those lucky enough to receive the good early may overstate its quality, thereby increasing demand among the second wave of consumers.

As noted above, a misattributor systematically exaggerates the disutility from unlikely losses. Hence, she may naturally over-pay to insure against them. We return to this binary-outcome example in Section 6 to explore such situations where a person can take measures to reduce the probability of the bad outcome (i.e., increase \( p \)).

4 Long-Run Beliefs and Behavior

The previous examples highlighted how stochastic outcomes can distort a misattributor’s beliefs. This section analyzes more generally the beliefs and the behavior of a misattributor following ample experience with actions that yield stochastic outcomes. This section describes a misattributor’s beliefs and behavior following ample experience with actions that yield stochastic outcomes. For instance, imagine a person learning about the typical quality of an oft-used service, the returns from an investment, or the productivity of colleagues with whom she regularly works. We characterize a misattributor’s steady-state beliefs about the distribution of such outcomes and highlight how these beliefs depend on the true distribution and the person’s preferences. A main result is that a misattributor will underestimate an action’s mean outcome and overestimate its variability. Importantly, these misperceptions undervalue a prospect in proportion to its true variance, leading the agent to turn down some risky-but-optimal actions. Additionally, since only those outcomes with utility consequences distort a misattributor’s beliefs, she may perpetually cycle between actions in settings where she observes the counterfactual outcomes of foregone choices.

4.1 Long-Run Pessimism and Exaggerated Variance

Suppose each action \( a \in A \) returns unidimensional consumption utility \( x_t(a) = \theta(a) + \sigma(a)z_t \), where each \( z_t \) is an i.i.d. realization of a mean-zero, unit-variance random variable \( Z \) with a continuously differentiable distribution \( F_Z \) and density \( f_Z \). Parameters \( \theta(a) \) and \( \sigma(a) \) denote the true mean and standard deviation of outcomes, respectively. The learner begins with correct priors \( \pi_1^q \) about \( (\theta(a), \sigma(a)) \) on \( \mathbb{R} \times \mathbb{R}_+ \) and updates these beliefs based on experienced outcomes. While we turn to choice implications in the next subsection, we first examine biased learning when a person
takes the same action \( a \) each round. As such, we momentarily drop reference to \( a \) from parameters \( \theta(a) \) and \( \sigma(a) \).

How does a misattributor’s estimate of \( \theta \) evolve in response to outcomes \( x_t \)? Given an estimated mean \( \hat{\theta}_{t-1} \) entering round \( t \), a misattributor encodes an outcome \( \hat{x}_t = x_t + \kappa_t(x_t - \hat{\theta}_{t-1}) \), where \( \kappa_t \equiv \kappa^G \mathbb{1}_t \{ x_t \geq \hat{\theta}_{t-1} \} + \kappa^L \mathbb{1}_t \{ x_t < \hat{\theta}_{t-1} \} \). To describe the dynamics of \( \hat{\theta}_t \), we define the function \( G^a : \mathbb{N} \times \mathbb{R} \to \mathbb{R} \) as the expectation of \( \hat{x}_t - \hat{\theta}_{t-1} \) conditional on the true parameters:

\[
G^a(t, \hat{\theta}_{t-1}) = \mathbb{E}[\hat{x}_t | \theta] - \hat{\theta}_{t-1}.
\] (11)

\( G^a(t, \hat{\theta}_{t-1}) \) corresponds to the misattributor’s expected “surprise” resulting from outcome \( x_t \) as measured by an outside observer who knows both the true parameters and the misattributor’s subjective beliefs. Steady-state beliefs are thus a zero of \( G^a(\hat{\theta}) \equiv \lim_{t \to \infty} G^a(t, \hat{\theta}) \), corresponding to a belief at which the agent’s average surprise is zero. Intuitively, the misattributor cannot hold a long-run belief \( \hat{\theta} \) outside of \( \Gamma^a \equiv \{ \hat{\theta} : G^a(\hat{\theta}) = 0 \} \), as such a belief would generate encoded outcomes that push her systematically away from \( \hat{\theta} \): \( G^a(\hat{\theta}) > 0 \) implies her estimate of \( \theta \) drifts upward, while \( G(\hat{\theta}) < 0 \) implies it drifts downward. We call \( \hat{\theta} \in \Gamma^a \) stable when encoded outcomes at beliefs near \( \hat{\theta} \) push a misattributor’s expectations toward \( \hat{\theta} \). Formally, \( \hat{\theta} \in \Gamma^a \) is stable if there exists \( \varepsilon > 0 \) such that \( G^a(\theta') < 0 \) for \( \theta' \in (\hat{\theta}, \hat{\theta} + \varepsilon) \) and \( G^a(\theta') > 0 \) for \( \theta' \in (\hat{\theta} - \varepsilon, \hat{\theta}) \). Finally, the condition \( G^a(\hat{\theta}) = 0 \) requires that, while holding expectation \( \hat{\theta} \), the misattributor’s average encoded outcome equals \( \hat{\theta} \). Hence, \( \hat{\theta} = \theta + \kappa^G \mathbb{P}(x \geq \hat{\theta})(\mathbb{E}[x | x \geq \hat{\theta}] - \hat{\theta}) + \kappa^L \mathbb{P}(x < \hat{\theta})(\mathbb{E}[x | x < \hat{\theta}] - \hat{\theta}) \), which yields

\[
\hat{\theta} = \theta - \left( \frac{(\lambda - 1)(\eta - \hat{\eta})}{(1 + \eta)(1 + \hat{\eta} \lambda)} \right) \mathbb{P}(x < \hat{\theta}) \left( \hat{\theta} - \mathbb{E}[x | x < \hat{\theta}] \right).
\] (12)

Our next result first establishes that there exists a unique steady-state belief and it is stable. It then presents key comparative statics on the mean and variance of perceived outcomes in the steady state.

**Proposition 1.** For all \( \lambda \geq 1 \) and \( \eta > 0 \), if \( \hat{\eta} < \eta \) and \( \sigma > 0 \), then

1. \( \Gamma^a \) is a singleton and the unique steady-state mean belief \( \hat{\theta} \) is stable.

2. (Underestimated mean.) \( \hat{\theta} \leq \theta \), and the inequality is strict if and only if \( \lambda > 1 \). If \( \lambda > 1 \), then \( \hat{\theta} \) is strictly decreasing in \( \sigma \), \( \lambda \), and \( \eta \), and is strictly increasing in \( \hat{\eta} \).

3. (Overestimated variation.) Let \( \hat{\sigma} \) be the standard deviation of encoded outcomes at the steady-state mean belief, \( \hat{\theta} \). \( \hat{\sigma} > \sigma \) and \( \hat{\sigma} \) is strictly increasing in \( \sigma \), \( \lambda \), and \( \eta \), and is strictly decreasing in \( \hat{\eta} \).

Appendix A shows that beliefs indeed converge to these steady-state values for a class of distributions \( F_Z \) under which mean beliefs update in the direction of each encoded outcome (e.g., \( x_t \).
are normally distributed). Although outcomes are truly i.i.d., convergence does not follow directly from a basic law of large numbers. This is because encoded outcomes are not i.i.d.: prior outcomes shift a misattributor’s reference point and thus influence the current encoded outcome. As such, we use techniques from stochastic-approximation theory to establish convergence. For brevity, we relegate these details to Appendix A, and the results in this section apply to those situations where beliefs do converge.\footnote{This includes the familiar setting where outcomes are normally distributed and the agent is learning about both $\theta$ and $\sigma$. Assuming the canonical environment where priors over $\theta$ and $\sigma$ follow Normal and Inverse-Gamma distributions, respectively, then $\hat{\theta}_t$ converges almost surely to the steady-state beliefs characterized in Proposition 1.}

Part 2 of Proposition 1 shows that a loss averse misattributor underestimates the mean outcome. Intuitively, loss aversion causes the misattributor to experience a distribution of outcomes that is negatively skewed relative to the true distribution—she underestimates bad experiences more than she overestimates good ones. While this force drives down a misattributor’s perception of $\theta$, it is not immediate that such a pessimistic belief will persist. When a misattributor underestimates $\theta$, she is pleasantly surprised more often than she ought to be, which tends to increase her perception of $\theta$. The steady-state belief balances these two forces: a misattributor underestimates $\theta$ to an extent that the resulting excess of positive surprises exactly offsets the downward bias in outcomes stemming from loss aversion.

Furthermore, the misattributor’s (wrong) beliefs form a consistent theory of the world: while holding her steady-state expectation $\hat{\theta}$, the average outcome she encodes exactly matches $\hat{\theta}$. Stability of $\hat{\theta}$ follows from the belief-based nature of utility. Specifically, if she had low expectations $\bar{\theta} < \hat{\theta}$, then outcomes would typically be encoded as elations and drive her expectations upward. Conversely, if she had expectations above $\hat{\theta}$, then an increased frequency of disappointments would push her expectations downward.

Part 2 of Proposition 1 demonstrates an important result: greater variability in the underlying consumption utility of outcomes, $\sigma$, causes a misattributor to underestimate $\theta$ by a larger amount. That is, a misattributor develops more pessimistic beliefs about actions that are riskier. A higher variance implies a greater average magnitude of “surprise”. And since surprises are negative on average due to loss aversion, encoded outcomes tend to decrease in $\sigma$. To illustrate the logic, imagine a misattributing consumer learning about two services, $A$ and $B$, that have identical mean quality but different variability (e.g., Uber vs. Lyft). Suppose that $A$ exhibits large fluctuations in quality, while $B$ is consistent. Accordingly, the misattributor experiences primarily small gains and losses while using $B$, which translate to small distortions in $B$’s encoded outcomes. But $A$’s greater variability results in larger disappointments, and hence greater distortions in $A$’s encoded outcomes. As we emphasize below, this may lead to poor decisions when trading off risk and return, biasing the decision maker towards safer actions.
Not only does the true variance negatively influence beliefs about $\theta$, but a misattributor also overestimates the variance in outcomes (Part 3 of Proposition 1). Given expectation $\hat{\theta}$, the long-run distribution of perceived outcomes, denoted $\tilde{F}^a$, is readily derived from Equation 6:

$$
\tilde{F}^a(x) = \begin{cases} 
F_Z \left( \frac{x - \left( \theta + \kappa^L \left( \theta - \hat{\theta} \right) \right)}{(1 + \kappa^L)\sigma} \right) & \text{if } x \geq \hat{\theta} \\
F_Z \left( \frac{x - \left( \theta + \kappa^G \left( \theta - \hat{\theta} \right) \right)}{(1 + \kappa^G)\sigma} \right) & \text{if } x < \hat{\theta}.
\end{cases}
$$

(13)

For any value $v < \hat{\theta}$, a misattributor overestimates the probability of an outcome $x_t < v$. Likewise, for any $v > \hat{\theta}$, she overestimates the probability of $x_t > v$. Relative to the true distribution, the perceived distribution has fatter tails and is more negatively skewed. Figure 1 displays the results described above, depicting both the long-run path of beliefs and the density of perceived outcomes for two different variances: $\sigma^2 = 1$ (top panels) and $\sigma^2 = 5$ (bottom panels).

We now consider the misattributor’s (distorted) expected utility of an action given the statistical misperceptions described in Proposition 1. This will help us analyze decisions and welfare in the steady state. With misattribution, this analysis requires three distinct notions of (per-period) expected utility. First, \textit{forecasted utility} is what a misattributor expects to earn from action $a$ given her biased estimates $\hat{\theta}$ and $\hat{\sigma}$, and is the object that directs choice. Second, \textit{experienced utility} is what she actually receives, on average, from action $a$ given $\hat{\theta}$. Because the misattributor holds biased beliefs, her experienced utility will diverge from her forecasted utility. Finally, because expectations directly influence utility in this model, the average experienced utility from a fixed action $a$ differs depending on whether the agent holds biased versus rational expectations. Thus, to facilitate comparisons with the rational benchmark, we refer to the average utility received absent misattribution as \textit{hypothetical utility}.

Our next result compares forecasted and experienced utility to hypothetical utility. Importantly, a misattributor comes to underestimate the payoff she would earn from an action that yields variable outcomes. Let $\hat{v}_a$, $\bar{v}_a$ denote the person’s per-period forecasted and expected experienced utility, respectively, in the steady state characterized by Proposition 1, and let $v_a$ denote per-period hypothetical utility.

**Proposition 2.** Suppose $\hat{\eta} < \eta$, $\lambda > 1$, and that there exists a period $t^*$ such that the person chooses $a$ for all $t > t^*$.

1. Forecasted utility $\hat{v}_a$ is strictly less than hypothetical utility $v_a$. The difference $|\hat{v}_a - v_a|$ is

$\text{Recall that the true c.d.f. of outcomes is } F^a(x) = F_Z \left( \frac{x - \theta}{\sigma} \right)$.  

$\text{Formally, forecasted utility is the expectation of } u(x | \hat{\theta}) \text{ w.r.t. the misattributor’s perceived distribution of } x \text{ given } \hat{\theta} \text{ and } \hat{\sigma}. \text{ Experienced utility is the expectation of } u(x | \hat{\theta}) \text{ w.r.t. the true distribution of } x. \text{ Hypothetical utility is the expectation of } u(x | \theta) \text{ w.r.t. the true distribution.}$
Figure 1: The top-left panel depicts a simulated path of estimates ($\hat{\theta}_t$) for both a rational and biased agent. The top-right panel shows the true and perceived density of outcomes. The simulation assumes normally-distributed outcomes and priors with $\theta = 0$, $\sigma^2 = 1$, $\eta = 1$, $\lambda = 3$, and $\hat{\eta} = 1/3$. The bottom two panels are analogous except with an increased variance of $\sigma^2 = 5$.

2. Experienced utility $\bar{v}_a$ is strictly greater than hypothetical utility $v_a$. The difference $|\bar{v}_a - v_a|$ is strictly increasing in $\sigma$.

Part 1 of Proposition 2 follows immediately from the fact that a misattributor underestimates the mean and overestimates the variance of outcomes: due to loss aversion, both of these erroneous beliefs diminish her perceived benefit of action $a$. Part 2 shows that, fixing the long-run action, a misattributor experiences an average payoff that exceeds her hypothetical utility: since she forms overly pessimistic expectations $\hat{\theta} < \theta$, she realizes pleasant “surprises” more often than she would absent misattribution. It would be wrong, however, to conclude that the misattributor necessarily benefits from her mislearning. The proposition above holds the long-run action fixed, but a misattributor’s biased beliefs can cause her to settle on an action different from (and inferior to) the one she would choose if rational. In fact, Proposition 2’s comparative statics on $\sigma$ suggest that those
actions providing the greatest additional benefit to a misattributor relative to the rational benchmark (those with high \( \sigma \)) are precisely those that the misattributor undervalues the most. Indeed, the next section characterizes when a misattributor’s biased beliefs lead to suboptimal long-run choices.

### 4.2 Choice Implications

We now show that a misattributor is systematically biased against risky actions: she may continually choose a relatively safe action even when superior (but riskier) actions are available. This bias against risk is on top of the agent’s intrinsic risk preferences, and stems from pessimistic long-run beliefs. To illustrate, we consider a person who experiments with option \( A \) for \( t^* \) periods, and then commits (once and for all) to either \( A \) or \( B \) for all remaining periods \( t > t^* \). To abstract from bandit problems—where insufficient experimentation can lead to incomplete learning even among rational agents—suppose the person experiments with option \( A \) for an arbitrarily long horizon before deciding whether to switch (i.e., \( t^* \to \infty \)). While inconsequential to our basic results, we also assume \( B \) yields fixed, known consumption utility \( v_B \).

Corollary 1 (below) shows that, fixing \( v_B \), there always exists some risky action \( A \) that dominates \( B \) yet is wrongly abandoned in favor of \( B \). For example, imagine a farmer learning the payoffs from farming her land (option \( A \)), which are inherently stochastic due to factors such as weather. Eventually, she must decide whether to continue farming herself or sell the land for a fixed price (option \( B \)). Unless the returns to farming are sufficiently large, the misattributor will choose to sell the land even when it is optimal to continue farming. As Proposition 2 shows that the misattributor’s forecasted benefit from \( A \) falls below the benefit it actually provides, she will erroneously sell the land whenever \( v_B \) falls in the range between these two values. Furthermore, this range is larger when returns have greater variance: fixing the true expected utility of an action, the more variable is its output, the more likely the decision maker wrongly abandons it.

**Corollary 1.** Suppose \( \sigma(A) > 0 \) and \( \lambda > 1 \) and fix \( v_B \). Let \( \theta^*(\sigma(A)) \) be the value of \( \theta(A) \) that leaves the decision maker indifferent between \( A \) and \( B \) under full information. There exists a threshold \( \tilde{\theta}(\sigma(A)) > \theta^*(\sigma(A)) \) such that if \( \theta(A) \in (\theta^*(\sigma(A)), \tilde{\theta}(\sigma(A))) \), then the misattributor wrongly switches to action \( B \) almost surely as \( t^* \to \infty \). The difference in the misattributor’s threshold and the rational threshold, \( \tilde{\theta}(\sigma(A)) - \theta^*(\sigma(A)) \), is strictly increasing in \( \sigma(A) \).

Our next proposition demonstrates that the type of error identified in Corollary 1—an unwarranted bias toward safe actions—can result in arbitrarily costly mistakes each period.

**Proposition 3.** Suppose \( \lambda > 1 \) and fix \( v_B \). For any \( v_A > v_B \), there exist parameters \( (\theta(A), \sigma(A)) \) with \( \sigma(A) > 0 \) such that
1. Under rational expectations, $A$ yields (per-period) expected utility $v_A$.

2. The misattributor chooses $B$ almost surely as $t^* \to \infty$.

No matter how much the expected utility of $A$ exceeds that of $B$ (given rational beliefs), a misattributor ultimately settles on the safe option $B$ whenever $A$ is sufficiently volatile. Intuitively, the excessive weighting of occasional losses swamps the benefit of a high mean outcome as the magnitude of these losses grows large. Such biased learning may help explain why individuals tend to overly avoid risk based on their personal experiences, as shown by Malmendier and Nagel (2011).\footnote{According with our model, that paper argues that investors exit the stock market due to the formation unduly pessimistic beliefs rather than changes in risk attitudes.}

The type of mistaken choices characterized here would also arise in experimentation settings, though the analysis of such bandit problems is substantially more involved. To provide some intuition on how our results extend, suppose the agent is uncertain about both $A$ and $B$ and can switch between them over time. Fixing $(\theta(B), \sigma(B))$, there exist parameters $(\theta(A), \sigma(A))$ such that option $A$ would yield arbitrarily greater experienced utility than $B$ on average, yet a fully patient misattributor would fail to settle on $A$. The proof of Proposition 3 shows that while holding the hypothetical utility of $A$ fixed, $A$’s forecasted utility decreases unboundedly in $\sigma(A)$. Hence, no matter how large a benefit $A$ may provide, high variability can cause a misattributor to persistently take an inferior action.\footnote{While Corollary 1 and Proposition 3 do not address issues that arise from insufficient experimentation, the basic point—that a misattributor comes to excessively avoid risk—extends to more complicated bandit settings. Imagine a misattributor with a discount factor less than one who faces several uncertain options $a \in \{1, \ldots, N\} \equiv \mathcal{A}$ and a known outside option, $B$. It is well known that the “exploit-explore” trade-off in this $N$-armed bandit problem may lead a rational decision maker to settle on an inferior option (relative to the full-information benchmark). In these settings, misattribution increases the likelihood of settling on an inferior action, and this likelihood grows larger as the optimal action increases in variance.}

The result above—that a misattributor will permanently abandon optimal-but-risky actions—stems in part from the assumption that once an action is abandoned, the misattributor receives no additional information about that action. Under “full observability”, where the person observes the counterfactual outcomes of actions she did not take, our result changes in an interesting way. When a misattributor continues to get feedback on an action she abandoned, she may fall into a perpetual cycle of alternating between the optimal action and an inferior one. Since the misattributor distorts only those outcomes with utility consequences, she will update correctly about the (truly) optimal action once it is abandoned. This tends to correct her overly-pessimistic beliefs, and she may periodically return to the optimal action while never settling on it.

To illustrate this point, consider a variant of the workplace example: each period, the agent chooses between two potential collaborators, $A$ and $B$, and she learns about their productivity over time. Suppose that she observes the output of both colleagues no matter which she is currently
working with. Given this full observability—and hence no experimentation motives—the agent’s optimal strategy is to simply choose whichever collaborator provides the highest expected benefit given her current beliefs. Suppose in truth $A$ is the superior option by a small margin, so (assuming correct priors) the agent begins by selecting $A$. First-hand experience with $A$ carries utility consequences, and thus variability in $A$’s output will cause her to grow pessimistic about $A$’s productivity. Contrastingly, information about $B$ does not carry utility consequences, so she correctly updates about $B$’s productivity. When pessimism about $A$ is sufficiently strong, the misattributor will switch to working with $B$.\(^{34}\) At this point, $B$’s productivity will appear to deteriorate, while $A$’s will appear to improve. This will go on until the misattributor switches back to $A$, when the logic above will repeat. As such, the decision maker persistently vacillates between the two options.\(^{35}\)

The intuition above accords with the psychology that “the grass is always greener on the other side of the fence”: regardless of what the person chooses today, a different option will grow more appealing. This resembles, for instance, the common (mis)perception that a person’s chosen queue moves more slowly than an alternative, no matter which is chosen. And as soon as the person switches queues, her previous choice starts to appear better.

### 4.3 Stochastic Reference Points

Most of the results in Section 4 do not require us to assume deterministic reference points (Equation 3). In Appendix B, we present results analogous to Proposition 1 for the case of stochastic reference points (Equation 4) assuming normally-distributed outcomes. As above, a loss-averse misattributor will underestimate the mean and overestimate the variance of an action’s outcomes. Additionally, she underestimates the mean in proportion to the true variance in outcomes, again implying a bias toward less risky actions.

