Privacy in Bargaining: The Case of Endogenous Entry

Isaías Chaves Villamizar †

Abstract

I study the role of privacy in bargaining. A seller makes offers every instant, without commitment, to a privately informed buyer. There are potential competing buyers (entrants) who observe something about the negotiation and can choose to interrupt it by triggering a bidding war. As entrants learn about ongoing disagreement, they update their beliefs about the type of the buyer. The seller’s lack of commitment reverses the seemingly intuitive effects of publicity. If learning that the buyer’s type is lower encourages entrants, so that the seller “should” want to make publicly observable offers that lure in entrants against the incumbent buyer, then in equilibrium the seller typically prefers private bargaining. If learning that the buyer’s type is lower discourages entrants, so that the seller “should” want to hide her offers to avoid frightening away entrants, then in equilibrium the seller prefers public bargaining.

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†Department of Economics, Stanford University. ichaves@stanford.edu
1 Introduction

This paper studies the effects of privacy on bargaining. How do agents negotiate differently when they know their offers are being observed? While there are many possible reasons for agents to care about privacy, I focus on one particular force: as they bargain, agents are concerned about strategic audiences who observe (something about) the negotiation and can choose to interrupt it at a cost.[1]

For example, many mergers-and-acquisitions deals start as bilateral negotiations between a company and an acquirer, but they endogenously become an auction when another acquirer chooses to trigger a bidding war after its interest is piqued. Consider the recent acquisition of Straight Path, a telecom company, by Verizon[2] Straight Path owned spectrum licenses in the 28 and 39 gigahertz frequencies, which industry observers think will be crucial for 5G networks. In early 2017, AT&T approached Straight Path privately, and after months of negotiations (in mid-April of that year), the two companies announced their intention to sell Straight Path to AT&T for $1.25 billion, a 160% premium on Straight Path’s stock price. The transaction was scheduled to close within 12 months (a long deadline that allowed for further due diligence and clearing of regulatory hurdles).

Since Straight Path was publicly traded, the terms of the deal were released in press statements at the time. The long closing window provided an opening for AT&T’s competitors. Within a couple of weeks of the terms of the deal being released released, Verizon approached Straight Path with a counter-offer. This triggered a fast bidding war between AT&T and Verizon, with Verizon nearly doubling its bid within a two-week period. By early May, Verizon offered $3.1 billion, a %500 premium on the original stock price, and AT&T folded. Even though the original AT&T deal had a 12 month closing window, Straight Path shareholders approved the merger with Verizon two months after Verizon’s winning bid[3].

This concern about “endogenous interruption” is at work in many other bargaining interactions beyond the M&A example. Endogenous interruption seems especially relevant

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[1] More generally, parties to a negotiation would have two reasons to care for privacy: they are worried strategic audiences who come into play while the two parties are disagreeing, or they are worried strategic audiences who come into play only once an agreement has been reached. This paper covers the first reason for privacy, while concerns about reputation or releasing proprietary information that hurts one’s competitive edge even after a successful deal is struck would correspond to the second reason.


[3] To my knowledge, this process of negotiations endogenously becoming auctions has not been tabulated explicitly in the corporate finance literature, but there is some indicative evidence that it is common. Schwert (1996) shows that, in the 1975-1991 period, 30% of tender offers for US-listed companies in his sample eventually drew multiple bidders.
in the realm of private politics (Egorov and Harstad [2017]). When a labor union bargains with a firm over a wage contract, both parties understand that there are regulators following the negotiation who can choose to intervene depending on how the bargaining evolves—regulators can impose an arbitration agreement, they can force injunctions that prevent lockouts, or they can require the union to continue working before an agreement is reached. For example, the early 2000’s strike by ILWU, the longshoremen’s union in the US, essentially ended after the intervention of President Bush. At the time, both parties were clearly aware of the possibility of endogenous interruption, and in fact press reports suggest that the shippers’s bargaining strategies were tailored explicitly to induce the president’s intervention. According to a New York Times article on the dispute, “[t]he union said the lockout was a management ploy intended to have the president intervene.”

The contribution of this paper is to characterize negotiating parties’ induced preferences for privacy when they face strategic audiences of this kind, and relatedly, to characterize the equilibrium effects of disclosure rules, of the kind imposed on publicly traded companies, that prevent players from bargaining in private (or as privately as they would like).