That said, stochastic reference points greatly complicate the analysis of long-run beliefs. With deterministic reference points, the person’s perceived distribution in a steady state is entirely specified by her perceived mean outcome. This is because her reference point depends solely on this single moment of the outcome distribution. With stochastic reference points, however, gain-loss utility depends on the entire perceived distribution—her belief about all moments influence the gains and losses she feels. As such, solving for the steady-state perception with stochastic reference points in general would entail finding a “fixed point” in the space of distributions: the

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\(^{34}\)The cycling behavior described here emerges when the two options are sufficiently similar in value, so that the biased agent undervalues $A$ enough to switch actions. If instead $A$ dominates $B$ by a significant margin, then the misattributor’s pessimistic steady-state impression of $A$ will not incite a switch. Fixing $\sigma(A)$ and the hypothetical utility of option $B$, the range of $\theta(A)$ that lead to cycling behavior matches the interval derived in Corollary 1.

\(^{35}\)Although the decision maker’s behavior does not converge, she may fail to reconsider her model. Since any outcome path realizes with positive probability, such cycles do not contradict the decision maker’s subjective model despite being unlikely ex ante. Additionally, the decision maker may ignore information on how often she switches actions, which is a form of “channeled attention” discussed in Footnote 24.
steady-state distribution is such that, if believed, the person’s encoded outcomes follow that distribution.\footnote{Relatedly, our result that experienced utility exceeds hypothetical utility (Proposition 2) is not straightforward with stochastic reference points. In that case, realized utility depends explicitly on the agent’s perceived variance. Fixing the true variance in outcomes, a person who faces normally-distributed outcomes receives a lower utility on average when she anticipates exaggerated variance. This force counters the increase in experienced utility earned from additional “pleasant surprises” generated by overly-pessimistic beliefs about the mean.}

5 Order Effects and Extrapolative Beliefs

This section explores how the order in which a misattribution experiences outcomes can influence her perceived benefit of an action. Even when outcomes are i.i.d., misattribution can generate a “recency bias”: expectations weight recent outcomes more heavily than older ones. This result further implies that—so long as the extent of misattribution is not too extreme—a misattribution forms the highest estimate of an action’s benefit following an increasing sequence of outcomes.\footnote{These basic insights form the premise of Section 7, where we explore how a sophisticated agent may strategically manipulate outcomes to exploit a misattribution’s errors.}

Moreover, we show that when outcomes are autocorrelated, a misattribution forms extrapolative forecasts of future payoffs even when there is nothing to learn about the data-generating process. Importantly, unlike the long-run results above, these short-run biases arise even without loss aversion (i.e., $\lambda = 1$).

Both parts of this section follow a similar setup. We consider a person’s beliefs when she takes the same action $a \in \mathcal{A}$ for $T$ periods; from this benchmark, the intuition easily extends to settings with multiple actions available. Each period, the action yields unidimensional consumption utility $x_t \in \mathbb{R}$. For tractability, we assume $x_t = \theta + \epsilon_t$, where $\theta$ is the parameter of interest and $\epsilon_t$ are i.i.d. mean-zero Gaussian shocks with known variance $\sigma^2$. The misattribution sequentially updates her beliefs about $\theta$ given each encoded outcome $\hat{x}_t$ and her prior $\theta \sim N(\theta_0, \rho^2)$.\footnote{Our qualitative results extend beyond the case of normally distributed outcomes and priors. For many of results in Sections 5 and 7, it is sufficient to assume that these distributions are symmetric and quasiconcave (i.e., unimodal). These assumptions imply that a rational agent’s updated estimate of $\theta$ always falls between her previous estimate and the most recent observation. See Chambers and Healy (2012) for a complete characterization of when beliefs “update toward the signal”.}

Consequently, her posterior belief over $\theta$ after $t$ observations is normally distributed with mean $\hat{\theta}_t$ and variance $\rho_t$, where

$$\hat{\theta}_t \equiv \frac{\rho^2}{i \rho^2 + \sigma^2} \sum_{k=1}^{t} \hat{x}_k + \frac{\sigma^2}{i \rho^2 + \sigma^2} \theta_0 \quad \text{and} \quad \rho_t \equiv \frac{\sigma^2 \rho^2}{i \rho^2 + \sigma^2}. \quad (14)$$

Section 5.1 examines how misattribution influences the evolution of $\hat{\theta}_t$, and Section 5.2 extends this analysis to cases where outcomes are autocorrelated.
5.1 Order Effects in Belief Updating

Although outcomes are i.i.d. and should therefore be treated exchangeably, the order of outcomes can distort a misattributor’s beliefs. Outcomes that are experienced early determine whether later outcomes are assessed as elations—and hence overestimated—or as disappointments—and thus underestimated. To build intuition, consider a misattributing consumer who experiments with Uber for one week. Suppose the rides are of high quality on each day aside from one “off day” that has poor quality. If that off day comes first, it will lower the misattributor’s expectations and the remaining rides may seem surprisingly good. Alternatively, if the off day comes last—after the rider has come to expect high quality—it may seem surprisingly bad. Even though the two sequences are permutations of the same outcomes, the fact that one generates subsequent gains whereas the other ends with a loss can cause the misattributor to reach different beliefs following the two sequences.

Each period, the misattributor updates her estimate $\hat{\theta}_{t-1}$ based on an encoded outcome $\hat{x}_t$. From Equation 14, her updated belief is

$$\hat{\theta}_t = \alpha_t \hat{x}_t + (1 - \alpha_t) \hat{\theta}_{t-1},$$

where $\alpha_t \equiv \rho^2 / (t \rho^2 + \sigma^2)$ is the proper weight a Bayesian would attach to a new observation. While the misattributor properly weights her encoded outcome $\hat{x}_t$, a rational Bayesian assigns weight $\alpha_t$ to the true outcome, $x_t$, not $\hat{x}_t$. Because a misattributor encodes $\hat{x}_t = x_t + \kappa_t (x_t - \hat{\theta}_{t-1})$ (Equation 6), she reaches a biased estimate

$$\hat{\theta}_t = \alpha_t (1 + \kappa_t) x_t + [1 - \alpha_t (1 + \kappa_t)] \hat{\theta}_{t-1}.\quad (15)$$

Since rational beliefs put weight $\alpha_t$ on $x_t$, Equation 15 immediately reveals that, relative to her rational counterpart, a misattributor “overreacts” to a new piece of evidence. Furthermore, the misattributor’s estimate $\hat{\theta}_t$ improperly weights each of the $t$ outcomes she experienced. Iterating Equation 15, we can express $\hat{\theta}_t$ as a weighted sum of past outcomes:

**Lemma 1.** Following a sequence of outcomes $(x_1, \ldots, x_t)$, a misattributor forms an estimate

$$\hat{\theta}_t = \alpha_t \sum_{\tau=1}^{t} \xi^\tau_t x_\tau + \xi^0_0 \theta_0,\quad (16)$$

where

$$\xi^\tau_t = \begin{cases} 
(1 + \kappa_t) & \text{if } \tau = t, \\
(1 + \kappa_t) \prod_{j=\tau}^{t-1} [1 - \alpha_j (1 + \kappa_j)] & \text{if } \tau \in \{1, \ldots, t-1\}, \\
\prod_{j=1}^{t-1} [1 - \alpha_j (1 + \kappa_j)] & \text{if } \tau = 0.
\end{cases}$$

More formally, consider both a misattributing and rational learner who share a common prior with mean $\theta_0$. Letting $\hat{\theta}$ and $\hat{\theta}^r$ be the biased and rational estimates of $\theta$, respectively, following outcome $x \in \mathbb{R}$, then $|\hat{\theta} - \theta_0| \geq |\hat{\theta}^r - \theta_0|$. Furthermore, the misattributor’s reaction $|\hat{\theta} - \theta_0|$ is increasing $\eta$ and $\lambda$ and decreasing in $\eta$—she overreacts more as the extent of misattribution increases.
The updating rule in Lemma 1 stands in clear contrast with rational updating, which assigns an equal weight \( \xi_t^t = 1 \) to each outcome.

To characterize the implications of a misattributor’s mis-weighted estimates, we introduce two definitions:

**Definition 1.** Beliefs are *convex in period* \( t \) if, given any prior estimate \( \hat{\theta}_{t-1} \in \mathbb{R} \) and any \( x_t \in \mathbb{R} \), there exists \( \alpha \in [0, 1] \) such that \( \hat{\theta}_t = \alpha x_t + (1 - \alpha) \hat{\theta}_{t-1} \). We call beliefs convex if they are convex in period \( t \) for all \( t \geq 1 \).

Convexity means that a misattributor’s posterior does not overreact by too much, so that her posterior falls between the true outcome \( x_t \) and her prior \( \hat{\theta}_{t-1} \).

Our results in this section largely assume convexity, which is equivalent to \( \kappa^L < \sigma^2 / \rho^2 \): the extent of misattribution is not too extreme relative to the informativeness of outcomes.

A related, weaker concept is “monotonicity”:

**Definition 2.** Let \( x'_{t-k} \) denote \((x_1, \ldots, x_t)\) excluding the \( k^{th} \) element. Beliefs are *monotonic* if for all \( t = 1, 2, \ldots \), all \( k \leq t \), and all \( x'_{t-k} \in \mathbb{R}^{t-1} \), \( \hat{\theta}_t \) is increasing in \( x_k \) conditional on \( x'_{t-k} \).

While convexity ensures that a misattributor’s estimate is increasing in her prior, monotonicity implies \( \hat{\theta}_t \) is increasing in each of her outcomes. Monotonicity, which is implied by convexity, is equivalent to \( \kappa^L < 1 + \sigma^2 / \rho^2 \).

If beliefs are monotonic, a misattributor exhibits a *recency bias*: her beliefs weight a recent gain more than any of her preceding gains and a recent loss more than any preceding loss.

**Proposition 4.** Consider a sequence of outcomes \((x_1, \ldots, x_T)\), and suppose beliefs are monotonic. For any two outcomes \( x_t, x_{\tau} \) that are both gains or both losses (i.e., \( \kappa_t = \kappa_{\tau} \)), the more recent one receives greater weight: \( \xi_{T-t}^T > \xi_{T-\tau}^T \) if and only if \( t > \tau \). Furthermore, the most recent outcome is overweighted and early outcomes are underweighted: \( \xi_{T-t}^T > 1 \) and for all \( t < T \), \( \xi_{T-t}^T \to 0 \) as \( T - t \to \infty \).

Since losses influence beliefs more than gains (Observation 2), it is possible that a loss in period \( t - 1 \) has a larger influence on beliefs than a gain in period \( t \). However, for any two outcomes that fall in the same domain (i.e., gain or loss), the more recent outcome has a greater influence on

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40 From Equation 15, if beliefs are not convex, then the person’s posterior \( \hat{\theta}_t \) overweights \( x_t \) and is decreasing in her prior. For instance, a higher prior implies that \( x_t \) may come as a larger loss. If beliefs overreact enough, then this sensation of loss more than offsets the obvious positive effect of a higher prior on the person’s posterior. Note that rational beliefs in this setting are convex given that \( \theta \) and \( \varepsilon_t \) are independently drawn from normal distributions. This holds more generally assuming the distributions of \( \theta \) and \( \varepsilon_t \) are symmetric and quasiconcave.

41 More specifically, when outcomes are very informative about \( \theta \)—the variance in outcomes \( \sigma^2 \) is low relative to the variance of the prior \( \rho^2 \)—then the extent of misattribution must be low to ensure convexity. Very informative outcomes will be used more heavily, and thus overreaction will be extreme if the extent of misattribution is large. For a sense of what this condition means in terms of the underlying parameters, if \( \eta = 1, \lambda = 3 \) and \( \sigma^2 / \rho^2 = 1 \), then beliefs are convex if \( \eta > 1/3 \).
beliefs than the earlier one. Additionally, the weight on that early outcome vanishes relative to the recent one as the time between the two grows large.

The proposition above highlights an important consequence of misattribution: it naturally leads to “contrast effects”, or a negative relationship between early outcomes and the perceptions of those that come later. If beliefs are monotonic, low initial outcomes reduce expectations in later rounds, causing later outcomes to be seen in better light. Conversely, high initial outcomes increase expectations, causing later outcomes to be judged more harshly. Thus, in terms of their influence on final beliefs, early outcomes are “self limiting”: high early outcomes raise the person’s expectations and subsequent outcomes are thus underestimated by more (or overestimated by less), offsetting the positive effect of the high initial outcomes. Similarly, when early outcomes lower expectations, later outcomes are overestimated by more (or underestimated by less), offsetting the initial reduction in beliefs. As the horizon grows long, an early outcome $x_t$ exerts this countervailing force on a larger number of subsequent outcomes, which pushes its weight $\xi_t$ to zero.

To illustrate the contrast effects described above, consider the Uber example where the misattributor faces quality $(x_1, x_2)$ on her first two rides. Without loss, normalize her prior about Uber’s average quality to $\theta_0 = 0$. Suppose $x_1 > 0 > x_2$ so that her first experience is a gain and her second is a loss. Following $x_1$, her belief about Uber’s average quality increases to $\hat{\theta}_1 = \alpha_1 \hat{x}_1 = \alpha_1 (x_1 + \kappa^G x_1)$ (see Equation 14). Her final estimate of $\theta$ is then $\hat{\theta}_2 = \alpha_2 [\hat{x}_1 + \hat{x}_2] = \alpha_2 [x_1 + \kappa^G x_1 + x_2 + \kappa^L (x_2 - \hat{\theta}_1)]$. Combining the expression above for $\hat{\theta}_1$ yields

$$\hat{\theta}_2 = \alpha_2 \left[(1 + \kappa^G)(1 - \kappa^L \alpha_1)x_1 + (1 + \kappa^L)x_2\right].$$

(17)

The factor $(1 - \alpha_1 \kappa^L) < 1$ multiplied on $x_1$ captures the contrast effect discussed above. Increasing $x_1$ increases expectation $\hat{\theta}_1$ going into period 2, which means $x_2$ seems worse when evaluated against these higher expectations. As such, the misattributor underestimates $x_2$ by an amount proportional to the increase in expectations. This provides intuition for how the “self-limiting” multiplier $(1 - \alpha_1 \kappa^L)$ on $x_1$ depends on the parameters: (1) increasing the influence $x_1$ has on expectations entering day two—that is, increasing $\alpha_1$—decreases the person’s assessment of $x_2$ and overall estimate of $\theta$, and (2) increasing $\kappa^L$—the extent to which the person fails to account for gain-loss utility—similarly decreases the person’s estimate. That said, the overall effect of $x_1$ on $\hat{\theta}_2$ is positive unless $\alpha_1$ and $\kappa^L$ are sufficiently large. Indeed, $\kappa^L < 1 + \frac{\sigma^2}{\rho^2}$ ensures such monotonicity.

This recency bias suggests that the misattributor will form a higher estimate of $\theta$ when her best experiences happen near the end of the horizon. To characterize when this is the case, suppose the misattributor will experience some arbitrary set of outcomes $\mathcal{X} = \{x_1, \ldots, x_T\}$. We ask: among
all possible orderings of the outcomes in $\mathcal{X}$, when does the sequence with an increasing profile maximize the misattributor’s perception of $\theta$?

We first consider sequences with just two outcomes, where we can provide the weakest conditions sufficient for such order effects. Fixing outcomes $a, b \in \mathbb{R}$ with $a > b$, the misattributor necessarily reaches a higher estimate of $\theta$ when she receives the worse outcome, $b$, first so long as $\lambda$ is not too large. Roughly put, an increasing sequence minimizes disappointments while maximizing elations.

**Proposition 5.** Suppose $a, b \in \mathbb{R}$ such that $a > b$. Let $\hat{\theta}^d_2$ denote the mean belief following the decreasing sequence $(a, b)$ and let $\hat{\theta}^i_2$ denote that following the increasing sequence $(b, a)$.

1. If beliefs are convex, then $\hat{\theta}^i_2 > \hat{\theta}^d_2$ for all $a, b \in \mathbb{R}$.

2. If beliefs are not convex, then there exists a threshold $\bar{\lambda} > 2$ such that $\hat{\theta}^d_2 > \hat{\theta}^i_2$ for some $a, b \in \mathbb{R}$ only if $\lambda > \bar{\lambda}$.

Building on Part 2 of Proposition 5, even when beliefs are not convex and $\lambda > \bar{\lambda}$, the range of values $(a, b)$ such that the decreasing sequence maximizes beliefs is quite limited. In such cases, $\hat{\theta}^d_2 > \hat{\theta}^i_2$ only if both outcomes come as a loss relative to the person’s prior—$\theta_0 > a > b$—and $b$ is fairly close to $a$ (see Equation C.21 for a precise condition). To see the rationale, notice that if beliefs are not convex—that is, they overreact excessively to new observations—and $b$ is close to $a$, then $b$ is perceived as a gain when experienced after $a$: beliefs become so pessimistic after the initial loss $a$ that the truly worse outcome $b$ feels like a gain. If in addition losses distort beliefs sufficiently more than gains (i.e., $\lambda > \bar{\lambda}$), then $\hat{\theta}_2$ is maximized (roughly) by minimizing experienced losses. Since in this case the first outcome necessarily comes as a loss while the second comes as a gain, losses are minimized when the better outcome happens first. In light of this caveat, the following corollary provides a sufficient condition for a recency effect that may be useful for empirical tests.

**Corollary 2.** Suppose $T = 2$. For all $\lambda \geq 1$ and $\hat{\eta} < \eta$, if at least one of the two realized outcomes beats initial expectations, then a misattributor forms the highest estimate of $\theta$ when the two outcomes are experienced in increasing order.

The logic of Proposition 5 extends to sequences of any length so long as beliefs are convex.\(^\text{42}\)

**Proposition 6.** Consider a fixed set of $T$ distinct outcomes, $\mathcal{X}$. If beliefs are convex, then among all possible orderings of the outcomes in $\mathcal{X}$, the misattributor’s estimate $\hat{\theta}_T$ is highest following the sequence in which the elements are ordered from least to greatest.

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\(^{42}\)Requiring that beliefs are convex is stronger than necessary, but it ensures that our result holds for any collection of outcomes—we do not need to tailor conditions to the realized set of outcomes.
If beliefs are convex, then the person’s expectations following any string of outcomes \( (x_t, \ldots, x_T) \) is increasing in the prior \( \hat{\theta}_{t-1} \) she carries into round \( t \). The recency effect in Proposition 6 thus follows by applying the result of Proposition 5 to any adjacent pair of outcomes in a longer sequence. Even when beliefs are initially non-convex, there is still a sense in which increasing outcomes maximize the misatributor’s expectations. Because each observation has less impact on beliefs when the total number of observations increases (which is true for both rational and biased beliefs), there exists a period \( t^* = \left\lfloor 1 + \kappa \Delta - \sigma^2/\rho^2 \right\rfloor \) beyond which beliefs are always convex. Hence, no matter the parameter values, \( \hat{\theta}_T \) is maximized by having an increasing profile of payoffs beyond period \( t^* \). We explore this prediction in an application in Section 7.

The results above accord with a large literature that explores the role of improving sequences on perceptions. Fixing the outcomes a person experiences, evidence demonstrates that people both prefer improving sequences and retrospectively form the most optimistic evaluations thereafter. For example, Ross and Simonson (1991) allow participants to sample two video games and find that willingness to pay for the bundle is significantly higher for those who sampled the better game second. Similarly, Haisley and Loewenstein (2011) demonstrate that advertising promotions are most effective when sequenced in increasing order of value—that is, the high-value promotional item is given last. Several authors argue that such assessments follow a mechanism like ours, stemming from adaptation and subsequent contrast (e.g., Tversky and Griffin 1990; Loewenstein and Prelec 1993; Baumgartner, Sujan, and Padgett 1997).

That said, such order effects stand at odds with confirmation bias: new evidence is wrongly interpreted as conforming to one’s expectations (e.g., Rabin and Schrag 1999; Fryer, Harms, and Jackson 2015). Under confirmatory bias, outcomes deviating from expectations are encoded as closer to expectations, while misattribution makes the opposite prediction. Despite this tension, the mechanisms are not mutually exclusive. Confirmation bias may stem from a reliance on priors to make sense of ambiguous evidence, whereas misattribution follows from a failure to separate outcomes from the hedonic feelings of surprise they induce. Indeed, empirical tests of order effects in belief updating find support for both confirmatory/primacy effects and recency effects—see Hogarth and Einhorn (1992) for a meta analysis. Which effect prevails typically depends on the nature of the learning problem: confirmatory effects tend to dominate as evidence becomes more ambiguous and difficult to interpret.\(^{43}\)

\(^{43}\)Furthermore, although our recency bias suggests a diminished importance of first impressions, our model with stochastic reference points captures an element of this intuition: the first outcome is, on average, more distorted than any following outcome. Hence, the initial outcome may impose a greater weight on final beliefs than subsequent outcomes. This follows from the fact that with a stochastic reference point, the magnitude of gain-loss utility from any outcome \( x \) is increasing in the variance of the person’s subjective distribution of outcomes. Since such variance is largest in the first round before receiving any feedback, this outcome naturally generates the greatest gain-loss utility (in expectation) and thus generates relatively large distortions in perceptions.
5.1.1 False Perceptions of Trends and the Role of Priors

A misattributor will perceive a fictitious trend in outcomes when, in fact, she faces exactly the same outcome repeatedly. Whether outcomes appear to improve or deteriorate depends on how the true outcome compares to initial expectations: an outcome that is better than average appears to deteriorate, while one that is worse than average appears to improve.

Imagine a consumer who tries a new product that yields consumption utility \( x \) each time she consumes it, and suppose \( x \) surpasses her prior expectations. Due to the initial pleasant surprise, a misattributor perceives a quality \( \hat{x}_1 > x \) following her first trial. This perception lifts her expectations, so facing the same quality she previously experienced on the second trial feels less impressive relative to these higher beliefs. Consequently, she perceives a lower quality the second time: \( \hat{x}_2 < \hat{x}_1 \). As her expectations grow closer to the true outcome over time, she experiences diminishing sensations of surprise—she adapts to the experience—and misattributes this sensation of adaptation to a declining quality. Similarly, when the initial experience is worse than expected, disappointment fades over time and experiences appear to increase in quality. The next proposition formalizes these intuitions, first outlined in Section 3.

**Proposition 7.** Suppose \( x_t = x \) for each \( t = 1, \ldots, T \) and that beliefs are convex. If \( x > \theta_0 \), then \( \hat{x}_t \) is strictly decreasing in \( t \). If \( x < \theta_0 \), then \( \hat{x}_t \) is strictly increasing in \( t \). In both cases, \( \lim_{T \to \infty} x_t = x \).

Note the role of prior beliefs on a misattributor’s perceived trend: those options that turn out better than expected seem to deteriorate with time, while worse-than-expected options seem to improve.\(^{44}\) Of course, such perceptions can lead to poor decisions. A misattributing consumer, for instance, may abandon high-quality products due to a false perception that they are on the decline.

5.2 Over-Extrapolation and Belief Reversals in Autocorrelated Environments

The contrast effects discussed above extend in interesting ways when outcomes are autocorrelated. In such settings, a misattributor forms overly-extrapolative forecasts of future payoffs even when she (rightly) thinks she has nothing to learn about the data generating process. And unlike the i.i.d. case above, this pattern of excessively extrapolative and volatile forecasts persists over time. Furthermore, the misattributor makes a predictable forecasting error: if she overestimates today’s outcome, she tends to underestimate tomorrow’s outcome (and vice versa).

To introduce autocorrelation to the Gaussian setup above, we assume outcomes follow the process \( x_t = \theta + \phi x_{t-1} + \epsilon_t \), where the parameter \( \phi \in [0, 1] \) measures the extent of autocorrelation.

\(^{44}\)The basic psychology described here may play a role in perceptions of a “sophomore slump”—the idea that impressive debuts are often followed by seemingly less-stellar performances. While regression to the mean surely contributes to this phenomenon, our mechanism amplifies the perceived drop in quality between the debut and follow-up performances, making the “slump” more pronounced.
and $\mathcal{E}_t$ are again i.i.d. Gaussian shocks with mean zero and variance $\sigma^2$. For simplicity, we assume the person knows parameters $\varphi$ and $\sigma$. Furthermore, we focus on the limit where there is no uncertainty about $\theta$, and thus set $\theta = 0$. This highlights the persistent error introduced by autocorrelation independent of the order effects generated by uncertainty over $\theta$ (Section 5.1).

Hence, the person’s forecast in period $t$ about tomorrow’s outcome, denoted $\hat{\mathbb{E}}_t[x_{t+1}]$, is simply $\hat{\mathbb{E}}_t[x_{t+1}] = \varphi \hat{x}_t$.

For an example of this environment, imagine a person who suffers a chronic disease (e.g., arthritis) that varies in severity over time. Let $x_t$ measure the severity of pain on day $t$. The shocks $\mathcal{E}_t$ represent incremental improvements or setbacks, and parameter $\varphi$ measures the persistence of her day-to-day condition. When $\varphi > 0$, the decision maker forms expectations about tomorrow’s condition based on her condition today, which may inform important decisions such as whether to preemptively call in sick to work.

To illustrate the effect of misattribution in such settings, suppose that on a Monday the person experiences worse pain than expected. As usual, misattribution causes her to exaggerate this pain—she incorrectly attributes her overall disappointment to the pain itself. She thus forms overly-pessimistic expectations about Tuesday, meaning that, on average, her condition on Tuesday comes as a pleasant surprise. This leads her to overestimate the quality of her condition on Tuesday and form an overly-optimistic forecast going forward. In turn, her condition on Wednesday will likely come as a disappointment, again leading to an overly-pessimistic forecast. This oscillating pattern will continue over time: the person forms exaggerated forecasts in the direction of the most recent outcome, which subsequently lead to “surprises” in the opposite direction.

Formalizing these intuitions, our main result in this section shows that a misattributor’s forecast (a) continues to be influenced by prior outcomes that have no bearing on the rational forecast, and (b) over-responds to the most recent outcome. As such, these forecasts are more variable than rational forecasts and exhibit predictable errors. Let $d_t \equiv \hat{x}_t - \hat{\mathbb{E}}_{t-1}[x_t]$ denote the misattributor’s forecast error experienced on date $t$.

**Proposition 8.** Let $\varphi > 0$. A misattributor’s forecast formed on date $t$ is $\hat{\mathbb{E}}_t[x_{t+1}] = \varphi \hat{x}_t$ where

$$\hat{x}_t = (1 + \kappa_t)x_t + \sum_{j=1}^{t-1}(1 + \kappa_j) \left( (-\varphi)^{t-j} \prod_{i=j+1}^{t} \kappa_i \right) x_j. \quad (18)$$

---

$^{45}$We restrict attention to the case of positive autocorrelation solely for the sake of exposition. Analogous results hold for the case of $\varphi \in [-1, 0]$.

$^{46}$This assumption corresponds either to scenarios late in the horizon—meaning that the person has already developed confident beliefs about $\theta$—or those in which the person knows $\theta$ from the start ($\rho^2 \to 0$).

$^{47}$We define the forecast error as the difference between the person’s perceived outcome and her expectations. Our prediction of a negative relationship between today’s forecast error and tomorrow’s continues to hold if we alternatively define the forecast error as the difference between the true outcome and expectations.
Hence, forecasts exhibit:

1. Excessive extrapolation and volatility: \( \hat{E}_t[x_{t+1}] \) overweights the outcome on date \( t \) by a factor \( (1 + \kappa_t) \), and conditional on \( (x_1, \ldots, x_{t-1}) \), \( \text{Var}(\hat{E}_t[x_{t+1}]) = (1 + \kappa_t)^2 \text{Var}(E_t[x_{t+1}]) \).

2. Predictable errors and reversals: forecast errors follow a negatively-correlated process given by

\[
d_t = (1 + \kappa_t) \left\{ -\varphi \left( \frac{\kappa_{t-1}}{1 + \kappa_{t-1}} \right) d_{t-1} + \epsilon_t \right\}.
\] (19)

The most recent outcome is overweighted by a factor \( 1 + \kappa_t \), implying that forecasts overreact to recent losses more than gains. Additionally, a misattributor’s predictions are wrongly influenced by all past outcomes—after accounting for \( x_t \), the rational forecast does not depend on any outcome prior to \( t \). As demonstrated by Equation 18, more recent outcomes are given greater weight. Furthermore, consistent with the oscillating logic in the example above, the current forecast negatively weights outcomes that occurred an odd number of periods ago, and positively weights those that happened an even number of periods ago. For instance, outcome \( x_{t-1} \) has a negative effect on \( \hat{x}_t \), since expectations about \( x_t \) are increasing in \( x_{t-1} \). But outcome \( x_{t-2} \) has a positive effect on \( \hat{x}_t \) because an increase in \( x_{t-2} \) raises expectations in \( t-1 \), thereby lowering \( \hat{x}_{t-1} \) and hence lowering expectations entering round \( t \).

While rational predictions generate uncorrelated forecast errors, Part 2 of Proposition 8 highlights the negative relationship between a misattributor’s errors: overly optimistic forecasts are typically followed by overly pessimistic forecasts. The strength of this relationship is increasing in both the extent of misattribution and the extent of autocorrelation. Figure 2 below uses a simulated time series to depict a misattributor’s overly extrapolative forecasts (top panel) and the negative relationship in her forecast errors (bottom panel).

Our basic prediction of overly extrapolative, volatile forecasts accords with a range of evidence. For instance, Gennaioli, Ma, and Shleifer (2015) and Greenwood and Shleifer (2014) find that managers and investors form extrapolative, volatile predictions of their future earnings and that these forecasts exhibit errors that negatively correlate with past performance. While many models give rise to this general pattern of extrapolative beliefs and systematic reversals—e.g., Bordalo, Gennaioli, and Shleifer’s (2016) model of diagnostic expectations based on Kahneman and Tversky’s (1972) representativeness heuristic—our model provides additional predictions which may help empirically disentangle these mechanisms. In particular, we predict that these patterns are more pronounced when outcomes carry utility consequences for the forecaster and that beliefs respond more to bad news than good.
Figure 2: The top panel displays both the rational and biased forecasts for a simulated process with $\phi = 0.7$, $\sigma = 5$, $\eta = 1$, $\lambda = 3$ and $\hat{\eta} = 1/3$. Using that same data, the bottom panel depicts the negative relationship between forecast errors on date $t$ and $t + 1$.

6 Application: Insuring Against Undesirable Outcomes

In our first application, we consider a decision maker who can pay (through costly effort, money, etc.) to insure herself against bad states of the world. With misattribution, initial effort to decrease the chance of a bad outcome endogenously generates unforeseen additional demand for such insurance, causing the person to settle on inefficiently high payments. Roughly put, increased effort raises her expectations, which causes the bad outcome to seem even worse when it happens. This perception further increases her desire to avoid that bad outcome.

For clarity and continuity, we consider a binary-outcome example similar to the one in Section 3: each period, the person receives consumption utility $\theta(H)$ with probability $p$ and $\theta(L) < \theta(H)$. 

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with probability $1 - p$. Suppose additionally that the person can act at the start of each round to increase $p$. To use a concrete example, imagine a farmer who may invest each season in technologies to reduce the impact of adverse weather or disease on her crops (e.g., improved irrigation practices, crop rotation, silos to prevent post-harvest loss, etc.). In good conditions, she receives $\theta(H)$ and in bad conditions she receives $\theta(L)$. Suppose that absent any investment, her crops survive with probability $p_0 = 1/2$. However, she can increase this probability with investment in agricultural technologies and effectively choose a survival probability $p_t$. We assume that implementing $p_t \in (p_0, 1)$ incurs a cost $c(p_t - p_0)$ in round $t$, where $c(\cdot)$ is twice continuously differentiable, convex, and admits a marginal cost $c'(\cdot)$ that is weakly convex and satisfies $c'(0) = 0$.

The misattributor’s perceived benefit to increasing $p$ will depend on her past choice of $p$. If she faced a fixed probability $\bar{p}$ in the past, her perceptions of $\theta(H)$ and $\theta(L)$ will approach the values $\hat{\theta}(H; \bar{p})$ and $\hat{\theta}(L; \bar{p})$ derived in Equation 9 of Section 3, respectively. Given these beliefs (which depend on $\bar{p}$), the person’s perceived expected benefit from switching to a new probability $p$, denoted $\hat{U}(p; \bar{p})$, is

$$\hat{U}(p; \bar{p}) = p\hat{\theta}(H; \bar{p}) + (1 - p)\hat{\theta}(L; \bar{p}) - p(1 - p)\eta(\lambda - 1)[\hat{\theta}(H; \bar{p}) - \hat{\theta}(L; \bar{p})],$$ (20)

Herein lies the misattributor’s mistake when choosing her effort to avoid the bad outcome: she ignores how changing $p$ will affect her future perceptions of $\hat{\theta}(H)$ and $\hat{\theta}(L)$. This is evident from Equation 20—she treats $\hat{\theta}(H)$ and $\hat{\theta}(L)$ as independent of her planned choice, $p$. The misattributor thus naively maximizes $\hat{U}(p; \bar{p})$ with respect to $p$, disregarding how this new level of effort will influence her subsequent valuations.

Once the misattributor pays to increase $p$, her seemingly optimal action will eventually feel inadequate. Roughly put, exerting effort to protect her crops—which raises expectations about her yield—makes a bad yield seem even worse and hence increases her perceived marginal benefit of the good state. Recall from Section 3 that the more a misattributor expects the good state (i.e., higher $p$), the less she overestimates $\theta(H)$. Additionally, the more she expects the good state, the more she underestimates $\theta(L)$. Because of loss aversion, these distortions are stronger for the bad outcome than the good one, so overall increasing $p$ causes a misattributor to overestimate the payoff difference between the good and bad states by more. Since the optimal action is increasing in her perceived marginal benefit of the good state, these new perceptions inspire her to further increase $p$.

How will the misattributor’s choice of effort $p_t$ evolve over time? To provide an intuition for

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48As in Section 3, we assume that, conditional on event $a \in \{H, L\}$, consumption utility is normally distributed with $x_t = \theta(a) + \varepsilon_t$ and $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$ with $\sigma \rightarrow 0$.

49The assumptions on $c'(\cdot)$ simplify the exposition, as they guarantee a unique long-run choice. The assumption that $p_t \in (p_0, 1)$ rules out paths where the person reaches $p_t = 1$ and remains there due to lack of feedback about $\theta(L)$.  

the dynamics, imagine the scenario where each time she chooses a new \( p \), she remains at that value until (nearly) reaching new steady-state beliefs. At that point, she reevaluates her desired choice of \( p \). In this case, we refer to a “period” \( \tau \) as the unit of time the person sticks with a fixed value of \( p \). Hence, at the start of each “period”, she chooses the optimal value \( p^*_\tau \) based on her perceptions of \( \hat{\theta}(H) \) and \( \hat{\theta}(L) \) that she formed under value \( p^*_{\tau-1} \). More specifically, \( p^*_\tau \) satisfies 
\[
\hat{U}'(p^*_\tau; p^*_{\tau-1}) = c'(p^*_\tau - p_0) \quad \text{and thus Equation 20 implies a path defined by}
\]
\[
\hat{\Delta}(p^*_{\tau-1})\left(2\eta \lambda p^*_\tau + 1 - \eta \lambda\right) = c'(p^*_\tau - p_0),
\]
where \( \hat{\Delta}(p) \equiv \hat{\theta}(H; p) - \hat{\theta}(L; p) \).\(^{50}\)

How does this process play out? Before the person has any experience, she chooses \( p^*_1 \) based on her priors (which we take to be correct). Thus, \( p^*_1 > p_0 \) solves \( \Delta(2\eta \lambda p^*_1 + 1 - \eta \lambda) = c'(p^*_1) \), where \( \Delta = \theta(H) - \theta(L) \). Note that \( p^*_1 \) corresponds with the rational level of effort: a rational decision maker would choose \( p = p^*_1 \) in all periods, as her choice of \( p^*_1 \) would not change her perceptions of the outcomes. For a misattributor, however, his initial increase in \( p \) will fuel greater subsequent effort. This insight relies on two simple facts: (1) \( \lambda > 1 \) implies \( \hat{\Delta}(p) \) is strictly increasing in \( p \), and (2) \( p^*_\tau \) is increasing in \( \hat{\Delta}(p^*_{\tau-1}) \). The first fact, as discussed above, is due to greater distortion in beliefs about the bad outcome than the good one. The second fact follows directly from Equation 21. Together, these properties imply that \( p^*_\tau \) is increasing in \( p^*_{\tau-1} \). Initial effort to raise \( p_0 \) to \( p^*_1 \) spawns an increasing profile of subsequent effort, \( p^*_\tau > p^*_{\tau-1} \) for all \( \tau = 1, 2, \ldots \). Figure 3 depicts this process, demonstrating how an increase in investment shifts the perceived marginal benefit \( \hat{U}'(p^*_\tau; p^*_{\tau-1}) \) upward, leading to a new optimal action higher than the previous one.

This pattern of escalating effort will go on indefinitely as the person converges to a steady state action \( \hat{p} \equiv p^*_\infty > p^*_1 \). Although we described dynamics above using a simplified, deterministic argument, the misattributor’s long-run action in the setting where she selects \( p_t \) each round converges to this same value, \( \hat{p} \). This steady-state action is characterized by \( \hat{\Delta}(\hat{p})(2\eta \lambda \hat{p} + 1 - \eta \lambda) = c'(\hat{p} - p_0) \) so long as the solution is less than \( 1 \); otherwise, \( p^*_\infty \rightarrow 1 \).\(^{51}\)

**Proposition 9.** Consider the setting described above. If \( \lambda > 1 \), then a misattributor perpetually exerts excessive effort to reduce the chance of the bad state: \( p_t \) converges almost surely to a long-run value \( \hat{p} \) that strictly exceeds the rational optimal action.

This excess effort may be costly to a misattributor: her average experienced utility at the long-run

\(^{50}\)Notice that \( \hat{U}'(p; \hat{p}) \) is always positive for \( p \geq 1/2 \). Hence, given a “default” value \( p_0 = 1/2 \), the marginal benefit of increasing \( p \) is always positive. That said, there exists some \( \bar{p} < 1/2 \) such that for \( p_0 < \bar{p} \), the decision-maker actually prefers a smaller chance of the preferred outcome. This stems from the large aversion to risk that choice-acclimating personal equilibrium imposes in the Kőszegi and Rabin model. We focus on \( p_0 = 1/2 \) to avoid such issues.

\(^{51}\)The solution is necessarily interior whenever \( c'(1) > \hat{\Delta}(1)(1 + \eta(\lambda - 1)) = \frac{1 + \eta \lambda}{1 + \eta \lambda^2} (1 + \eta(\lambda - 1)) \).
level \( \hat{\rho} \) is strictly less than what she would earn if she correctly inferred \( \theta(H) \) and \( \theta(L) \). However, the fact that misattribution leads to increased demand for preventative measures may be beneficial when farmers would otherwise do too little for reasons beyond misattribution. Indeed, our prediction of experience-driven demand is consistent with Cai and Song’s (2017) field evidence, which demonstrates that vivid personal experience with losses (albeit hypothetical) increases farmers’ take-up of weather insurance. They find that this increased demand cannot be explained by changes in risk attitudes or the perceived probability of losses, and is generally helpful given the farmers’ tendency to under-insure.

7 Application: Expectations Management and Reputation

Our model suggests that a sophisticated agent may be able to strategically manipulate outcomes in order to bias the evaluations of a misattributor.\textsuperscript{52} To explore this issue, we consider a career-

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\textsuperscript{52}Research in marketing examines similar problems with different motivation. For instance, Kopalle and Lehmann (2006) study how a firm should optimally set quality expectations through advertisements when consumers have expectations-based reference-dependent preferences (known as “disconfirmation” or the “gap model” in marketing; see, for example, Anderson 1973, Oliver 1977, or Ho and Zheng 2004). Fixing the realized quality of a good, a
concern setting where a misattributing (but otherwise rational) principal sequentially updates her beliefs about an agent’s ability and offers wages based on those inferences. For instance, imagine an investor who hires a financial advisor to manage her wealth, or a homeowner who repeatedly hires a contractor (the agent) to remodel parts of her home. An agent who is aware of the principal’s misattribution faces new incentives that push against the classical intuitions of models like Holmström (1999) that predict declining effort over time. A sophisticated agent chooses an effort path that initially under-performs relative to the principal’s expectations, but consistently beats them thereafter. Essentially, he sends the most optimistic signal of his ability when he initially sets the bar low and subsequently supplies a series of positive surprises.

While we frame the problem in terms of a familiar career-concern model, our analysis directly applies to a variety of settings where a party is rewarded for building a positive reputation. In particular, our results accord with forms of expectations management used in diverse settings ranging from politics, to marketing, to finance. Politicians and firms often strategically “walk down” expectations only to later surpass them. For example, a longstanding empirical finding in finance is that firms attempt to lower investors’ expectations prior to earnings announcements. Bartov, Givoly and Hayn (2002) demonstrate that meeting or beating analyst expectations yields significant excess stock returns. Similarly, Teoh, Yang and Zhang (2009) find that firms are rewarded for beating expectations even when they actively manage expectations by walking down analyst forecasts.

7.1 Reputation-Building Environment

We consider the environment described in Holmström (1999). A principal (she) hires an agent (he) with uncertain ability to exert effort over $T$ periods. Each period $t$, the agent supplies effort $e_t \in \mathbb{R}_+$ leading to output $x_t = \theta + e_t + \varepsilon_t$, where $\theta \in \mathbb{R}$ is the agent’s ability and $\varepsilon_t$ is mean-zero noise. Importantly, we take $x_t$ (net of paid wages) as the principal’s consumption value in period $t$—she directly cares about the agent’s performance. Additionally, the principal cannot observe effort, so she updates her beliefs about the agent’s ability based on output. These beliefs determine the agent’s wage in the following period. Continuing the normal-normal setup in Section 5, we assume the two parties share a common prior $\theta \sim \mathcal{N}(0, \rho^2)$ and that $\varepsilon_t$ are i.i.d. with $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$.\(^{54}\)

We explore effort provision and equilibrium wages when the principal suffers misattribution consumer in their model who is pleasantly surprised by this quality in the first period is more likely to buy again in the second period. This logic seems very much in line with our model of misattribution.

\(^{53}\)This management can come through strategic accounting of working capital and cash flow from operations (Burgstahler and Dichev 1997), real activities such as sales (Roychowdhury 2006), or through indirect channels such as managing analyst forecasts (e.g., Richardson, Teoh, and Wyckoki 2004).

\(^{54}\)As in Holmström, we assume that both parties are initially uncertain about the agent’s ability. This rules out signaling motives for the agent, which both simplifies the analysis and allows us to clearly highlight the effects of biased evaluations on effort.
while inferring the agent’s ability, and the agent—aware of this error—best responds to it. A naive principal who is unaware of her mistake will neglect the agent’s incentive to exploit it. Thus, she may form incorrect beliefs about the agent’s strategy. We denote the principal’s anticipated effort in period $t$ by $\hat{e}_t$ and $\hat{\theta}_t$ (as usual) denotes her expectation of $\theta$. Hence, the principal enters round $t$ expecting output $\hat{E}[x_t] = \hat{\theta}_{t-1} + \hat{e}_t$. Given these reference points, the principal’s encoded outcomes and beliefs follow the mechanics highlighted Section 5. While the principal overestimates output $x_t$ when it beats expectations and underestimates it when it falls short, the agent uses the true values of output and effort to update his beliefs.