In the foregoing, I abstract somewhat from institutional detail, and I study these issues in a model where a long-lived seller and a long-lived buyer bargain over an indivisible asset. The seller makes frequent offers to the buyer, who has private information about his value for the asset. Time is continuous, on an infinite horizon. The seller can adjust her offers every instant, and she cannot commit to future prices. The buyer and the seller bargain while waiting for the possible entry of short-lived competing buyers—“entrants”—who observe (something about) the negotiation. Entrants can choose to interrupt the negotiation at a cost and trigger a bidding war that ends the game. I study how outcomes differ when bargaining happens in private, so offers are hidden from entrants, versus when it happens in public, so the entrants can see the entire history of rejected offers. The buyer and the seller realize that their continued disagreement conveys different information to the entrants when the offers are public than when the offers are private, so they bargain differently in the two cases.

The main force in the model is that, as the entrants learn that the buyer has rejected

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4 The article quotes the AFL-CIO secretary-treasurer as follows: “We’re absolutely furious … The P.M.A. [ports management] locked the workers out, contrived a phony crisis and then gets rescued by the administration. They’re getting their way and have the weight of the government behind them.” See [http://www.nytimes.com/2002/10/09/us/president-invokes-taft-hartley-act-to-open-29-ports.html](http://www.nytimes.com/2002/10/09/us/president-invokes-taft-hartley-act-to-open-29-ports.html) Another classic case that follows this pattern is the strike by PATCO, the air traffic controllers’ union, against the FAA, which ended when President Reagan fired all striking workers after they ignored his deadline to return to work.

5 For example, “serious” offers are considered material information for shareholders, and must therefore be disclosed by management. In the private politics examples, where regulators can always “ask to see the books” when they intervene, the negotiating parties could still hide their offers from the regulators by making them verbally and not directly verifiable.

6 Throughout I refer to the seller as “she,” while both the incumbent buyer and the entrants are “he” whenever this would not cause confusion. “The buyer” always refers to the incumbent buyer.
an offer, they become more optimistic about the outcome of an eventual bidding war and are more likely to enter: high types of the buyer dislike delay relatively more and are willing to accept higher prices sooner, so when the buyer rejects an offer, the entrants conclude that the buyer’s type was probably low and that the competition they will face if they enter is probably weak. Therefore, after a rejection by the buyer, entrants expect higher payoffs and enter faster. This force applies both when offers are public, so that entrants know the particular offer that was rejected, and when offers are private, so that they only that an offer was rejected, but not its particular value.

The main result is that the seller often prefers to bargain in private, even though when she bargains in public, she can use her offers to increase competition against the incumbent buyer. When offers are public, if the seller makes a high offer and this offer gets rejected, entrants hardly update their beliefs or the rate at which they enter, since they conclude that only very high types would have accepted such an offer; in contrast, if the seller makes a low offer that gets rejected, entrants conclude the buyer’s type must be very low and accelerate their entry. Meanwhile, when offers are private, entry rates are endogenously determined in equilibrium, but the seller takes them as exogenous when she is choosing what prices to offer—at any point in the game, entrants never know whether a rejected offer is high or low, so they behave in the same way regardless of which offer is made at that point.

To see why this distinction leads the seller to prefer private bargaining, note that, as first noted by Coase (1972), the issue faced by the seller when she cannot commit to future prices is that she is tempted to lower her offers too quickly for her own good: she would do better to commit not to revise a rejected offer for a while, but in practice, after a rejection, she reduces her offer to avoid additional delay. The buyer anticipates these future price drops, and in response reduces his willingness to pay high prices in the present, which hurts the seller’s profit. Making offers public gives the seller an additional incentive to lower her offers quickly in the future (since this increases the competition faced by the buyer), which only exacerbates her commitment problem. I show that even in those cases where the seller does not necessarily prefer private to public bargaining, the seller would always strictly prefer bargaining in public only in the first instant, and then bargaining in private thereafter, to bargaining only in public. Having the ability to lure in competition by making public offers can therefore hurt the seller.