Like Holmström (1999), we assume the market perfectly competes for the agent’s labor. Hence, the principal pays a wage $w_t$ at the start of each round equal to her current expectation of $x_t$: $w_t = \hat{\theta}_{t-1} + \hat{e}_t$. As such, the agent maximizes the principal’s perception of his ability subject to effort costs. We assume these costs are separable across periods, and we denote the flow disutility of effort by $c(e_t)$. We further assume that $c(\cdot)$ is strictly increasing, convex, and has a first derivative denoted by $c_e(\cdot)$, and we impose the normalization $c(0) = 0$.

### 7.2 Optimal Effort Provision Under Biased Evaluations

The agent exerts effort to maximize the sum of discounted expected utility given discount factor $\delta \in (0, 1]$. Hence, in period $t$, the agent faces an objective function

$$U_t \equiv \sum_{\tau=t}^{T} \delta^{\tau-1} [w_{\tau} - c(e_{\tau})] = \sum_{\tau=t}^{T} \delta^{\tau-1} [\hat{\theta}_{\tau-1} + \hat{e}_{\tau} - c(e_{\tau})]. \quad (22)$$

For ease of exposition, we restrict attention to $\lambda = 1$ and relegate the case of $\lambda > 1$—which yields qualitatively similar results—to Appendix C. From Lemma 16, $\hat{\theta}_t = \alpha_t \sum_{k=1}^{T} \xi_k \tau (x_k - \hat{e}_k)$, where $\xi_k \tau = (1 + \kappa^G)$ and $\xi_k \tau = (1 + \kappa^G) \prod_{j=k}^{T-1} [1 - \kappa^G \alpha_j]$ for all $k < t$. Thus, we can isolate the parts of

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55 Our results hold for two natural specifications for the anticipated effort profile, $(\hat{e}_1, \ldots, \hat{e}_T)$. In the first, the principal expects the agent to behave just as he would in the rational Bayesian Nash Equilibrium. In this case, the principal consistently mispredicts the agent’s behavior and misattributes this discrepancy to ability and noise. In the second, the principal correctly predicts the agent’s effort profile despite lacking a good theory as to why the agent deviates from the Bayesian-Nash strategy.

56 The effect of today’s effort on future wages ($M_t$, derived below in Equation 23) is particularly simple when $\lambda = 1$. Since gains and losses have the same impact on encoded outcomes, today’s effort has a deterministic effect on later wages. When $\lambda > 1$, this effect is uncertain since it depends on whether a future outcome comes as a gain or loss. In that case, the agent chooses $e_t$ to maximize its expected effect on future wages, taking into account how $e_t$ influences the likelihood that future outcomes will come as gains versus losses. The predicted behavior in that case is qualitatively similar to the one we explore here, but significantly more tedious to analyze. Specifically, when $\lambda > 1$, effort satisfies an analog of Condition 23 below that differs only in that all $\kappa^G$’s are replaced by the expected value of $\kappa$, which lies in the interval $[\kappa^L, \kappa^H]$. This change will not alter the qualitative pattern of the effort profile in expectation.

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U_t that depend on \( e_t \) as follows:

\[
-c(e_t) + \sum_{\tau=t+1}^{T} \delta^{\tau-t} \alpha_{\tau-1} \xi_{\tau-1} e_{\tau} = -c(e_t) + e_t (1 + \kappa^G) \left\{ \delta \alpha_t + \sum_{\tau=t+2}^{T} \delta^{\tau-t} \alpha_{\tau-1} \left( \prod_{j=t}^{\tau-2} [1 - \kappa^G \alpha_j] \right) \right\}.
\]

The optimal effort in period \( t \) is thus \( e_t^* = c_e^{-1}(M_t) \), where

\[
M_t \equiv (1 + \kappa^G) \left\{ \delta \alpha_t + \sum_{\tau=t+2}^{T} \delta^{\tau-t} \alpha_{\tau-1} \left( \prod_{j=t}^{\tau-2} [1 - \kappa^G \alpha_j] \right) \right\}. \tag{23}
\]

\( M_t \) is the marginal effect of period-\( t \) effort on all future wages. More specifically, the \( \tau \)th term in the sum is the marginal effect of period-\( t \) effort on the principal’s beliefs—and thus the agent’s wage—\( \tau \) periods later.

To understand how the agent adjusts his effort in response to biased evaluations, we first consider the solution under rational evaluations. In that case, the marginal impact of period-\( t \) effort is \( M_t^r = \sum_{\tau=t+1}^{T} \delta^{\tau-t} \alpha_{\tau-1} \), implying the optimal effort in period \( t \) is the value \( e_t^r = c_e^{-1}(M_t^r) \). Since \( \alpha_{\tau} \) is decreasing in \( \tau \)—the weight one observation receives decreases as the total number of observations increases—the agent exerts decreasing effort over time and \( e_T = 0 \). He optimally exerts the more effort in early periods for two reasons: (1) early effort has a relatively high impact on the principal’s beliefs prior to the accrual of a large sample of outputs, and (2) early effort will play into the principal’s beliefs over a longer horizon. In the rational equilibrium, the principal perfectly anticipates the agent’s effort strategy, leading to a signal-jamming equilibrium where the strategic provision of effort has no effect on the principal’s beliefs.

When the principal suffers misattribution, early effort by the agent imposes a cost on his future selves: hard work in period one increases the principal’s expectations in all future periods, which means that subsequent output will be judged more harshly.\(^{57}\) This force is seen clearly in the marginal effects comprising \( M_t \) (Equation 23). Assuming \( \delta = 1 \), the impact of increasing period-\( t \) effort on period \( \tau + 1 \)’s wage \( w_{\tau+1} = \hat{\theta}_{\tau} + \hat{e}_{\tau+1} \) for \( \tau > t \) is

\[
\alpha_{\tau} (1 + \kappa^G) (1 - \kappa^G \alpha_{\tau-1}) (1 - \kappa^G \alpha_{\tau-2}) \ldots (1 - \kappa^G \alpha_t).
\]

In the rational case, this product is simply \( \alpha_{\tau} \). Under biased evaluations, this effect is initially scaled upward by factor \( 1 + \kappa^G \), which comes from the initial surprise of deviating from expectations. But for each period \( j \) between \( t \) and \( \tau \), the influence of \( e_t \) is diminished by a factor \( 1 - \kappa^G \alpha_j < 1 \) resulting from a contrast effect. This creates a tension for the agent when deciding

\(^{57}\)This intuition shares similarities with “ratcheting effects” studied in the literature on contracts and regulation. In those settings, the agent is reluctant to reveal private information about his efficiency early in the relationship so that he can demand higher compensation.
how to provide effort early in rounds: on the one hand, first-period effort influences the principal’s expectations of θ for the largest number of rounds, suggesting a benefit to exerting high effort early. On the other hand, first-period effort has the longest influence on the principal’s expectations against which all subsequent effort is judged. This force provides incentive for the agent to reduce early effort.

**Proposition 10.** If beliefs are convex, then there exists a period \( t^* \), \( 1 \leq t^* < T \), such that the agent’s optimal effort falls short of the rational benchmark \( e^*_t < e^*_r \) for all \( t < t^* \) and exceeds this benchmark \( e^*_t > e^*_r \) for all \( t \geq t^* \).

Proposition 10 shows that the agent strategically falls short of expectations early, but consistently beats the principal’s expectations later in the relationship. The proposition highlights that, while beating expectations is beneficial, the agent prefers to do so sufficiently late in the horizon as to not set the bar too high too early. However, given that this result describes the agent’s deviations from the rational path, it does not mean that \( e^*_t \) increases over time. In fact, convexity—which implies monotonicity (see Definition 2)—rules out an increasing effort path.\(^{58}\)

Intuitively, the longer is the horizon of the relationship, the greater is the externality that early effort imposes on future selves. Indeed, the extent to which the agent under-performs at the onset is increasing in the horizon.

**Corollary 3.** If beliefs are convex, then there exists a horizon \( \bar{T} \) such that \( T > \bar{T} \) implies \( e^*_1 < e^*_1 \). Furthermore, \( e^*_1 - e^*_1 \) is decreasing in \( T \) among \( T > \bar{T} \).

While moral hazard leads the agent to supply inefficiently little effort when the principal is rational, misattribution can lead the agent to provide greater effort in total. As in the rational case, an agent working for a misattributing principal still has some incentive to provide high initial effort. However, since misattribution implies that the agent is unduly penalized for falling short of expectations, he has greater incentive to maintain this high level of effort. In a sense, the principal’s biased evaluations impose an informal contract under which the agent feels committed to uphold the precedent for high effort he sets early in the relationship. Hence, these additional incentives can help improve efficiency. To see this concretely, suppose \( c(e) = e^2 / 2 \), \( \eta = \delta = 1 \), and \( \sigma = \rho \). These assumptions imply that \( e^*_t = M_t \), and assuming \( \hat{\eta} = 0 \) yields per-period incentives \( M_t = 2 - 2t / T \) (see Equation 23). With a rational principal, however, effort is given by \( e^*_t = M^*_t = \sum_{\tau=t+1}^T (1 / \tau) \).

\(^{58}\)This result is specific to our particular setting where the agent earns rewards each round. Even if beliefs are convex, there are natural settings where a misattributing principal will inspire the agent to supply an increasing profile of effort. The most stark example is a setting with a single payment period that follows multiple periods of effort and evaluation. If the principal in this setting were Bayesian, the agent would smooth her effort over the evaluation periods, as each round is a perfect substitute for another in terms of the principal’s posterior beliefs. In contrast, when the principal suffers from misattribution, the optimal pattern of effort follows a low-to-high profile. This result follows directly from Proposition 6.
In this case, total effort under misattribution, $\sum_{t=1}^{T} e^*_t$, exceeds total effort given a rational principal for all horizons $T > 2$.

Which party benefits from the principal’s mistake depends on the principal’s degree of loss aversion. When $\lambda = 1$, the agent is surely better off: he could achieve the same expected payoff as the case with a rational principal if he simply follows the rational effort profile. Contrastingly, when $\lambda > 1$, this assessment becomes ambiguous. Recall from Section 4 that $\lambda > 1$ causes the principal to grow pessimistic about $\theta$ over time. Thus, while the agent could attempt to counter this force by inflating the principal’s perceptions, doing so clearly requires more effort than the $\lambda = 1$ case. As such, the agent’s payoffs diminish in $\lambda$. Furthermore, unlike the case where the principal is rational, she may enjoy some of the surplus when $\lambda > 1$: due to loss aversion, the principal underestimates the agent’s effort on average, which drives down the total wages she pays.

8 Discussion

This paper models a potentially important mistake in learning from experience that stems from a decision maker neglecting how her expectations shaped her evaluation of outcomes. In this section, we discuss other models of mistaken learning and contrast our approach with those works. We also highlight ways that future empirical work could explore the implications of our model. Finally, we discuss two natural extensions of our framework: (1) incorporating misattribution of news utility (Kőszegi and Rabin 2009), and (2) extending misattribution to social-learning environments.

A growing literature explores forms of mistaken learning when agents hold misspecified models of the world.\textsuperscript{59} Esponda and Pouzo (2016) provide a general framework for assessing the long-run beliefs and behavior of agents with misspecified models, and our approach can be seen as a special case of that framework. Other papers have examined specific instances of learning with misspecified models. For example, Heidhues, Kőszegi, and Strack (2016) study agents who are overconfident about their ability and who learn about the value of a fundamental that influences how their effort translates to output. Similar to our results above, they document a form of long-run pessimism where agents systematically underestimate the fundamental. Unlike our explanation—which is driven by an overweighting of bad outcomes—theirs stems from self-defeating choices. This provides a potential route for empirical distinction.

We also build on an emerging literature that examines errors in memory or recall. Mullainathan (2002) provides a model of limited rationality which can generate a form of over-reaction to infor-

\textsuperscript{59}In addition to the models discussed throughout the paper, examples include Rabin (2002) and Rabin and Vayanos (2010) on the law of small numbers; DeMarzo, Vayanos, Zweibel (2003), Eyster and Rabin (2010) and Bohren (2016) on misinferring from others’ behavior in social-learning contexts; Madarász (2012) on information projection; Schwartzstein (2014) on selective attention; Spiegler (2016) on biases in causal reasoning; and Fudenberg, Romanyuk, and Strack (2016) on experimentation with misspecified models.
mation through memory associations. Our model predicts a similar effect—surprisingly good or bad experiences “stand out”—but we predict this effect even for new experiences that simply stand out relative to expectations. Relatedly, Bordalo, Gennaioli and Shleifer (2016) consider a model of limited attention whereby events are more memorable when they are “salient”. In some settings, this model shares our property that outcomes that differ from expectations are overweighted. Wilson (2014) follows a rational approach with bounded memory and examines the optimal coarsening of information given this constrained memory. This approach yields predictions distinct from misattribution, as Wilson finds that first impressions dominate subsequent evaluations.

A natural avenue for empirical exploration is our prediction of belief-based contrast effects—an outcome will seem better when the previous outcome was worse. We predict such effects when (a) people compare their realized outcomes against other outcomes they were expecting, and (b) those expectations are based on previous experience. Additionally, we predict that contrast effects increase when the perceived correlation between today’s outcome and tomorrow’s is stronger.\footnote{Relatedly, Hartzmark and Shue (2016) find that contrast effects among investors stemming from prior-day earnings announcements are larger for within-sector peers than across industries.}

In order to separate effects generated by our mechanism from other potential explanations—e.g. the Gambler’s Fallacy (Chen, Moskowitz, and Shue 2016)—we suggest that researchers examine circumstances where decisions have utility consequences versus those without. Our model predicts that contrast effects will be enhanced the more that a person cares about the outcomes she receives. A similar empirical strategy could help distinguish our mechanism from either base-rate neglect (e.g. Benjamin, Bodoh-Creed, and Rabin, in progress) or the representativeness heuristic (e.g. Bordalo, Gennaioli, and Shleifer 2016), which both predict recency effects and extrapolative beliefs. For instance, one could test whether investors’ forecasts are more extrapolative about companies they held a stake in relative to those they did not. Additionally, testing whether such forecasts respond differently to losses versus gains may provide yet another way to distinguish these mechanisms.

Finally, we note two potential extensions of our model. In this paper, we have omitted the notion of “news utility” from Kőszegi and Rabin (2009), in which a person experiences sensations of gain or loss based solely on changes in beliefs about future consumption. As discussed in that paper, news utility provides a channel for monetary outcomes to directly influence contemporaneous experienced utility, and thus incorporating misattribution of news utility would naturally extend our predictions to settings involving money or earnings. Additionally, such an extension introduces interesting comparative statics on the length of commitments. To illustrate, consider a worker who agrees to a new position or project for a prespecified amount of time, and imagine that her first encounter with the new job is worse than expected. With misattribution of news utility, her evaluation of that first experience will be worse the longer she committed to the job—essentially, that
first episode provides worse news about the future as the duration of her contract grows longer.

Throughout this paper, we explore the implications of an agent who retrospectively misunderstands the source of her own utility. However, our model can be reframed as an interpersonal bias where an observer neglects how expectations shape the experiences of others. For instance, a person reading online reviews (e.g., Yelp) for a product may fail to appreciate that a bad rating could simply reflect the reviewer’s high expectations rather than poor quality. In scenarios where consumers form their expectations based on predecessors’ reviews, misattribution—taking others’ ratings at face value without accounting for their expectations—might hinder social learning. Additionally, social-learning settings may be data-rich environments to explore the empirical implications of our model. If such social misattribution occurs, we would expect ratings to demonstrate the dynamic patterns, such as contrast effects, that we have described in this paper.

References


A Convergence of Mean Beliefs

This section describes sufficient conditions for the convergence of the sequence mean beliefs \( \langle \hat{\theta}_t \rangle \) studied in Section 4. Our convergence arguments, which rely on stochastic approximation theory, are similar to those used by Esponda and Pouzo (2016) and Heidhues, Kószegi, and Strack (2016). Stochastic approximation theory implies that, roughly speaking, the asymptotic behavior of beliefs is described by a deterministic ordinary differential equation. While encoded outcomes are not independent (\( \hat{x}_t \) is a function of \( \hat{\theta}_{t-1} \), which depends on \( x_1, \ldots, x_{t-1} \)), they become approximately independent as \( t \) grows large and \( \hat{\theta}_t \) changes a small amount (on average) in response to any new outcome. Intuitively, we could consider a new process \( \tilde{\theta}(\tau) \) on a redefined time scale \( \tau \) where (1) the person experiences a large number \( t \) of outcomes holding a fixed expectation \( \tilde{\theta}(\tau) \), and then (2) updates her belief \( \tilde{\theta}(\tau) \) based on those \( t \) independent observations. Under such a rescaling that keeps expected steps in \( \tilde{\theta}(\tau) \) constant, the process \( \tilde{\theta}(\tau) \) corresponds to a continuous-time, “sped up” analog of \( \hat{\theta}_t \). As \( \tau \) becomes large, a length of time in the process \( \tilde{\theta}(\tau) \) corresponds to many i.i.d. outcomes, which means the resulting change in \( \tilde{\theta}(\tau) \) is nearly deterministic. The limiting behavior of \( \tilde{\theta}(\tau) \) is thus approximated by the deterministic ordinary differential equation \( \tilde{\theta}'(\tau) = G(\tilde{\theta}(\tau)) \), where \( G \) is the average deviation of \( \hat{x}_t \) from \( \hat{\theta}_t \) (Equation 11) in the limit \( t \to \infty \).
We apply this method to demonstrate convergence of \( \langle \hat{\theta}_t \rangle \) in the specific case where (1) \( x_t = \theta + \sigma z_t \) and \( z_t \) are i.i.d. normally-distributed mean-zero random variables with unit variance, and (2) the person begins with a prior \( \theta \sim N(\theta_0, \rho^2) \). While convergence obtains more generally whenever the conditions below are met, it is particularly straightforward to verify these conditions for the normal case given our derivation of \( \hat{\theta}_t \) for normally-distributed outcomes in Section 5. From Equation 15, the misattributor’s beliefs update according to \( \hat{\theta}_t = \hat{\theta}_{t-1} + \alpha_t [\hat{x}_t - \hat{\theta}_{t-1}] \), where \( \alpha_t = (1 + \kappa_t)\alpha_t \) and \( \alpha_t = \rho^2/(t\rho^2 + \sigma^2) \). In this case, we can appeal to Kushner and Yin (2003), who provide sufficient conditions for the convergence of dynamic systems that take this form. More specifically, when the 4 conditions below are met, \( \langle \hat{\theta}_t \rangle \) converges almost surely to the unique element of \( \Gamma \) characterized in Proposition 1:

A1. \( \sum_{t=1}^{\infty} \alpha_t = \infty \) and \( \lim_{t \to \infty} \alpha_t = 0 \).

A2. \( \sum_{t=1}^{\infty} (\alpha_t)^2 < \infty \).

A3. \( \sup_t \mathbb{E}[|\hat{x}_t - \hat{\theta}_{t-1}|^2] < \infty \), where the expectation is taken at time \( t = 0 \).

A4. Finally, we require the existence of a continuous function \( G \) and a sequence of random variables \( (\gamma_t)_t \) such that \( \mathbb{E}[\hat{x}_t - \hat{\theta}_{t-1} | \theta, \hat{\theta}_{t-1}] = G(\hat{\theta}_t) + \gamma_t \) and \( \sum_{t=1}^{\infty} \alpha_t |\gamma_t| < \infty \) w.p. 1. We take \( G \) to be the function defined in Equation 11, and \( \gamma_t = G(t, \hat{\theta}_{t-1}) - G(\hat{\theta}_{t-1}) \). Given this definition, \( \sum_{t=1}^{\infty} \alpha_t |\gamma_t| < \infty \) w.p. 1, as required. Furthermore, from Equation 11, it is straightforward that \( G(\cdot) \) is continuous given that \( F_Z \) and \( f_Z \) are continuous.

The following proposition establishes that the 4 sufficient conditions hold for any arbitrary collection of distributional parameters \((\theta_0, \rho, \theta, \sigma)\).

**Proposition A.1.** Let \( G \) be defined as in Equation 11, and let \( \hat{\theta}_\infty \in \mathbb{R} \) be the unique value derived in Proposition 1 such that \( G(\hat{\theta}_\infty) = 0 \). For all \( \eta, \lambda, \) and \( \bar{\eta} \in [0, \eta] \), \( \hat{\theta}_t \) converges a.s. to \( \hat{\theta}_\infty \).

**Proof.** Since Condition 4 is argued in the text above, we need only show that Conditions 1-3 hold.

**Condition 1.** Note that

\[
\sum_{t=1}^{\infty} \alpha_t = \sum_{t=1}^{\infty} (1 + \kappa_t)\alpha_t > (1 + \kappa^G) \sum_{t=1}^{\infty} \alpha_t = (1 + \kappa^G) \sum_{t=1}^{\infty} \frac{\rho^2}{t\rho^2 + \sigma^2}. \tag{A.1}
\]

Since the final sum diverges to \( \infty \), \( \sum_{t=1}^{\infty} \alpha_t \) diverges. Furthermore, it is clear that \( \lim_{t \to \infty} \alpha_t = 0 \).

**Condition 2.** Note that

\[
\sum_{t=1}^{\infty} (\alpha_t)^2 = \sum_{t=1}^{\infty} (1 + \kappa_t)^2 \alpha_t^2 < (1 + \kappa^G)^2 \sum_{t=1}^{\infty} \alpha_t^2. \tag{A.2}
\]

From the definition of \( \alpha_t \), \( \sum_{t=1}^{\infty} (\alpha_t)^2 < \sum_{t=1}^{\infty} \frac{1}{t^2} < \infty \). Thus, \( \sum_{t=1}^{\infty} (\alpha_t)^2 < \infty \).

---

\footnote{This result trivially extends to the case where the person is also uncertain about the variance of outcomes, \( \sigma^2 \), so long as her beliefs about the variance do not influence how she updates about \( \theta \). This is the case, for instance, in the canonical example where priors over \( \theta \) and \( \sigma^2 \) follow Normal and Inverse-Gamma distributions, respectively.}
Condition 3. We must show \( \sup_{t} \mathbb{E}[(\hat{\theta}_{t} - \hat{\theta}_{t-1})^{2} | \theta] < \infty \). Note that \( \hat{\theta}_{t} - \hat{\theta}_{t-1} = x_{t} - \kappa_{t}(x_{t} - \hat{\theta}_{t-1}) - \hat{\theta}_{t-1} = (1 + \kappa_{t})(x_{t} - \hat{\theta}_{t-1}) \). Letting \( \theta_{t-1} \) be the rational estimate of \( \theta \) following \( t-1 \) rounds, we have

\[
\sup_{t} \mathbb{E}[(\hat{\theta}_{t} - \hat{\theta}_{t-1})^{2} | \theta] \leq (1 + \kappa_{t}) \sup_{t} \mathbb{E}[(x_{t} - \theta_{t-1}) + (\theta_{t-1} - \hat{\theta}_{t-1})^{2} | \theta].
\]  

(A.3)

From Minkowski’s Inequality,

\[
\sqrt{\mathbb{E}[(x_{t} - \theta_{t-1}) + (\theta_{t-1} - \hat{\theta}_{t-1})^{2} | \theta]} \leq \sqrt{\mathbb{E}[(x_{t} - \theta_{t-1})^{2} | \theta]} + \sqrt{\mathbb{E}[(\theta_{t-1} - \hat{\theta}_{t-1})^{2} | \theta]}.
\]  

(A.4)

Since \( \mathbb{E}[(x_{t} - \theta_{t-1})^{2} | \theta] \) is finite, we need only examine the second term on the right-hand side of Equation A.4. Using Lemma 1, we can write

\[
\theta_{t-1} - \hat{\theta}_{t-1} = \alpha_{t-1} \sum_{k=1}^{t-1} \xi_{k} - \alpha_{t-1} \sum_{k=1}^{t-1} \xi_{k} x_{k} = \alpha_{t-1} \sum_{k=1}^{t-1} \xi_{k} (1 - \xi_{k}^{-1}) x_{k}.
\]  

(A.5)

where \( \xi_{k}^{-1} \), defined in Lemma 1, is a function of \( \kappa_{j} \) and \( \alpha_{j} \) for \( j \in \{k, \ldots, t-1\} \). Thus

\[
\sqrt{\mathbb{E}[(\theta_{t-1} - \hat{\theta}_{t-1})^{2} | \theta]} \leq \alpha_{t-1} \sum_{k=1}^{t-1} \sqrt{\mathbb{E}[x_{k}^{2} | \theta]}
\]  

(A.6)

We now argue that for all \( t \geq 2 \) and all \( k \leq t-1 \), the value \( |1 - \xi_{k}^{-1}| \) is bounded from above by some finite constant \( Q \). Given that \( \kappa_{t} \in \{\kappa^{L}, \kappa^{U}\} \) and the definition of \( \alpha_{j} \), it is clear that such a \( Q \) exists for any finite \( t \). Thus, we need only consider the case where \( t \to \infty \). In this case, we have

\[
\lim_{t \to \infty} \xi_{k}^{-1} = (1 + \kappa_{k}) \lim_{t \to \infty} \prod_{j=1}^{t-2} [1 - \alpha_{j} \kappa_{j+1}].
\]

For sufficiently large \( j \), \( |1 - \alpha_{j} \kappa_{j+1}| < 1 \), which means that, fixing \( k \), \( |\xi_{k}^{-1}| \) is decreasing in \( t \). Thus, given that \( |1 - \xi_{k}^{-1}| \) is bounded by some finite \( Q \),

\[
\sqrt{\mathbb{E}[(\theta_{t-1} - \hat{\theta}_{t-1})^{2} | \theta]} \leq Q \alpha_{t-1} \sum_{k=1}^{t-1} \sqrt{\mathbb{E}[x_{k}^{2} | \theta]}
\]

\[
= Q \alpha_{t-1} \sum_{k=1}^{t-1} \sqrt{\sigma^{2} + \theta^{2}}
\]

\[
= Q \frac{\rho^{2}}{\rho^{2} + \sigma^{2}/(t-1)} \sqrt{\sigma^{2} + \theta^{2}}
\]

\[
\leq Q \sqrt{\sigma^{2} + \theta^{2}},
\]  

(A.7)

where the first equality follows from the fact that \( \mathbb{E}[x_{k}^{2}] = \text{Var}(x_{k}) - \mathbb{E}[x_{k}]^{2} \), and the second equality follows from the fact that for all \( t \geq 2 \), \( \alpha_{t-1} = \rho^{2}/((t-1)\rho^{2} + \sigma^{2}) \). Thus \( \sqrt{\mathbb{E}[(\theta_{t-1} - \hat{\theta}_{t-1})^{2} | \theta]} \) is finite, which completes the proof.
B Stochastic Reference Points

This section considers a misattributor with a stochastic reference point (Equation 4) who is learning the parameters of an action with normally-distributed outcomes. In truth, suppose that \( x_t \sim \mathcal{N}(\theta, \sigma^2) \). Throughout, we let \( \Phi(\cdot) \) and \( \phi(\cdot) \) denote the standard-normal c.d.f. and p.d.f., respectively. The following lemma specifies the gain-loss utility function (Equation 4) assuming the person believes \( x \) is normally distributed. When the person believes \( x \sim \mathcal{N}(\hat{\theta}, \hat{\sigma}^2) \), we denote gain-loss utility by \( n(x|\hat{\theta}, \hat{\sigma}) \).

Lemma B.1. Suppose that the person believes \( x \sim \mathcal{N}(\hat{\theta}, \hat{\sigma}^2) \). With a stochastic reference point,

\[
    n(x|\hat{\theta}, \hat{\sigma}) = \hat{\sigma} \left( \lambda z - (\lambda - 1) [z\Phi(z) + \phi(z)] \right),
\]

where \( z = (x - \hat{\theta})/\hat{\sigma} \).

We now assess how beliefs about \( \theta \), denoted \( \hat{\theta} \), evolve over time when a misattributor faces outcomes \( x_t \). For simplicity, we restrict attention to the case where \( \hat{\eta} = 0 \). What beliefs does a misattributor reach when she is uncertain about both \( \theta \) and \( \sigma \)? Unlike the case with deterministic reference points, stochastic reference points imply that perceptions of \( \theta \) depend on perceptions of \( \sigma \). This implies that the steady-state perceptions, \( \hat{\theta} \) and \( \hat{\sigma} \), must satisfy the following system of equations:

\[
\begin{align*}
\hat{\theta} &= \mathbb{E}[-1] \hat{\theta}, \hat{\sigma} \tag{B.1} \\
\hat{\sigma}^2 &= \text{Var}[-1] \hat{\theta}, \hat{\sigma} \tag{B.2}
\end{align*}
\]

where \( \mathbb{E}[-1] \hat{\theta}, \hat{\sigma} \) and \( \text{Var}[-1] \hat{\theta}, \hat{\sigma} \) are with respect to the true data generating process fixing the person’s perceptions \( \hat{\theta} \) and \( \hat{\sigma} \). The remainder of this section solves these steady-state equations, demonstrating a unique solution in which a misattributor overestimates \( \sigma \) and underestimates \( \theta \) in proportion to \( \hat{\sigma} \)—her perception of \( \sigma \).

We first derive the first-moment condition, Equation B.1. With \( \hat{\eta} = 0 \), \( \hat{x} = x + \hat{\eta} n(x|\hat{\theta}, \hat{\sigma}) \), where \( n(\cdot|\hat{\theta}, \hat{\sigma}) \) is given in Lemma B.1. As in that lemma, \( z = (x - \hat{\theta})/\hat{\sigma} \) and let \( \bar{z} \equiv (\theta - \hat{\theta})/\hat{\sigma} \). The expectation of \( \hat{x} \) with respect to the true distribution, which has density \( \frac{1}{\hat{\sigma}} \phi \left( \frac{x - \theta}{\hat{\sigma}} \right) \), is

\[
\mathbb{E}[x + \hat{\eta} n(x|\hat{\theta}, \hat{\sigma})] = \theta + \hat{\sigma} \eta \left( \lambda \bar{z} - (\lambda - 1) \left[ \int z \Phi(z) \frac{1}{\sigma} \phi((x - \theta)/\sigma) \phi((x - \theta)/\sigma) dx \right] \right),
\]

Let \( w = \frac{x - \theta}{\sigma} \), which implies \( z = a + bw \) where \( a = \bar{z} = \frac{\theta - \hat{\theta}}{\hat{\sigma}} \) and \( b = \frac{\bar{\sigma}}{\hat{\sigma}} \). Hence,

\[
\mathbb{E}[x + \hat{\eta} n(x|\hat{\theta}, \hat{\sigma})] = \theta + \hat{\sigma} \eta \left( \lambda a - (\lambda - 1) \left[ \int (a + bw) \Phi(a + bw) \phi(w) dw + \int \phi(a + bw) \phi(w) dw \right] \right)
\]
Thus,

\[ \mathbb{E}[x + \eta n(x|\hat{\theta}, \hat{\sigma})] = \theta + \hat{\sigma} \eta \left( \lambda a - (\lambda - 1) [aI_1 + bI_2 + I_3] \right), \quad \text{(B.3)} \]

where

\[ I_1 \equiv \int \Phi(a + bw)\phi(w)dw = \Phi \left( \frac{a}{\sqrt{1+b^2}} \right), \quad \text{(B.4)} \]
\[ I_2 \equiv \int w\Phi(a + bw)\phi(w)dw = \frac{b}{\sqrt{1+b^2}} \phi \left( \frac{a}{\sqrt{1+b^2}} \right), \quad \text{(B.5)} \]
\[ I_3 \equiv \int \phi(a + bw)\phi(w)dw = \frac{1}{\sqrt{1+b^2}} \phi \left( \frac{a}{\sqrt{1+b^2}} \right). \quad \text{(B.6)} \]

Hence, the first equation of the steady-state system, Equation B.1, amounts to

\[ 0 = a + \eta \left\{ \lambda a - (\lambda - 1) \left[ a\Phi \left( \frac{a}{\sqrt{1+b^2}} \right) + \sqrt{1+b^2} \phi \left( \frac{a}{\sqrt{1+b^2}} \right) \right] \right\} \quad \text{(B.7)} \]

We now turn to the second-moment equation of the steady-state system, Equation B.2. Note that \( \text{Var}[x|\hat{\theta}, \hat{\sigma}] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 \), where \( \mathbb{E}[x^2] = \sigma^2 + \theta^2 + 2\eta \mathbb{E}[x n(x|\hat{\theta}, \hat{\sigma})] + \eta^2 \mathbb{E}[n(x|\hat{\theta}, \hat{\sigma})^2] \). We derive \( \mathbb{E}[n(x|\hat{\theta}, \hat{\sigma})^2] \) and \( \mathbb{E}[x n(x|\hat{\theta}, \hat{\sigma})] \) in turn (\( \mathbb{E}[\bar{s}] \) is derived above). From Lemma B.1,

\[ n(x|\hat{\theta}, \hat{\sigma})^2 = \hat{\sigma}^2 \left\{ \lambda^2 \bar{s}^2 - [2\lambda (\lambda - 1)](z^2\Phi(z) + z\phi(z)) + (\lambda - 1)^2(z^2\Phi(z)^2 + 2z\Phi(z)\phi(z) + \phi(z)^2) \right\} \quad \text{(B.8)} \]

We now take the expectation of each of these terms with respect to the true distribution. To do so, we first rewrite \( n(x|\hat{\theta}, \hat{\sigma})^2 \) in terms of several Gaussian integrals, and then evaluate those integrals.

\[ n(x|\hat{\theta}, \hat{\sigma})^2 = \hat{\sigma}^2 \left\{ \lambda^2 (a^2 + b^2 I_4) - [2\lambda (\lambda - 1)] \left( a^2 I_1 + 2abI_2 + b^2 I_5 + aI_3 + bI_6 \right) + (\lambda - 1)^2 \left( a^2 I_{10} + 2abI_{11} + b^2 I_{12} + 2(aI_8 + bI_9) + I_7 \right) \right\} \quad \text{(B.9)} \]

where

\[ I_1 = \int \Phi(a + bw)\phi(w)dw \quad I_2 = \int w\Phi(a + bw)\phi(w)dw \]
\[ I_3 = \int \phi(a + bw)\phi(w)dw \quad I_4 = \int w^2\phi(w)dw \]
\[ I_5 = \int \phi(a + bw)\phi(w)dw \quad I_6 = \int w^2\Phi(a + bw)\phi(w)dw \]
\[ I_7 = \int \Phi(a + bw)^2\phi(w)dw \quad I_8 = \int \Phi(a + bw)^2\phi(w)dw \]
\[ I_9 = \int \Phi(a + bw)^2\phi(w)dw \quad I_{10} = \int \Phi(a + bw)^2\phi(w)dw \]
\[ I_{11} = \int \Phi(a + bw)^2\phi(w)dw \quad I_{12} = \int w^2\Phi(a + bw)^2\phi(w)dw \]

We now evaluate each of these integral terms. Note that \( I_1, I_2, \) and \( I_3 \) were derived above, and \( I_4 = \mathbb{E}[w^2] = \sigma^2 - \mathbb{E}[w]^2 = 1 \). We now turn to the remaining terms:

\( I_5 \): Letting \( f(-|m,s) \) denote a generic normal p.d.f. with mean \( m \) and standard deviation \( s \), the following identity will be useful:

\[ f(w|m_1,s_1)f(w|m_2,s_2) = Sf(w|\bar{m},\bar{s}) \quad \text{(B.10)} \]
where
\[ \bar{m} = \frac{m_1 s_2 + m_2 s_1}{s_1^2 + s_2^2} \quad \text{and} \quad \bar{s} = \sqrt{\frac{s_1^2 s_2^2}{s_1^2 + s_2^2}}, \]
and \( S \equiv f(m_1 | m_2, \sqrt{s_1^2 + s_2^2}) \) is a scaling factor. Using this identity with \( \bar{m} = -a/b \left( \frac{1}{b^2} + 1 \right) \),

\[
I_5 = \frac{1}{b} \frac{1}{\sqrt{1 + b^2}} \phi \left( \frac{-a/b}{\sqrt{1 + b^2}} \right) \int w f(w | \bar{m}, \bar{s}) dw
= \frac{1}{\sqrt{1 + b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) - \frac{a b}{1 + b^2},
\]
which follows from the fact that \( \phi \) is symmetric: \( \phi(-y) = \phi(y) \).

\( I_6 \): Using integration by parts,

\[
I_6 = \int w^2 \Phi(a + bw) \phi(w) dw = \int w \Phi(a + bw) [w \phi(w)] dw
= -w \Phi(a + bw) \phi(w) \bigg|_{-\infty}^{\infty} + \int \phi(w) [bw \phi(a + bw) + \Phi(a + bw)] dw
= b \int w \phi(a + bw) \phi(w) dw + \int \Phi(a + bw) \phi(w) dw
= b I_5 + I_1.
\]

\( I_7 \): Using our identity for the product of two normal densities above (Equation B.10), \( \phi(z)^2 = S f(z | \bar{m}, \bar{s}) \), where \( \bar{m} = 0 \), \( \bar{s} = 1/\sqrt{2} \) and \( S = \phi(0)/\sqrt{2} \). Hence, \( I_7 = \phi(0) \int \phi(\bar{a} + \bar{b}z) \phi(w) dw \), where \( \bar{a} = \sqrt{2}a \) and \( \bar{b} = \sqrt{2}b \), thus, using the derivation of \( I_5 \):

\[
I_7 = \frac{\phi(0)}{\sqrt{1 + b^2}} \phi \left( \frac{\bar{a}}{\sqrt{1 + b^2}} \right) = \frac{\phi(0)}{\sqrt{1 + 2b^2}} \phi \left( \frac{\sqrt{2}a}{\sqrt{1 + 2b^2}} \right).
\]

\( a I_8 + b I_9 \): Note that \( a I_8 + b I_9 = \int (a + bw) \Phi(a + bw) \Phi(a + bw) \phi(w) dw \), which can be simplified using our formula for the product of two normal density functions: \( \phi(a + bw) \phi(w) = S f(w | \bar{m}, \bar{s}^2) \)
where \( f(\cdot | \bar{m}, \bar{s}^2) \) is the p.d.f. of a normal random variable with mean \( \bar{m} = -ab/(1 + b^2) \), standard deviation \( \bar{s} = 1/(\sqrt{1 + b^2}) \), and scaling factor \( S = \frac{1}{\sqrt{1 + b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \). Thus

\[
a I_8 + b I_9 = S \left( (a + b\bar{m}) \int \Phi(a + bw) f(w | \bar{m}, \bar{s}^2) dw + b\bar{s} \int y \Phi(a + bw) f(w | \bar{m}, \bar{s}^2) dw \right)
\]
where \( y = \frac{w - \bar{m}}{\bar{s}} \). Finally, using the derivation of \( I_1 \) (Equation B.4), \( \int \Phi(a + bw) f(w | \bar{m}, \bar{s}^2) dw = \int \Phi(a' + b'y) \phi(y) dy = \Phi \left( \frac{a'}{\sqrt{1 + b'^2}} \right) \), where \( a' = a + b\bar{m} = a/(1 + b^2) \) and \( b' = b\bar{s} = b/\sqrt{1 + b^2} \).

Likewise, using the derivation of \( I_2 \) (Equation B.5), \( \int y \Phi(a' + b'y) \phi(y) dy = \frac{b'}{\sqrt{1 + b'^2}} \phi \left( \frac{a'}{\sqrt{1 + b'^2}} \right) \).
Combining all the terms above yields

\[ aI_8 + bI_9 = \frac{1}{\sqrt{1 + b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \left[ \frac{a}{1 + b^2} \Phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) + \frac{b^2}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) \right]. \]  

(B.14)

\( I_{10} \): Note that

\[ I_{10} = \int \Phi(a + bw)^2 \phi(w) dw = \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right) - 2T \left( \frac{a}{\sqrt{1 + b^2}}, \frac{1}{\sqrt{1 + 2b^2}} \right), \]

where \( T(h, q) = \phi(h) \int_0^q \frac{\phi(hx)}{1 + x^2} dx \) is Owen’s T function.

\( I_{11} \): Note that

\[ I_{11} = \int w\Phi(a + bw)^2 \phi(w) dw = \frac{2b}{\sqrt{1 + b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \Phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right). \]

(B.15)

\( I_{12} \): Integration by parts yields

\[ I_{12} = \int \left[ \Phi(a + bw)^2 + 2bw\Phi(a + bw)\Phi(a + bw) \right] \phi(w) dw = I_{10} + 2b \int w\Phi(a + bw)\phi(a + bw)\phi(w) dw. \]

The integral \( \int w\Phi(a + bw)\phi(a + bw)\phi(w) dw \) was calculated in the derivation of \( aI_8 + bI_9 \) above.

It follows that

\[ I_{12} = I_{10} + \frac{2b^2}{\sqrt{1 + b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \left[ \frac{-a}{1 + b^2} \Phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) + \frac{1}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) \right]. \]  

(B.16)

Using \( I_1 \) through \( I_{12} \) derived above, tedious algebra allows us to write \( \mathbb{E}[n(x|\hat{\theta}, \hat{\sigma})^2] \) (Equation B.9) as \( \hat{\sigma}^2 N_2(a, b) \) where

\[ N_2(a, b) = \lambda^2 (a^2 + b^2) \]

\[- [2\lambda (1 - \lambda)] \left[ (a^2 + b^2) \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right) + a \sqrt{1 + b^2} \phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \right] + (\lambda - 1)^2 \left\{ (a^2 + b^2) \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right) - 2(a^2 + b^2) T \left( \frac{a}{\sqrt{1 + b^2}}, \frac{1}{\sqrt{1 + 2b^2}} \right) \right\]  

\[ + 2(1 + b^2) S \left[ a \Phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) + \frac{b^2}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \phi \left( \frac{a}{\sqrt{1 + b^2} \sqrt{1 + 2b^2}} \right) \right] + \phi(0) \frac{\sqrt{2a}}{\sqrt{1 + 2b^2}} \phi \left( \frac{\sqrt{2a}}{\sqrt{1 + 2b^2}} \right) \right\}. \]  

(B.18)
We now derive $\mathbb{E}[x_\eta(x|\hat{\theta}, \hat{\sigma})]$. Using Lemma B.1 and the change of variables introduced above,

\[
x_\eta(x|\hat{\theta}, \hat{\sigma}) = \hat{\sigma} \lambda (\theta a + [\theta b + \sigma a] w + \sigma b w^2)
- \hat{\sigma}(\lambda - 1)(\theta a + [\theta b + \sigma a] w + \sigma b w^2) \Phi(a + bw) - \hat{\sigma}(\lambda - 1)(\theta - \sigma w) \phi(a + bw).
\]  

(B.19)

Taking expectations with respect to $w$ and using the integral identities above yields

\[
\mathbb{E}[x_\eta(x)] = \hat{\sigma} \{ \lambda [\theta a + \sigma b] - (\lambda - 1)[\theta aI_1 + bI_2 + I_3] + \sigma bI_1 \}.
\]  

(B.20)

Putting these components together,

\[
\text{Var}[\hat{x}|\hat{\theta}, \hat{\sigma}] = \sigma^2 + \theta^2 + 2\eta \mathbb{E}[x_\eta(x|\hat{\theta}, \hat{\sigma})] + \eta^2 \hat{\sigma}^2 N_2(a, b) - (\theta + \hat{\sigma} \eta N_0(a, b))^2,
\]  

(B.21)

where the last term follows from Equation B.3 with $N_1(a, b) \equiv \lambda a - (\lambda - 1)[aI_1 + bI_2 + I_3]$. Combining Equations B.20 and B.21 yields

\[
\text{Var}[\hat{x}|\hat{\theta}, \hat{\sigma}] = \sigma^2 + 2\eta \hat{\sigma} (\mathbb{E}[x_\eta(x|\hat{\theta}, \hat{\sigma})] - N_1(a, b)) + \eta^2 \hat{\sigma}^2 [N_2(a, b) - N_1(a, b)]
\]  

\[
= \sigma^2 + 2\eta \sigma^2 (\lambda - (\lambda - 1)I_1) + \eta^2 \hat{\sigma}^2 [N_2(a, b) - N_1(a, b)]
\]  

\[
= \sigma^2 + 2\eta \sigma^2 N_0(a, b) + \eta^2 \hat{\sigma}^2 [N_2(a, b) - N_1(a, b)],
\]  

(B.22)

where $N_0(a, b) \equiv \lambda - (\lambda - 1)I_1$. Thus $\hat{\sigma}^2 = \text{Var}[\hat{x}|\hat{\theta}, \hat{\sigma}]$ is equivalent to

\[
0 = b^2 + 2\eta b^2 N_0(a, b) + \eta^2 [N_2(a, b) - N_1(a, b)].
\]  

(B.23)

Finally, from Equations B.7 and B.23, $\hat{\theta}$ and $\hat{\sigma}$ are implicitly defined by the values $a$ and $b$ that solve the system

\[
0 = a + \eta N_1(a, b)
\]  

(B.24)

\[
0 = b^2 + 2\eta b^2 N_0(a, b) + \eta^2 [N_2(a, b) - N_1(a, b)].
\]  

(B.25)

One can show that this system has a unique solution. Notice that both of the equations in the system above depend solely on the p.d.f. and c.d.f. of the normal distribution and parameters $\eta$ and $\lambda$. Thus, the solution $(a^*, b^*)$ characterizes $\hat{\theta}$ and $\hat{\sigma}$ as follows: $\hat{\sigma} = \sigma / b^*$ and $\hat{\theta} = \theta - a^* \hat{\sigma} = \theta - \frac{a^*}{b^*} \sigma$. Again, since $a^*$ and $b^*$ are independent of $\sigma$, it’s clear that the perceived mean is linearly decreasing in the true variance of outcomes.