Beyond the main payoff result, I characterize the effects of privacy on bargaining. First, I show that the seller’s preference for private bargaining is the composition of two effects that push her payoffs in opposite directions. On the one hand, when offers are private, the seller and the buyer reach agreement strictly later, and entry happens

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7 Below, I also address the opposite scenario, where the buyer’s type being low is bad news for entrants and learning that the buyer rejected an offer discourages entry. This case is covered in detail in Section 8.1.
strictly later in the usual stochastic order, uniformly in the buyer’s type—this hurts the seller. On the other, when offers are private, the buyer and the seller reach agreement at strictly higher prices, uniformly in the buyer’s type—this benefits the seller, and in fact it overwhelms the first effect. Second, I show that hiding the offers from entrants makes high-valuation buyers worse off in expectation, and it makes all buyer types worse off on histories with no entry. Third, I show there are differences in kind between bargaining in public and bargaining in private. When the seller’s offers are public, prices have an “ex post regret” property: to lure the entrant, the seller makes offers so low that, conditional on their being accepted, she wishes she had waited for entry instead. Prior to entry, the equilibrium can alternate between bursts of trade, when a positive mass of buyer types accepts in the same instant, and dribs and drabs of trade, during which the seller screens buyer types one by one. When the offers are private (hidden from the entrant), trade happens in dribs and drabs only and there is no ex post regret.

Finally, I show that, in more general models of bargaining with endogenous entry, the direction of the effects of privacy depends on whether entrants are encouraged or discouraged by disagreement. Recall that, as the entrants learn that an offer is rejected, they can rule out or “skim” some types of the informed player. In the benchmark model, the informed player and the entrants are all buyers with independent private values, so this skimming discourages entry, but in more general models, the skimming of informed player types can encourage or discourage entry. For example, when values are common and entrants learn that the buyer’s value is lower, they become more optimistic about the amount of competition they will face in the bidding war, but more pessimistic about the value they will get if they win. If entrants become pessimistic about their value faster than they become optimistic about their competition, then entry would be discouraged by disagreement. Hold constant the fact that higher types of the informed player are better for the uninformed player upon entry. I show that, if skimming encourages entry, publicizing offers will tend to hurt the uninformed player, lower prices, and speed up trade. If skimming discourages entry, publicizing offers will tend to help the uninformed player, raise prices, and slow down trade. The direction of these effects holds regardless of whether the model of entry involves competing buyers triggering a bidding war, or an inattentive regulator forcing a surplus split. The direction holds regardless of whether values are private or common in a bidding war: all that matters is whether on net, entry is encouraged or discouraged by the skimming of informed player types. Indeed, when disagreement (skimming) encourages entry, this exacerbates the uninformed player’s commitment problem coming from the Coase-Conjecture forces; when disagreement discourages entry, this lessens the uninformed player’s commitment problem.

The paper is organized as follows. In the remainder of this section, I provide a graphical intuition behind the paper’s main results. Section 2 briefly discusses related literature. Section 3 then presents the benchmark model the equilibrium concepts for
public and private offers. Writing down the model directly in continuous time greatly simplifies the analysis, but it requires me to introduce some technicalities and use an equilibrium concept that, while not being fully game-theoretic, captures the key features of a discrete-time game-theoretic analysis. Section 4 formally states the main results on the seller’s induced preferences over privacy, which are then developed in the rest of the paper. Section 5 characterizes equilibrium dynamics with public offers. It explains the source of “regret” pricing and bursts of agreement in equilibrium. Section 6 characterizes bargaining dynamics with private offers. This section highlights the differences in kind across the two regimes: with private offers, there are no bursts of agreement, and pricing has a “no-regret” property. Using the characterizations in the previous two sections, Section 7 then ranks the public and private bargaining games in terms of equilibrium prices and the equilibrium delay incurred before agreement, and it presents the formal arguments behind the main result. Section 8 considers extensions to the model: how the model can incorporate common values, and how the analysis extends to different models of privacy in bargaining, such as when a firm and a labor union bargain over a wage contract under the shadow of an inattentive regulator. Section 9 concludes. All omitted proofs are in Appendix A.

**Graphical Intuition: Coasean Force and Main Result** The intuition behind the main result (the seller’s induced preference for privacy) can be seen through a sequence of diagrams reminiscent of traditional monopoly theory. Recall that higher types of the buyer dislike delay relatively more, so they accept sooner. In particular, this means that at any point in the game, there is a cutoff such that all buyer types above the cutoff would have already accepted. Therefore, players who have observed the history of offers till that point have beliefs about the buyer’s value that are right-truncations of the prior at that cutoff. I focus on equilibria where strategies are Markov in that belief cutoff.