### C Reputation Model with Loss Aversion

This section explores the career-concern model of Section 7 when the principal is loss averse ($\lambda > 1$). In this case, because gains and losses affect the principal’s beliefs asymmetrically, the agent faces uncertainty over whether any future outcome will be weighted as a gain or as a loss. Given that the agent’s current choice of effort influences this uncertainty, he must account for this when forming his optimal policy. We derive the agent’s optimal effort policy when $T = 3$, highlighting that it is qualitatively similar to the case without loss aversion ($\lambda = 1$) analyzed in the main text. We also note how the intuition behind Proposition 10 continues to hold. In particular,
in the three-period model, \( e_1 \) will fall below the rational benchmark, while \( e_2 \) exceeds it. Finally, we argue that these qualitative similarities between the \( \lambda = 1 \) and \( \lambda > 1 \) cases hold more generally for any arbitrary \( T \).

Suppose \( T = 3 \), and for ease of exposition, let \( \delta = 1 \). Since \( e_3^* = 0 \), the agent initially seeks to maximize \( \Pi_1 \equiv \mathbb{E}_{x_1}[\Theta_1] - c(e_1) + \mathbb{E}_{(x_1,x_2)}[\hat{\Theta}_2 - c(e_2)] \). Note that \( \hat{\Theta}_2 = \hat{\Theta}_1 + \alpha_2(1 + \kappa_2)[x_2 - \hat{e}_2 - \hat{\Theta}_1] = \hat{\Theta}_1 + \alpha_2(1 + \kappa_2)d_2 \), where \( d_2 \equiv x_2 - \hat{e}_2 - \hat{\Theta}_1 \). Hence, \( \Pi_1 = \mathbb{E}_{(x_1,x_2)}[2\hat{\Theta}_1 + \alpha_2(1 + \kappa_2)d_2 - c(e_2)] - c(e_1) \). Of course, \( e_2 \) is a function of \( e_1 \). Hence, to optimize \( \Pi_1 \) with respect to \( e_1 \), we first derive how \( e_2 \) depends on \( e_1 \) at the optimum.

Note that the period-2 objective is \( \Pi_2 = \mathbb{E}_{x_2}\hat{\Theta}_2 - c(e_2) = \hat{\Theta}_1 + \alpha_2\mathbb{E}_{x_2}[1 + \kappa_2]d_2|x_1] - c(e_2) \), implying \( e_2 \) must satisfy \( c'(e_2) = \alpha_2\frac{\partial}{\partial e_2}\mathbb{E}_{x_2}[1 + \kappa_2]d_2|x_1] \). Note that \( d_2 \sim N(\hat{\Theta}_1 + e_2 - \hat{\Theta}_1 - \hat{e}_2, \sigma_1^2) \), where \( \hat{\Theta}_1 \) is the rational estimate of \( \Theta \) following \( x_1 \) and \( \sigma_1^2 \) is the variance of \( \hat{\Theta}_1 + e_2 \), which is independent of \( e_1 \). Let \( p_2 \equiv \text{Pr}(d_2 > 0|x_1) \), and note that \( p_2 = 1 - \Phi\left(-\frac{\hat{e}_2}{\sigma_1}\right) \), where \( \Phi \) is the standard normal c.d.f. Thus, \( \mathbb{E}_{x_2}(1 + \kappa_2)d_2|x_1] = \bar{d}_2 + p_2 \kappa^G \mathbb{E}_{x_2}[d_2|d_2 > 0,x_1] + (1 - p_2) \kappa^G \mathbb{E}_{x_2}[-d_2|d_2 < 0,x_1] \), where \( \bar{d}_2 = \mathbb{E}[d_2|x_1] \). Since \( \mathbb{E}_{x_2}[d_2|d_2 > 0] = \bar{d}_2 + \sigma_1 \phi\left(-\frac{\hat{e}_2}{\sigma_1}\right) / p_2 \) and \( \mathbb{E}_{x_2}[d_2|d_2 < 0] = \bar{d}_2 - \sigma_1 \phi\left(-\frac{\hat{e}_2}{\sigma_1}\right) / (1 - p_2) \), it follows that \( \mathbb{E}_{x_2}(1 + \kappa_2)d_2|x_1] = [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \bar{d}_2 - (\kappa^L - \kappa^G) \sigma_1 \phi\left(-\frac{\hat{e}_2}{\sigma_1}\right) \). This implies

\[
\frac{\partial}{\partial e_2}\mathbb{E}_{x_2}(1 + \kappa_2)d_2|x_1] = - (\kappa^L - \kappa^G) \frac{\partial p_2}{\partial e_2} \bar{d}_2 + [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \frac{\partial \bar{d}_2}{\partial e_2}
\]

\[
- (\kappa^L - \kappa^G) \sigma_1 \phi\left(-\frac{\hat{e}_2}{\sigma_1}\right) \left(\frac{\bar{d}_2}{\sigma_1}\right) \left(-\frac{1}{\sigma_1}\right) \frac{\partial \bar{d}_2}{\partial e_2} \]

From the definition of \( p_2 \), \( \frac{\partial p_2}{\partial e_2} = \phi\left(-\frac{\hat{e}_2}{\sigma_1}\right) \left(\frac{1}{\sigma_1}\right) \frac{\partial \bar{d}_2}{\partial e_2} \), and since \( \frac{\partial \bar{d}_2}{\partial e_2} = 1 \), Equation B.26 reduces to

\[
\mathbb{E}_{x_2}(1 + \kappa_2)d_2|x_1] = [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \bar{d}_2 - (\kappa^L - \kappa^G) \sigma_1 \phi\left(-\frac{\hat{e}_2}{\sigma_1}\right) \phi\left(-\frac{1}{\sigma_1}\right) \frac{\partial \bar{d}_2}{\partial e_2} \]

Hence, the first-order condition for \( e_2 \) amounts to

\[
c'(e_2) = \alpha_2[1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \]

Returning to the optimal effort choice in period 1, note that \( e_1 \) must satisfy first-order condition

\[
\mathbb{E}_{x_1}\left[2\frac{\partial \hat{\Theta}_1}{\partial e_1} + \alpha_2(1 + \kappa_2)d_2 - c(e_2)|x_1]\right] = c'(e_1). \quad (B.28)
\]

Analogous to the derivation of Equation B.26, \( \frac{\partial}{\partial e_1}\mathbb{E}_{x_2}(1 + \kappa_2)d_2|x_1] = [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \frac{\partial \bar{d}_2}{\partial e_1} \), where \( \frac{\partial \bar{d}_2}{\partial e_1} = \frac{\partial \bar{d}_2}{\partial e_1} - \frac{\partial \hat{\Theta}_1}{\partial e_1} \) given that \( \bar{d}_2 = \hat{\Theta}_1 + e_2 - \hat{\Theta}_1 - \hat{e}_2 \). Hence, the first-order condition for \( e_1 \) amounts to

\[
\mathbb{E}_{x_1}\left[2\frac{\partial \hat{\Theta}_1}{\partial e_1} + \alpha_2[1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \left[\frac{\partial e_2}{\partial e_1} - \frac{\partial \hat{\Theta}_1}{\partial e_1}\right] - c'(e_2) \frac{\partial e_2}{\partial e_1}\right] = c'(e_1) \quad (B.29)
\]
Since the agent’s choice of $e_2$ conditional on $x_1$ must satisfy Equation B.27, Equation B.28 yields
\[
\mathbb{E}_{x_1} \left[ 2 \frac{\partial \hat{\theta}_1}{\partial e_1} - \alpha_2 [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \frac{\partial \hat{\theta}_1}{\partial e_1} \right] = c'(e_1). \tag{B.30}
\]

Since \( \hat{\theta}_1 = \alpha_1(1 + \kappa_1) d_1 \), where \( d_1 = x_1 - \hat{e}_1, \frac{\partial \hat{\theta}_1}{\partial \sigma_1} = \alpha_1 (1 + \kappa_1) \), the first-order condition for \( e_1 \) reduces further to \( c'(e_1) = \alpha_1 \mathbb{E}_{x_1} [(1 + \kappa_1)(2 - \alpha_2 - \alpha_2 [p_2 \kappa^G + (1 - p_2) \kappa^L])] = \mathbb{E}_{x_1} [(1 + \kappa_1)(\alpha_1 + \alpha_2 - \alpha_1 \alpha_2 [p_2 \kappa^G + (1 - p_2) \kappa^L])], \) where the second equality follows from the fact that \( \alpha_1 (1 - \alpha_2) = \alpha_2 \). Note that \( [p_2 \kappa^G + (1 - p_2) \kappa^L] = \mathbb{E}[\kappa_2 | x_1] \) given the optimal policy. Hence, the first-order condition for \( e_1 \) can be written as \( c'(e_1) = \mathbb{E}_{(x_1, x_2)} [(1 + \kappa_1)(\alpha_1 + \alpha_2 [1 - \kappa_2 \alpha_2])]. \) Note that this first-order condition is equivalent to the one with \( \lambda = 1 \) (see Equation 23) aside from the expected value over \( \kappa_t \) on the right-hand side—with \( \lambda = 1 \), each \( \kappa_t = \kappa^G \) deterministically.

Given the similarity between the solutions with \( \hat{\lambda} = 1 \) and \( \lambda > 1 \), the predictions of Proposition 10 continue to hold with \( \lambda > 1 \). We can see this clearly in the analysis above. The first-order condition for \( e_2 \) requires that \( c'(e_2) = \alpha_2 [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] \). Rational effort, however, solves \( c'(e_2') = \alpha_2 \). Since \( \alpha_2 [1 + p_2 \kappa^G + (1 - p_2) \kappa^L] > \alpha_2 \), second-round effort under misattribution exceeds the rational benchmark. Contrastingly, first-round effort may fall short: rational effort solves \( c'(e_2') = \alpha_1 + \alpha_2 \), while effort under misattribution solves \( c'(e_1') = \mathbb{E}_{(x_1, x_2)} [(1 + \kappa_1)(\alpha_1 + \alpha_2 [1 - \kappa_2 \alpha_2])] \).

One could continue this backward induction argument for arbitrary \( T \) and show that, in general, \( e_t = c_e^{-1}(\mathbb{E}[M_t^e | e_t, h_{t-1}]) \), where \( M_t^e \equiv (1 + \kappa_j) \left\{ \delta \alpha_t + \sum_{t=1}^T \delta \tau^{-i} \alpha_{t-1} \left( \Pi_{j=1}^{t-1} [1 - \kappa_j] \right) \right\} \) and \( \mathbb{E}[M_t^e | e_t, h_{t-1}] \) is the expected value of \( M_t^e \) conditional on the history, the agent’s current choice, and her policy going forward. Recall that when \( \lambda = 1 \), \( e_t = c_e^{-1}(M_t) \), where
\[
M_t = (1 + \kappa^G) \left\{ \delta \alpha_t + \sum_{t=1}^T \delta \tau^{-i} \alpha_{t-1} \left( \Pi_{j=1}^{t-1} [1 - \kappa^G] \right) \right\} \tag{see Equation 23}.
\]
Hence, the key difference between the solution with \( \lambda > 1 \) and the one with \( \lambda = 1 \) is that the expected values of \( \kappa_t \) in \( M_t \) will fall in the interval \([\kappa^G, \kappa^L]\) rather than lie deterministically at \( \kappa^G \). This change will not alter the qualitative pattern of the effort profile in expectation.

**D Additional Proofs**

**Proof of Proposition 1.**

**Proof. Part 1.** Note \( \hat{\theta} \) solves \( G(\hat{\theta}; \theta, \sigma) = \hat{\theta} - \theta + kH(\hat{\theta}; \theta, \sigma) = 0 \) where
\[
H(\hat{\theta}; \theta, \sigma) \equiv \text{Pr}(x < \hat{\theta}) \left( \hat{\theta} - \mathbb{E}[x | x < \hat{\theta}] \right).
\]

Note that \( H(\cdot; \theta, \sigma) \) is a positive and strictly increasing function of \( \hat{\theta} \): \( H(\hat{\theta}; \theta, \sigma) = \hat{\theta} F_{Z} \left( \frac{\hat{\theta} - \theta}{\sigma} \right) - \).
\[ f_{\hat{\theta}} \frac{1}{\sigma} f_{Z}(\frac{x-\theta}{\sigma}) \] dx, hence
\[
\frac{\partial}{\partial \theta} H(\hat{\theta}; \theta, \sigma) = \theta \frac{1}{\sigma} f_{Z}(\frac{\theta - \theta}{\sigma}) + F_{Z}\left(\frac{\theta - \theta}{\sigma}\right) - \hat{\theta} \frac{1}{\sigma} f_{Z}\left(\frac{\hat{\theta} - \theta}{\sigma}\right) = F_{Z}\left(\frac{\hat{\theta} - \theta}{\sigma}\right) > 0. \tag{C.1}
\]

Hence \( G(\hat{\theta}; \theta, \sigma) \) is an increasing function of \( \hat{\theta} \) with range \( \mathbb{R} \). Since \( \hat{\theta} \) is defined by \( G(\hat{\theta}; \theta, \sigma) = 0 \), the solution \( \hat{\theta} \) exists and is unique.

**Part 2.** We first establish underestimation. For sake of contradiction, suppose \( \hat{\theta} > \theta \). Note that \( \hat{\theta} \) solves \( \hat{\theta} + kH(\hat{\theta}; \theta, \sigma) = \theta \). Because both \( H(\hat{\theta}; \theta, \sigma) \geq 0 \) and \( k > 0 \), \( \hat{\theta} + kH(\hat{\theta}; \theta, \sigma) \) exceeds \( \theta \) — a contradiction. Turning to comparative statics, since \( \hat{\theta} \) satisfies \( G(\hat{\theta}; \theta, \sigma) = 0 \), the implicit function theorem implies that, for any parameter \( w \in \{\eta, \hat{\eta}, \lambda, \sigma, \theta\} \),
\[
\frac{\partial \hat{\theta}}{\partial w} = -\left(\frac{\partial G(\hat{\theta}; \theta, \sigma)}{\partial \hat{\theta}}\right)^{-1}\frac{\partial G(\hat{\theta}; \theta, \sigma)}{\partial w}. \quad \text{Since } \frac{\partial G(\hat{\theta}; \theta, \sigma)}{\partial \theta} = 1 + k \frac{\hat{\theta}}{\partial \theta} H(\hat{\theta}; \theta, \sigma) > 0, \quad \frac{\partial \hat{\theta}}{\partial w} \text{ has the opposite sign as } \frac{\partial G(\hat{\theta}; \theta, \sigma)}{\partial \hat{\theta}}. \]
It thus follows from the definition of \( k \) that \( \hat{\theta} \) is decreasing in \( \lambda \) and \( \eta \), and increasing in \( \hat{\eta} \). To show that variance has a decreasing effect, note that
\[
\frac{\partial G(\hat{\theta}; \theta, \sigma)}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left( \hat{\theta} F_{Z}\left(\frac{\hat{\theta} - \theta}{\sigma}\right) - \int_{-\infty}^{\frac{\theta - \theta}{\sigma}} [\theta + \sigma u] f_{Z}(u) du \right)
\]
\[
= -\left(\frac{\hat{\theta} - \theta}{\sigma^{2}}\right) \hat{\theta} f_{Z}\left(\frac{\hat{\theta} - \theta}{\sigma}\right) + \left(\frac{\hat{\theta} - \theta}{\sigma^{2}}\right) \left[\theta + \sigma u\right] f_{Z}(u) \bigg|_{u=\frac{\theta - \theta}{\sigma}} - \int_{-\infty}^{\frac{\theta - \theta}{\sigma}} u f_{Z}(u) du
\]
\[
= -\int_{-\infty}^{\frac{\theta - \theta}{\sigma}} u f_{Z}(u) du
\]
\[
> 0,
\]
where the second line follows from Leibniz’s Rule and the final equality follows from the fact that the integral is negative given \( \hat{\theta} < \theta \). Finally, \( \frac{\partial}{\partial \theta} [\hat{\theta} - \theta] = -\left(1 + k \frac{\hat{\theta}}{\partial \theta} H(\hat{\theta}; \theta, \sigma)\right)^{-1} \frac{\partial G(\hat{\theta}; \theta, \sigma)}{\partial \theta} - 1. \]
Hence, from Equation C.1,
\[
\frac{\partial}{\partial \theta} [\hat{\theta} - \theta] = -\left(1 + k F_{Z}\left(\frac{\hat{\theta} - \theta}{\sigma}\right)\right)^{-1} \frac{\partial G(\hat{\theta}; \theta, \sigma)}{\partial \theta} - 1. \tag{C.2}
\]
Equation C.1 additionally implies \( \frac{\partial G(\hat{\theta}; \theta, \sigma)}{\partial \theta} = -1 - k F_{Z}\left(\frac{\hat{\theta} - \theta}{\sigma}\right) \). Hence, Equation C.2 implies \( \frac{\partial}{\partial \theta} [\hat{\theta} - \theta] = 0. \)

**Part 3.** Fix \( \hat{\theta} = \theta \), so \( \hat{x} = x_{i} + \kappa_{i}(x_{i} - \theta) \). Hence, \( \text{Var}(\hat{\theta}) = \text{Var}(x_{i}) + \text{Var}(\kappa_{i}(x_{i} - \theta)) + 2 \text{Cov}(x_{i}, \kappa_{i}(x_{i} - \theta)) \). Note that \( \text{Cov}(x_{i}, \kappa_{i}(x_{i} - \theta)) = \mathbb{E}[\kappa_{i}(x_{i} - \theta)] - \theta \mathbb{E}[\kappa_{i}(x_{i} - \theta)] \), where
\[
\mathbb{E}[\kappa_{i}(x_{i} - \theta)] = [1 - F(\theta)] \kappa^{G} \mathbb{E}[x_{i} - \theta | x_{i} \geq \theta] + F(\theta) \kappa^{L} \mathbb{E}[x_{i} - \theta | x_{i} < \theta]
\]
\[
= \kappa^{G}(\theta - \theta) + F(\theta)(\kappa^{L} - \kappa^{G}) [\mathbb{E}[x_{i} - \theta | x_{i} < \theta]], \tag{C.3}
\]
and
\[
\mathbb{E}[x, k_i(x_t - \hat{\theta})] = (1 - F(\hat{\theta}))\kappa^G \mathbb{E}[x_t(x_t - \hat{\theta}) | x_t \geq \hat{\theta}] + F(\hat{\theta})\kappa^L \mathbb{E}[x_t(x_t - \hat{\theta}) | x_t < \hat{\theta}] \\
= \kappa^G \mathbb{E}[x_t(x_t - \hat{\theta})] + F(\hat{\theta})(\kappa^L - \kappa^G) \mathbb{E}[x_t(x_t - \hat{\theta}) | x_t < \hat{\theta}] \\
= \kappa^G(\sigma^2 + \theta^2 - \theta \hat{\theta}) + F(\hat{\theta})(\kappa^L - \kappa^G) \mathbb{E}[x_t(x_t - \hat{\theta}) | x_t < \hat{\theta}],
\]
(C.4)
where the last line follows from the fact that \(\sigma^2 \equiv \text{Var}(x_t)\) and \(\theta \equiv \mathbb{E}[x_t]\). Hence,
\[
\text{Cov}(x_t, k_i(x_t - \hat{\theta})) = \kappa^G \sigma^2 + F(\hat{\theta})(\kappa^L - \kappa^G) \mathbb{E}[x_t(x_t - \hat{\theta}) - \theta(x_t - \hat{\theta}) | x_t \leq \hat{\theta}] \\
= \kappa^G \sigma^2 + F(\hat{\theta})(\kappa^L - \kappa^G) \mathbb{E}[(x_t - \theta)(x_t - \hat{\theta}) | x_t \leq \hat{\theta}]
\]
(C.5)
Since \(\hat{\theta} < \theta\) (Part 2), \(x_t < \hat{\theta}\) implies \(x_t < \theta\), meaning the expectation above is always positive. Thus, \(\text{Var}(\hat{x}_t) = \text{Var}(x_t) + \text{Var}(k_i(x_t - \hat{\theta})) + 2\text{Cov}(x_t, k_i(x_t - \hat{\theta})) > \text{Var}(x_t)\). The relevant comparative statics follow from the dependence of \(k_i\) on \(\eta, \lambda\), and \(\hat{\eta}\).

**Proof of Proposition 2.**

*Proof. Part 1. Hypothetical utility given \((\theta, \sigma)\) is*
\[
\mathbb{E}[u(x|\theta)] = \mathbb{E}[x] + \eta \Pr(x \geq \theta)(\mathbb{E}[x|x \geq \theta] - \theta) + \eta \lambda \Pr(x < \theta)(\mathbb{E}[x|x < \theta] - \theta) \\
= \theta - \eta(\lambda - 1)\Pr(x < \theta)(\theta - \mathbb{E}[x|x < \theta]) \\
= \theta - \eta(\lambda - 1)H(\theta; \theta, \sigma).
\]
(C.6)
Thus forecasted utility is \(\tilde{\mathbb{E}}[u(x|\hat{\theta})] = \hat{\theta} - \eta(\lambda - 1)H(\hat{\theta}; \hat{\theta}, \hat{\sigma})\). Notice that for any \(\hat{\theta}\) and \(\hat{\sigma}\), \(H(\hat{\theta}; \hat{\theta}, \hat{\sigma}) = \tilde{H}(\hat{\theta}, \hat{\sigma})\) where
\[
\tilde{H}(\hat{\theta}, \hat{\sigma}) = \hat{\theta} F_Z(0) - \int_{-\infty}^{0} [\hat{\theta} + \hat{\sigma} u] f_Z(u) du.
\]
(C.7)
Note that
\[
\frac{\partial}{\partial \theta} \tilde{H}(\hat{\theta}, \hat{\sigma}) = F_Z(0) - \int_{-\infty}^{0} f_Z(u) du = 0,
\]
and
\[
\frac{\partial}{\partial \hat{\sigma}} \tilde{H}(\hat{\theta}, \hat{\sigma}) = - \int_{-\infty}^{0} uf_Z(u) du > 0.
\]
Hence \(\tilde{H}(\hat{\theta}, \hat{\sigma}) > H(\theta, \sigma)\) given that \(\hat{\sigma} > \sigma\). Together with the fact that \(\hat{\theta} < \theta\), this implies that \(\mathbb{E}[u(x|\hat{\theta})] - \tilde{\mathbb{E}}[u(x|\hat{\theta})] = (\theta - \hat{\theta}) - \eta(\lambda - 1)(H(\theta, \sigma) - \tilde{H}(\hat{\theta}, \hat{\sigma})) > 0\).

*Part 2. Average experienced utility given \((\theta, \sigma)\) and \(\hat{\theta}\) is instead*
\[
\mathbb{E}[u(x|\hat{\theta})] = \mathbb{E}[x] + \eta \Pr(x \geq \hat{\theta})(\mathbb{E}[x|x \geq \hat{\theta}] - \hat{\theta}) + \eta \lambda \Pr(x < \hat{\theta})(\mathbb{E}[x|x < \hat{\theta}] - \hat{\theta}) \\
= \theta + \eta(\theta - \hat{\theta}) - \eta(\lambda - 1)\Pr(x < \hat{\theta})(\theta - \mathbb{E}[x|x < \hat{\theta}]) \\
= \theta + \eta(\theta - \hat{\theta}) - \eta(\lambda - 1)H(\hat{\theta}; \theta, \sigma).
\]
The difference in average utility experienced by a misattributing agent relative to hypothetical utility (Equation C.6) is thus
\[
\mathbb{E}[u(x|\hat{\theta})] - \mathbb{E}[u(x|\theta)] = \eta(\theta - \hat{\theta}) + \eta(\lambda - 1)(H(\theta; \theta, \sigma) - H(\hat{\theta}; \theta, \sigma)).
\] (C.8)

Given that \(\theta > \hat{\theta}\) and \(H(\cdot; \hat{\theta}, \sigma)\) is increasing, it is clear that experienced utility exceeds hypothetical utility. Because \(\hat{\theta}\) is decreasing in \(\lambda\) and \(\eta\), an increase in either of these values increases the difference in Equation C.8. Finally, we consider the effect of \(\sigma\) on this difference:

\[
\frac{d}{d\sigma}(\mathbb{E}[u(x|\hat{\theta})] - \mathbb{E}[u(x|\theta)]) = -\eta \frac{\partial \hat{\theta}}{\partial \sigma} + \eta(\lambda - 1)\left(\frac{\partial}{\partial \sigma} H(\theta; \theta, \sigma) - \frac{d}{d\sigma} H(\hat{\theta}; \theta, \sigma)\right)
= -\eta \frac{\partial \hat{\theta}}{\partial \sigma} \left(1 + (\lambda - 1)\frac{\partial}{\partial \sigma} H(\hat{\theta}; \theta, \sigma)\right)
> 0.
\]

\[\blacksquare\]

**Proof of Corollary 1.**

*Proof.* Suppose \(A\) has parameters \((\theta, \sigma)\), and let \(v_A(\theta, \sigma)\) denote the hypothetical (i.e., fully informed) expected utility from \(A\). Likewise, let \(\hat{v}_A(\theta, \sigma)\) denote the long-run forecasted utility after \(t^* \to \infty\) rounds of experimentation. From Proposition 2, \(\hat{v}_A(\theta, \sigma) < v_A(\theta, \sigma)\), and from Equation C.7

\[
\hat{v}_A(\theta, \sigma) = \hat{\theta} + \eta(\lambda - 1)\hat{\sigma} \int_{-\infty}^{0} uf_Z(u)du,
\] (C.9)

where \(\hat{\theta}\) and \(\hat{\sigma}\) are the long-run values characterized by Proposition 1. Let \(\hat{\theta}(\theta)\) be the perceived value as a function of the true parameter \(\theta\), and note from the proof of Proposition 1 that \(\hat{\theta}(\theta)\) is continuous (recall \(F_Z\) and \(f_Z\) are continuous) and strictly increasing in \(\theta\) whenever \(\lambda > 1\). Hence, let \(\hat{\theta}(\sigma)\) be defined by \(\hat{v}_A(\hat{\theta}(\sigma), \sigma) = v_B\). Clearly, the person selects \(A\) at \(t^*\) if and only if \(\theta > \hat{\theta}(\sigma)\) (assuming indifference is broken in favor of \(B\)). By definition of \(\theta^*(\sigma), v_A(\theta^*(\sigma), \sigma) = v_B\), which means \(\hat{v}_A(\theta^*(\sigma), \sigma) < v_A(\theta^*(\sigma), \sigma)\), implying \(\hat{\theta}(\sigma) > \theta^*(\sigma)\). Finally, by Part 1 of Proposition 2, \(\hat{v}_A(\theta, \sigma)\) is strictly decreasing in \(\sigma\), meaning \(\hat{\theta}(\sigma)\) must be strictly increasing in \(\sigma\) to maintain \(\hat{v}_A(\hat{\theta}(\sigma), \sigma) = v_B\).

\[\blacksquare\]

**Proof of Proposition 3.**

*Proof.* Let \(\mathcal{P}(v_A)\) denote the set of parameters \((\theta(A), \sigma(A))\) such that the hypothetical utility of \(A\) is equal to \(v_A \in \mathbb{R}\). For parameters \((\theta(A), \sigma(A)) \in \mathcal{P}(v_A)\), denote the perceived utility of the prospect defined by \((\theta(A), \sigma(A))\) by \(\hat{v}(\theta(A), \sigma(A))\). In truth all of these prospects have hypothetical utility \(v_A\), yet we’ll show that, constrained to \(\mathcal{P}(v_A), \lim_{\sigma(A) \to \infty} \hat{v}(\theta(A), \sigma(A)) = -\infty\).

First, we show that for all prospects \((\theta(A), \sigma(A)) \in \mathcal{P}(v_A)\), the steady-state perceived mean \(\hat{\theta}(A)\) of that prospect is a linearly decreasing function of \(\sigma(A)\). Recall that \(\hat{\theta}(A)\) solves \(\hat{\theta} - \theta + kH(\hat{\theta}; \theta, \sigma) = 0\). Since \(H(\hat{\theta}; \theta, \sigma) = \hat{\theta}F_Z\left(\frac{\hat{\theta} - \theta}{\sigma}\right) - \int_{-\infty}^{\hat{\theta}} \frac{1}{\sigma} f_Z\left(\frac{x - \theta}{\sigma}\right) dx\), we can define \(\hat{\theta} \equiv \hat{\theta}(A)\) as...
\((\hat{\theta} - \theta) / \sigma\) and rewrite \(H(\hat{\theta}; \theta, \sigma)\) as

\[
H(\hat{\theta}; \theta, \sigma) = \hat{\theta}F_Z(\hat{z}) - \int_{-\infty}^{\hat{z}} [\theta + \sigma z] f_Z(z) \, dz
\]  
\[
\quad = [\hat{\theta} - \theta] F_Z(\hat{z}) - \sigma \int_{-\infty}^{\hat{z}} z f_Z(z) \, dz.  \tag{C.10}
\]

Hence, the steady-state value \(\hat{\theta}\) is defined by the value \(\hat{z}\) that solves

\[
\hat{z} + k \left( \hat{z} F_Z(\hat{z}) - \int_{-\infty}^{\hat{z}} z f_Z(z) \, dz \right) = 0.  \tag{C.12}
\]

By virtue that \(\hat{\theta}\) is unique and finite, there exists a unique, finite \(\hat{z}\) that solves Equation C.12, which we denote by \(z^*\). Clearly \(z^*\) depends solely on \(F_Z, f_Z\), and \(k\), and is thus independent of \(\theta\) and \(\sigma\). As such, \(z^* = (\hat{\theta} - \theta) / \sigma\) implies the steady-state estimate is \(\hat{\theta} = \theta + z^* \sigma\). Furthermore, the fact that \(\hat{\theta} < \theta\) implies that \(z^* < 0\). Thus \(\hat{\theta} = \theta - |z^*| \sigma\).

Now consider \((\theta(A), \sigma(A)) \in \mathcal{P}(v_A)\). From Equations C.6 and C.7, the hypothetical and forecasted utilities from this prospect are

\[
v_A = \theta(A) + \eta(\lambda - 1)\sigma(A) \int_{-\infty}^{0} uf_Z(u) \, du  \tag{C.13}
\]

and

\[
\hat{v}(\theta(A), \sigma(A)) = \hat{\theta}(A) + \eta(\lambda - 1)\hat{\sigma}(A) \int_{-\infty}^{0} uf_Z(u) \, du.  \tag{C.14}
\]

Let \(\bar{z}^{-} = \int_{-\infty}^{0} uf_Z(u) \, du < 0\). Substituting the linear specification of \(\hat{\theta}\) into Equation C.14 and then using Equation C.13 yields

\[
\hat{v}(\theta(A), \sigma(A)) = \theta(A) - |z^*| \sigma(A) - \eta(\lambda - 1)\hat{\sigma}(A)|\bar{z}^{-}|
\]

\[
\quad = v_A - |z^*| \sigma(A) - \eta(\lambda - 1)[\hat{\sigma}(A) - \sigma(A)]|\bar{z}^{-}|
\]

\[
\quad < v_A - |z^*| \sigma(A).
\]

where the last line follows from \(\hat{\sigma}(A) > \sigma(A)\) using Proposition 1. Thus \(\hat{v}(\theta(A), \sigma(A)) - v_A\) diverges to \(-\infty\) as \(\sigma(A) \to \infty\) along the locus \(\mathcal{P}(v_A)\).

\[\blacksquare\]

**Proof of Lemma 1.**

*Proof.* Without loss of generality, suppose \(\theta_0 = 0\). Thus \(\hat{\theta}_1 = \alpha_1 \sum_{k=1}^{t} \hat{\xi}_k = \alpha_1 \sum_{k=1}^{t} [x_t + \kappa_t (x_t - \hat{\theta}_{t-1})]\). We prove the following claim by induction: \(\hat{\theta}_i = \alpha_t \sum_{k=1}^{i} \xi_k \xi_k'\) where \(\xi_k = (1 + \kappa_k) \prod_{j=k}^{t-1} [1 - \alpha_j \kappa_{j+1}]\) for \(k < t\) and \(\xi_t' = (1 + \kappa_t)\). To establish the base case, note \(\hat{\theta}_1 = \alpha_1 [x_1 + \kappa_1 (x_1 - \theta_0)] =\)
α_t(1 + κ_t)x_t. Now suppose the claim holds for period $t > 1$. Then

$$
\hat{\theta}_{t+1} = \alpha_{t+1} \sum_{k=1}^{t+1} \hat{x}_k
$$

$$
= \alpha_{t+1} \left\{ ([1 + \kappa_{t+1}]x_{t+1} - \kappa_{t+1} \hat{\theta}_t] + \frac{1}{\alpha_t} \right\}
$$

$$
= \alpha_{t+1} \left\{ (1 + \kappa_{t+1})x_{t+1} + [1 - \alpha_t \kappa_{t+1}] \sum_{k=1}^{t} (1 + \kappa_k) \left( \prod_{j=k}^{t-1} [1 - \alpha_j \kappa_{j+1}] \right) x_k \right\}
$$

Hence, the induction step holds, establishing the claim.

**Proof of Proposition 4.**

*Proof.* The results follow from Lemma 1. If beliefs are monotonic, then $\kappa^L < 1 + \sigma^2 / \rho^2$. This implies $[1 - \alpha_t \kappa_{t+1}]t \in (0, 1)$ for all $t \in \{1, \ldots, T - 1\}$. Consider $t, \tau \in \{1, \ldots, T\}$ such that $\tau < t$ and suppose $\kappa_t = \kappa_\tau$. From Lemma 1, $\xi_t^T / \xi_\tau^T = \prod_{j=\tau}^{t-1} [1 - \alpha_j \kappa_{j+1}] < 1$. Also, from Lemma 1, $\xi_t^T = 1 + \kappa_t > 1$ and $\lim_{T \to \infty} \xi_t^T = (1 + \kappa_t) \lim_{T \to \infty} \prod_{j=\tau}^{T-1} [1 - \alpha_j \kappa_{j+1}] \leq (1 + \kappa_t) \lim_{T \to \infty} \prod_{j=\tau}^{T-1} [1 - \alpha_j \kappa^G]$. Since $\sum_{j=\tau}^{\infty} \alpha_j$ diverges, $\prod_{j=\tau}^{\infty} [1 - \alpha_j \kappa^G] = 0$, completing the proof.

**Proof of Proposition 5.**

*Proof.* From Equation 14, we can write $\hat{\theta}_t^i = \alpha_2(\hat{b}_t^i + \hat{a}_t^i) + (1 - 2\alpha_2)\theta_0$ where $\hat{b}_t^i$ and $\hat{a}_t^i$ are the encoded values of $b$ and $a$ respectively when facing the increasing sequence $(b, a)$. Likewise, $\hat{\theta}_t^d = \alpha_2(\hat{a}_t^d + \hat{b}_t^d) + (1 - 2\alpha_2)\theta_0$, where $\hat{a}_t^d$ and $\hat{b}_t^d$ are the encoded values when facing the decreasing sequence $(a, b)$. Let $\kappa_t^i = \kappa^G \mathbb{I} \{b \geq \theta_0\} + \kappa^L \mathbb{I} \{b < \theta_0\}$, and $\kappa_t^d = \kappa^G \mathbb{I} \{a \geq \hat{\theta}_t^i\} + \kappa^L \mathbb{I} \{a < \hat{\theta}_t^i\}$ where $\hat{\theta}_t^i = \alpha_1(1 + \kappa_t^i)(b - \theta_0) + \theta_0$. Similarly, let $\kappa_t^i = \kappa^G \mathbb{I} \{a \geq \theta_0\} + \kappa^L \mathbb{I} \{a < \theta_0\}$, and $\kappa_t^d = \kappa^G \mathbb{I} \{b \geq \hat{\theta}_t^d\} + \kappa^L \mathbb{I} \{b < \hat{\theta}_t^d\}$ where $\hat{\theta}_t^d = \alpha_1(1 + \kappa_t^d)(a - \theta_0) + \theta_0$. Hence $\hat{a}_t^d = a + \kappa_t^d(a - \theta_0)$, $\hat{b}_t^i = b + \kappa_t^i(b - \theta_0)$, $\kappa_t^d = a + \kappa_t^d(a - \theta_0) - \alpha_1[1 + \kappa_t^d](b - \theta_0)$, and $\hat{b}_t^d = b + \kappa_t^d(b - \theta_0 - \alpha_1[1 + \kappa_t^d](a - \theta_0))$. This implies $\hat{\theta}_t^i > \hat{\theta}_t^d$ if and only if

$$
\kappa_t^i(b - \theta_0) + \kappa_t^d(a - \theta_0 - \alpha_1[1 + \kappa_t^d](b - \theta_0))
$$

$$
> \kappa_t^d(a - \theta_0) + \kappa_t^d(b - \theta_0 - \alpha_1[1 + \kappa_t^d](a - \theta_0)) \quad \text{(C.15)}
$$

Letting $\tilde{a} = (a - \theta_0)$ and $\tilde{b} = (b - \theta_0)$, Condition C.15 reduces to

$$
\kappa_t^i \tilde{b} + \kappa_t^d(\tilde{a} - \alpha_1[1 + \kappa_t^d] \tilde{b}) > \kappa_t^i \tilde{a} + \kappa_t^d(\tilde{b} - \alpha_1[1 + \kappa_t^d] \tilde{a}) \quad \text{(C.16)}
$$

There are three cases to consider, depending on whether $\tilde{a}$ and $\tilde{b}$ have the same sign. When $\tilde{a}$ and $\tilde{b}$ have the same sign, then $\kappa_t^i = \kappa_t^d$ and condition C.16 reduces as follows, which is useful for
checking the various cases: \( \hat{\theta}_1 > \hat{\theta}_2 \) if and only if
\[
\kappa_2^d (1 + \alpha_1 [1 + \kappa^d_1]) (\bar{a} - \bar{b}) - (\kappa_2^d - \kappa^d_1) (\bar{b} - \alpha_1 [1 + \kappa^d_1] \bar{a}) > \kappa^d_1 (\bar{a} - \bar{b}). \tag{C.17}
\]

**Case 1:** \( \theta_0 < b < a \). This implies \( \kappa_1^d = \kappa_1^d = \kappa^d \). There are 3 sub-cases to consider:

**Case 1.a** Suppose both \( a \) and \( b \) come as gains if received in period 2. This implies \( \kappa_2^d = \kappa_2^d = \kappa^d \). Hence, Condition C.17 amounts to \( \kappa^d (1 + \alpha_1 [1 + \kappa^d_1]) (\bar{a} - \bar{b}) > \kappa^d (\bar{a} - \bar{b}) \), which is true given \( \bar{a} > \bar{b} \).

**Case 1.b** Suppose both \( a \) and \( b \) come as losses if received in period 2. This implies \( \kappa_2^d = \kappa_2^d = \kappa^d \). Hence, Condition C.17 amounts to \( \kappa^d (1 + \alpha_1 [1 + \kappa^d_1]) (\bar{a} - \bar{b}) > \kappa^d (\bar{a} - \bar{b}) \), which is true given \( \bar{a} > \bar{b} \) and \( \kappa^d > \kappa^d \).

**Case 1.c** Suppose only \( a \) comes a gain if received in period 2. This implies \( \kappa_2^d = \kappa^d \) and \( \kappa_2^d = \kappa^d \). Hence, Condition C.17 amounts to \( \kappa^d (1 + \alpha_1 [1 + \kappa^d_1]) (\bar{a} - \bar{b}) - (\kappa^d - \kappa^d_1) (\bar{b} - \alpha_1 [1 + \kappa^d_1] \bar{a}) > \kappa^d (\bar{a} - \bar{b}) \), which reduces to \( \kappa^d (\alpha_1 [1 + \kappa^d_1]) (\bar{a} - \bar{b}) - (\kappa^d - \kappa^d_1) (\bar{b} - \alpha_1 [1 + \kappa^d_1] \bar{a}) > 0 \). Since we assumed \( \bar{b} \) comes as a loss, it must be that \( \bar{b} - \alpha_1 [1 + \kappa^d_1] \bar{a} < 0 \), meaning the condition above holds.

**Case 2:** \( b < a < \theta_0 \). This implies \( \kappa_1^d = \kappa_1^d = \kappa^d \). There are 3 sub-cases to consider:

**Case 2.a** Suppose both \( a \) and \( b \) come as losses if received in period 2. This implies \( \kappa_2^d = \kappa_2^d = \kappa^d \). Hence, Condition C.17 amounts to \( \kappa^d (1 + \alpha_1 [1 + \kappa^d_1]) (\bar{a} - \bar{b}) > \kappa^d (\bar{a} - \bar{b}) \), which is true given \( \bar{a} > \bar{b} \).

**Case 2.b** Suppose both \( a \) and \( b \) come as gains if received in period 2. This implies \( \kappa_2^d = \kappa_2^d = \kappa^d \). Hence, Condition C.17 amounts to \( \kappa^d (1 + \alpha_1 [1 + \kappa^d_1]) (\bar{a} - \bar{b}) > \kappa^d (\bar{a} - \bar{b}) \), which holds if and only if \( \kappa^d (1 + \alpha_1 [1 + \kappa^d_1]) > \kappa^d_1 \). Using the definitions of \( \kappa^d \) and \( \kappa^d_1 \) and simplifying reveals that this condition fails only if \( \lambda - 1 > \alpha_1 (1 + \eta \lambda) \). Furthermore, for both \( a \) and \( b \) to come as gains in period 2 implies that \( \alpha_1 (1 + \kappa^d_1) > 1 \). Thus, there exist values of \( (a, b) \) meeting the conditions of subcase 2.b for which recency fails only if \( \lambda - 1 > \alpha_1 (1 + \eta \lambda) \) and \( \alpha_1 (1 + \kappa^d_1) > 1 \).

**Case 2.c** Suppose only \( a \) comes as a gain if received in period 2. This implies \( \kappa_2^d = \kappa^d \) and \( \kappa_2^d = \kappa^d \). Hence, Condition C.17 amounts to \( \kappa^d (1 + \alpha_1 [1 + \kappa^d_1]) (\bar{a} - \bar{b}) - (\kappa^d - \kappa^d_1) (\bar{b} - \alpha_1 [1 + \kappa^d_1] \bar{a}) > \kappa^d (\bar{a} - \bar{b}) \). After some algebra, this condition reduces to
\[
\kappa^d \alpha_1 (1 + \kappa^d_1) (\bar{a} - \bar{b}) > (\kappa^d - \kappa^d_1) (1 - \alpha_1 (1 + \kappa^d_1) \bar{a}). \tag{C.18}
\]
The left-hand side of Condition C.18 is always positive, while the right-hand side is positive if and only if \( \alpha_1 (1 + \kappa^d_1) > 1 \) (i.e., beliefs are not convex). Thus, Condition C.18 always holds if \( \alpha_1 (1 + \kappa^d_1) < 0 \), but may fail otherwise. To see when it fails, notice that C.18 fails when
\[
\bar{b} > \frac{1}{\kappa^d} \left( \kappa^d - \frac{\kappa^d - \kappa^d_1}{\alpha_1 (1 + \kappa^d_1)} \right). \tag{C.19}
\]
Since this subcase assumes that $\hat{b}_i^L$ comes as a loss, it must be that $\hat{b} < \alpha_1(1 + \kappa^L)$. Hence, for there to exist a value $\hat{b} < 0$ that falls into case 2.c and satisfies Condition C.19, we require

$$
\frac{1}{\kappa^G} \left( \kappa^L - \frac{\kappa^L - \kappa^G}{\alpha_1(1 + \kappa^L)} \right) > \alpha_1(1 + \kappa^L).
$$

(C.20)

Using the definitions of $\kappa^G$ and $\kappa^L$ and simplifying reveals that Condition C.20 is equivalent to $\lambda - 1 > \alpha_1(1 + \eta \lambda)$. In summary, there exist values of $(a, b)$ meeting the conditions of subcase 2.c for which recency fails only if $\lambda - 1 > \alpha(1 + \eta \lambda)$ and $\alpha_1(1 + \kappa^L) > 1$. Assuming these conditions hold, values $(a, b)$ fail to generate recency only if

$$
\frac{1}{\kappa^G} \left[ \kappa^L - \frac{\kappa^L - \kappa^G}{\alpha_1(1 + \kappa^L)} \right] a < b < \alpha_1(1 + \kappa^L)a.
$$

(C.21)

Case 3: $b < \theta_0 < a$. This implies $\kappa^i_1 = \kappa^L$, $\kappa^i_1 = \kappa^G$, $\kappa^i_1 = \kappa^G$, and $\kappa^i_1 = \kappa^L$. Hence, Condition C.16 amounts to $\kappa^L \hat{b} + \kappa^G(\bar{a} - \alpha_1[1 + \kappa^L] \hat{b}) > \kappa^G \bar{a} + \kappa^L(\bar{b} - \alpha_1[1 + \kappa^G] \bar{a}) \iff -\alpha_1 \kappa^G[1 + \kappa^L] \hat{b} > -\alpha_1 \kappa^L[1 + \kappa^G] \bar{a}$. This condition always holds given $\bar{a} > 0 > \bar{b}$.

The only cases where recency may fail are 2.b and 2.c. As noted, recency holds in both of these ambiguous cases whenever $\alpha_1(1 + \kappa^L) < 1$, i.e., beliefs are convex. This completes Part 1 of the proposition. Turning to Part 2, if beliefs are not convex, then recency fails in cases 2.b and 2.c only when $\lambda > 1 + \alpha_1(1 + \eta \lambda) \equiv \hat{\lambda}$. Since beliefs are not convex in this case, $1 < \alpha_1(1 + \kappa^L) < \alpha_1(1 + \eta \lambda)$, implying $\hat{\lambda} > 2$.

Proof of Corollary 2.

Proof. The claim follows from Cases 1 and 3 of the proof of Proposition 5.

Proof of Proposition 6.

Proof. Let $\mathcal{X} = \{x_1, \cdots, x_T\}$ be an arbitrary set of $T$ elements of $\mathbb{R}$. Let $S(\mathcal{X})$ be the set of all distinct sequences formed from elements of $\mathcal{X}$. Consider $x \in S(\mathcal{X})$ and let $\theta_T(x)$ be the misattributor’s estimate following sequence $x$. We say $x$ is increasing if $x_i < x_{i+1}$ for all $i = 1, \ldots, T - 1$. Toward a contradiction, suppose $x$ is not increasing but $x = \arg\max_{x' \in S(\mathcal{X})} \theta_T(x')$. Hence, there must exist adjacent $x_i, x_{i+1}$ such that $x_i > x_{i+1}$. Fix $\hat{\theta}_{i-1}$ entering round $i$. From Proposition 5, permuting $x_i$ and $x_{i+1}$ leads to a larger round-$i+1$ estimate, $\hat{\theta}_{i+1}$, than if the person experiences $(x_i, x_{i+1})$. Hence, following this permutation, the person has a higher belief entering round $i + 2$ than under the original sequence. Again from Proposition 5, convexity implies that each $\hat{\theta}_{i+1}$ is increasing in $\hat{\theta}_{i-1}$, and hence $\hat{\theta}_T$ must increase in $\hat{\theta}_{i+1}$. Thus permuting $x_i$ and $x_{i+1}$ increases $\hat{\theta}_T$, implying a contradiction.

Proof of Proposition 7.

Proof. Suppose $x > \theta_0$. Since beliefs are convex, $\hat{\theta}_1 \in [\theta_0, x]$. Convexity again implies $\hat{\theta}_2 \in [\hat{\theta}_1, x]$. Continuing forward, it’s clear that $\hat{\theta}_t \in [\hat{\theta}_{t-1}, x]$ for all $t = 1, 2, \ldots$. Thus, $\hat{\theta}_t$ is increasing toward
x. Since \( \hat{x}_t = (1 + \kappa^G)x - \kappa^G\hat{\theta}_{t-1} \), \( \hat{x}_t \) is decreasing in \( t \). Conversely, if \( x < \theta_0 \), then \( \hat{\theta}_1 \in [x, \theta_0] \). Similar to the logic above, \( \hat{\theta}_t \in [x, \hat{\theta}_{t-1}] \) for all \( t = 1, 2, \ldots \). Thus, \( \hat{x}_t \) is increasing in \( t \).

### Proof of Proposition 8.

**Proof. Part 1.** Note that \( \hat{x}_t = x_t + \kappa_t \left( x_t - \hat{E}_{t-1}[x_t] \right) = x_t + \kappa_t(x_t - \varphi \hat{x}_{t-1}) \). Thus, recursively writing \( \hat{x}_t \) in terms of \((x_1, \ldots, x_t)\) yields

\[
\hat{x}_t = (1 + \kappa_t)x_t - \varphi \kappa_t \hat{x}_{t-1} \\
= (1 + \kappa_t)x_t - \varphi \kappa_t((1 + \kappa_{t-1})x_{t-1} - \varphi \kappa_{t-1}\hat{x}_{t-2}) \\
= (1 + \kappa_t)x_t - \varphi \kappa_t(1 + \kappa_{t-1})x_{t-1} + \varphi^2 \kappa_t \kappa_{t-1}\hat{x}_{t-2} \\
= (1 + \kappa_t)x_t - \varphi \kappa_t(1 + \kappa_{t-1})x_{t-1} + \varphi^2 \kappa_t \kappa_{t-1}(1 + \kappa_{t-1})x_{t-2} - \varphi^3 \kappa_t \kappa_{t-1} \kappa_{t-2}\hat{x}_3 \\
\cdots \\
= (1 + \kappa_t)x_t + \sum_{j=1}^{t-1} \left(-\varphi \right)^{t-j} \prod_{i=j+1}^{t} \kappa_i \right) (1 + \kappa_j)x_j. \tag{C.22}
\]

Hence, conditional on \((x_1, \ldots, x_{t-1})\), \( \text{Var}\left( \hat{E}_t[x_{t+1}] \right) = \varphi^2 \text{Var}\left((1 + \kappa_t)x_t\right) > \varphi^2 \text{Var}(x_t) = \text{Var}(\hat{E}_t[x_{t+1}]) \), where \( \hat{E}_t[x_{t+1}] \) denotes the rational expectation.

**Part 2.** Note that \( d_t = \hat{x}_t - \hat{E}_{t-1}[x_t] = \hat{x}_t - \varphi \hat{x}_{t-1} \). Thus

\[
d_t = (1 + \kappa_t)x_t - \varphi \kappa_t \hat{x}_{t-1} - \varphi \hat{x}_{t-1} \\
= (1 + \kappa_t)(x_t - \varphi \hat{x}_{t-1}) \\
= (1 + \kappa_t)(\varphi x_{t-1} + \epsilon_t - \varphi((1 + \kappa_{t-1})x_{t-1} - \varphi \kappa_{t-1}\hat{x}_{t-2})) \\
= (1 + \kappa_t)\epsilon_t - \varphi \kappa_{t-1}(x_{t-1} - \varphi \hat{x}_{t-2}) \\
= \left(1 + \kappa_t\right) \epsilon_t - \varphi \frac{\kappa_{t-1}}{1 + \kappa_{t-1}}d_{t-1}. \tag{C.23}
\]

### Proof of Proposition 9

**Proof.** Convergence follows along the lines of Proposition A.1. Mean beliefs \( \hat{\theta}_t(H) \) and \( \hat{\theta}_t(L) \) both follow dynamics \( \hat{\theta}_t(a) = \hat{\theta}_{t-1}(a) + \alpha_t(\hat{x}_t - \hat{\theta}_{t-1}(a)) \) if event \( a \in \{H, L\} \) occurs in \( t \); else \( \hat{\theta}_t(a) = \hat{\theta}_{t-1}(a) \). Thus, \( \hat{\theta}_j(a) \) follows precisely the dynamics considered in Proposition A.1 on the subsequence \((j_i)\) defined by periods where event \( a \) occurs. However, there are two differences between this setting and the one considered in Proposition A.1. Let \( \hat{x}_t(a) \) denote the encoded outcome in \( t \) conditional on event \( a \in \{H, L\} \) occurring in \( t \). First, note that \( \hat{x}_t(a) \) depends on both \( \hat{\theta}_t(H) \) and \( \hat{\theta}_t(L) \). Second, note that \( \hat{x}_t(a) \) depends on the person’s action \( p_t \) in \( t \). As such, we establish the convergence of this two-dimensional system of beliefs. The limiting values are given
by the solution of a two-dimensional system of ODEs analogous to the one-dimensional solution described in Appendix A.

We reassess conditions A1-A4 sufficient for convergence of both dimensions $\hat{\theta}_t(a)$, $a \in \{H, L\}$. Clearly A1 and A2 still hold. Now consider A3. Note that $\hat{x}_t(H) = x_t(H) + \kappa_t(x_t(H) - \hat{x}_{t - 1}(p_t))$ where $\hat{x}_{t - 1}(p_t) = p_t \hat{x}_{t - 1}(H) + (1 - p_t) \hat{x}_{t - 1}(L)$, meaning $\hat{x}_t(H) - \hat{\theta}_t(H) = x_t(H) + \kappa_t(x_t(H) - \hat{x}_{t - 1}(p_t)) - \hat{\theta}(H)$, and thus

$$\hat{x}_t(H) - \hat{\theta}_t(H) = (1 - p_t \kappa_t)x_t(H) + (1 - p_t) \kappa_t x_t(H) - (1 - p_t) \kappa_t \hat{x}_{t - 1}(H) - (1 - p_t) \kappa_t \hat{x}_t(H)$$

$$= (1 - \kappa_t)(x_t(H) - \theta_t(H)) + (1 - p_t) \kappa_t(\theta_t(H) - \hat{x}_t(H)) + (1 - p_t) \kappa_t(\theta_t(L) - \hat{x}_t(L)),$$

(C.24)

where $\theta_t(H)$ and $\theta_t(L)$ are the rational estimates following $t$ rounds. Thus,

$$\sup \mathbb{E}[|\hat{x}_t(H) - \hat{\theta}_t(H)|^2 | \theta(H), \theta(L)] < \infty$$

by an application of Minkowski’s Inequality and noting that the expected squared absolute value of each term in the final sum of Equation C.24 is finite, as established in the proof of Proposition A.1. The only difference between these terms and those addressed in Proposition A.1 is the presence of $p_t$. Since this action is bounded in $[0, 1]$, the inequalities from that previous result continue to hold. An analogous argument establishes

$$\sup \mathbb{E}[|\hat{x}_t(L) - \hat{\theta}_t(L)|^2 | \theta(H), \theta(L)] < \infty.$$

Turning to A4, the expected deviation function $G$ (analogous to Equation 11) will be two dimensional. For dimension $a \in \{H, L\}$, note that $\hat{x}_t - \hat{\theta}_{t - 1}(a)$ conditional on $a \in \{H, L\}$ and $\hat{\theta}_{t - 1} \equiv (\hat{\theta}_{t - 1}(H), \hat{\theta}_{t - 1}(L))$ is given by

$$G^a(t, \hat{\theta}_{t - 1}) = \mathbb{E}_{x_t(a)}[\hat{x}_t(a, p^*_t(t, \hat{\theta}_{t - 1}), \hat{\theta}_{t - 1}) | \theta(H), \theta(L)] - \hat{\theta}_{t - 1}(a),$$

(C.25)

where $\hat{x}_t(a, p^*_t(t, \hat{\theta}_{t - 1}), \hat{\theta}_{t - 1})$ is the encoded outcome given mean beliefs $\hat{\theta}_{t - 1}$ entering round $t$ and $p^*_t(t, \hat{\theta}_{t - 1})$ is the optimal action given those beliefs. From Equation 21, it is clear that $p^*$ is in fact independent of $t$ conditional on $\hat{\theta}_{t - 1}$, and hence we can simply write $p^*(\hat{\theta}_{t - 1})$. Thus

$$G^a(t, \hat{\theta}_{t - 1}) = \theta(a) + \kappa^G(1 - F^a(\hat{\theta}_{t - 1}(p^*(\hat{\theta}_{t - 1})))) \left( \mathbb{E}[X(a) | X(a) \geq \hat{\theta}_{t - 1} - \hat{\theta}_{t - 1}(p^*(\hat{\theta}_{t - 1}))] - \hat{\theta}_{t - 1}(a) \right)$$

$$+ \kappa^G F^a(\hat{\theta}_{t - 1}(p^*(\hat{\theta}_{t - 1}))) \left( \mathbb{E}[X(a) | X(a) < \hat{\theta}_{t - 1} - \hat{\theta}_{t - 1}(p^*(\hat{\theta}_{t - 1}))] - \hat{\theta}_{t - 1}(a) \right)$$

$$= \theta(a) + \kappa^G(\theta(a) - \hat{\theta}_{t - 1}(p^*(\hat{\theta}_{t - 1})))$$

$$+ (\kappa^G - \kappa^G)F(\hat{\theta}_{t - 1}(p_t)) \left( \mathbb{E}[X(a) | X(a) < \hat{\theta}_{t - 1}(p_t)] - \hat{\theta}_{t - 1}(p_t) \right) - \hat{\theta}_{t - 1}(a).$$

(C.26)

It is clear from Equation 21 that $p^*(\hat{\theta}_{t - 1})$ is continuous in $\hat{\theta}_{t - 1}(a)$, and $\hat{\theta}_{t - 1}(p_t)$ is continuous in $p_t$. Hence, $G(t, \hat{\theta}_{t - 1})$ is continuous in $\hat{\theta}_{t - 1}(a)$, as required. Finally, following the verification of A4 in Appendix A, we let the sequence of random variables $(\gamma_j)$ be again defined by $\gamma_j^a = G^a(t, \hat{\theta}_j) - G^a(\hat{\theta})$ for those rounds where event $a$ occurs, where $G^a(\hat{\theta}) = \lim_{t \to \infty} G^a(t, \hat{\theta})$. Since the time period is irrelevant for $G^a(t, \hat{\theta}_{t - 1})$ conditional on $\hat{\theta}_{t - 1}$, the condition $\sum_{j_t = 1}^{\infty} \hat{\theta}_{t - 1} | \gamma_j^a | < \infty$ holds on that subsequence $(j_t)$ of rounds where event $a$ occurs. The limiting beliefs $\hat{\theta} = (\hat{\theta}(H), \hat{\theta}(L))$ hence
satisfy
\[
\begin{bmatrix}
G^H(\hat{\theta}(H, \hat{\theta}(L))) \\
G^L(\hat{\theta}(H, \hat{\theta}(L)))
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

For simplicity, we focus on the case where \( \sigma \to 0 \). From Equation C.26, \( G^H \) and \( G^L \) reduce in this case to
\[
\begin{align*}
G^H(\hat{\theta}) &= \theta(H) + \kappa^G(\theta(H) - \hat{\theta}(p^*(\hat{\theta}))) - \hat{\theta}(H) \tag{C.27} \\
G^L(\hat{\theta}) &= \theta(L) + \kappa^L(\theta(L) - \hat{\theta}(p^*(\hat{\theta}))) - \hat{\theta}(L). \tag{C.28}
\end{align*}
\]

Fixing \( p^* \), the solution to this system is given by Equation 9 in Section 3. At this solution, \( \hat{\theta}(H) - \hat{\theta}(L) > \theta(H) - \theta(L) \) regardless of \( p^* \). Thus, Equation 21 implies that the long-run optimal action exceeds the rational action.

\[\square\]

**Proof of Proposition 10.**

**Proof.** From the text, the rational-benchmark effort in round \( t \), denoted \( e_t' \), satisfies \( c'(e_t') = M_t' \) where \( M_t' = \sum_{\tau=t+1}^{T} \delta^{\tau-t} \alpha_{\tau-1} \). Contrastingly, the optimal effort under biased evaluations in period \( t \), \( e_t^* \), satisfies
\[
c'(e_t^*) = M_t \equiv (1 + \kappa^G) \left\{ \delta \alpha_t + \sum_{\tau=t+2}^{T} \delta^{\tau-t} \alpha_{\tau-1} \left( \prod_{j=t}^{\tau-2} [1 - \kappa^G \alpha_j] \right) \right\}. \tag{C.29}
\]

Because \( c \) is strictly increasing, \( e_t^* < e_t' \iff M_t^* < M_t' \). Notice that one can write \( M_t \) in terms of \( M_t' \) as follows:
\[
M_t = M_t' + \kappa^G \delta \alpha_t - \sum_{\tau=t+2}^{T} \delta^{\tau-t} \alpha_{\tau-1} \left( 1 - (1 + \kappa^G) \prod_{j=t}^{\tau-2} [1 - \kappa^G \alpha_j] \right). \tag{C.30}
\]

It’s clear that \( M_t > M_t^* \) for \( t = T - 1 \). Furthermore, since \( \prod_{j=t}^{\tau-2} [1 - \kappa^G \alpha_j] \) decreases to 0 as \( \tau - 2 - t \) grows large, sufficiently large \( T \) implies there exists a period \( t^* < T \) such that \( M_t < M_t' \) for \( t < t^* \). If \( T \) is not sufficiently large, then \( t^* = 1 \). More formally, let \( D_t \equiv M_t - M_t' \). Equation C.30 implies
\[
D_t = \kappa^G \delta \alpha_t - \delta^2 \alpha_{t+1} \left[ 1 - (1 + \kappa^G)(1 - \kappa^G \alpha_t) \right] - \sum_{\tau=t+3}^{T} \delta^{\tau-t} \alpha_{\tau-1} \left( 1 - (1 + \kappa^G) \prod_{j=t+1}^{\tau-2} [1 - \kappa^G \alpha_j] \right) \tag{C.31}
\]

Hence \( D_t = \delta(D_{t+1} + \kappa^G \alpha_t[1 - M_{t+1}]) \). Thus, \( D_t > 0 \) implies \( D_{t+1} > 0 \) so long as \( D_{t+1} > -\kappa^G \alpha_t[1 - M_{t+1} = -\kappa^G \alpha_t[1 - D_{t+1} - M_t'] \), which is equivalent to \( [1 - \kappa^G \alpha_t]D_{t+1} > -\kappa^G \alpha_t[1 - M_{t+1}] \). By convexity, \( [1 - \kappa^G \alpha_t] > 0 \), so the preceding inequality holds so long as \( M_t' \) is sufficiently small. Since \( M_t' \) decreases to a value strictly less than one, there exists a \( t^* \) such that \( M_t' < 1 \) for \( t \geq t^* \). Thus, \( D_{t+1} \) remains positive for \( t \) sufficiently deep into the relationship.

\[\square\]

**Proof of Corollary 3.**

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Proof. The existence of $\bar{T}$ follows from the proof of Proposition 10, where we establish that there exists a value $t^*$ such that $M_t < M'_t$ whenever $t < t^*$ and $t^* > 1$ when $T$ is sufficiently large. Suppose $T > \bar{T}$ and let $M_1(T)$ and $M'_1(T)$ denote the biased and rational marginal benefit of effort in period 1 as a function of the horizon, $T$. From Equation C.30, $M'_1(T + 1) - M_1(T + 1) > M'_1(T) - M_1(T)$ if and only if

$$\delta^T \alpha_{T-1} \left[ 1 - (1 - \kappa^G) \prod_{j=1}^{T-1} (1 - \kappa^G \alpha_j) \right] > 0,$$

(C.32)

which holds iff $(1 - \kappa^G) \prod_{j=1}^{T-1} (1 - \kappa^G \alpha_j) < 1$. If this condition fails, then $M_1(T) > M'_1(T)$, contradicting $T > \bar{T}$. $\blacksquare$