Instead of thinking of the seller as acting in price space, making offers that, when rejected, induce different cutoffs, it is more convenient to think of the seller as choosing those cutoffs directly. That is, the seller chooses how fast to screen through or rule out buyer types, evaluating those choices of cutoffs according to the prices whose rejection would have ruled out the types above those cutoffs.

Keeping that in mind, let $P(v)$ be the equilibrium reservation price strategy of a buyer with value $v$, and let $F_B$ denote the distribution of buyer values. Because of the Markov assumption, $P(v)$ will also be the offer the seller makes when the state is $v$, that is, when she knows that, given the offers that she has made thus far, she can rule out all buyer types above $v$. Then one can plot $P(v)$ against $1 - F_B(v)$, the probability that $P(v)$ is

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8 I adopt this graphical interpretation of the Coase Conjecture from Fuchs and Skrzypacz (2010).
9 This equivalence only holds in continuous time. In discrete time, the price charged at a given state and the reservation price of the type equal to that state will always differ. Recall that, in discrete time, a positive mass of buyer types transacts in every period with trade. Suppose an interval $[v, \hat{v}]$ trades
accepted, to get a kind of “inverse demand” curve, where \( Q = 1 - F_B \) plays the role of quantity. Since cutoffs move from 1 to 0 over time, and \( P(v) \) is both \( v \)'s willingness to pay and the price offer at a cutoff \( v \), the x axis will represent not only quantity, but also calendar time.

Unlike a static monopoly pricing problem where the demand curve would be a primitive of the model, here \( P(v) \) is endogenous. In particular, buyer \( v \)'s willingness accept a given price now depends on some conjecture about how long he would have to wait for a lower price, and this conjecture must be correct in equilibrium. The seller’s lack of commitment to future prices is expressed in the fact that she takes the demand curve \( P \) as fixed. If she could commit to a sequence of prices, she would internalize the effect of that choice on the buyer’s equilibrium strategy; since she cannot commit, she takes that strategy, which is encoded in the reservation prices \( P \), as given.

Consider first the implications of the seller’s lack of commitment in a simpler model where the seller has a constant cost \( C \) from selling the good, but there is no entry of any in period \( k \) in discrete time. Then the price charged at state \( \bar{v} \) is \( P(\bar{v}) \) (so that exactly \([v, \bar{v}]\) want to accept), which is different from \( P(\bar{v}) \), the reservation price of type \( \bar{v} \). Meanwhile, in continuous time, when a positive mass of types \([v, \bar{v}]\) accepts in one instant, \( P \) must be constant on \([v, \bar{v}]\). In particular, \( P(\bar{v}) = P(\bar{v}) \), so that the price charged at state \( \bar{v} \), which induced a jump in the state, equals the reservation price of the type who is marginal at that state.
kind. This is the classic no-gap Coase Conjecture setup (Coase, 1972; Fudenberg et al., 1985; Gul et al., 1986). The thin solid curve of Panel (a) of Figure 1 depicts a possible equilibrium demand curve $\tilde{P}$, and the thick dashed curve depicts the (flat) cost curve $C$.

If $\tilde{P}$ is not perfectly flat, so that in equilibrium higher types of the buyer have higher reservation prices than lower types, it must be that the higher types expects some positive delay before the seller quotes lower prices, i.e., the buyer believes that the seller is not screening through types infinitely fast. As the seller screens through types from left to right, she collects the area between $\tilde{P}$ and $C$ in profits, but because of discounting, that area is “shrunk” by however long the seller took to collect it.

Recall, however, that when she cannot commit to future prices, the seller takes the willingness-to-pay curve $\tilde{P}$ as fixed in her best response problem. Hence, if the seller were to screen through types even faster than she is expected to in this putative equilibrium, she would still fetch prices from $\tilde{P}$ and collect the area between $\tilde{P}$ and $C$, but the area would be shrunk by less, since the seller incurs less discounting at this new screening speed. Therefore, this “speeding up” deviation makes the seller strictly better off, a contradiction. In fact, for any candidate demand curve $\tilde{P}$ that lay above $C$, one could apply this logic and derive a contradiction to equilibrium behavior. This suggests that in equilibrium, the actual reservation price curve $P$ must lie flat against cost, as shown in Panel (a), so that the seller’s payoffs are driven down to her outside option.

Below, I use the term “Coasean Force” to refer to the idea that, whenever $P$ lies above cost, the seller’s lack of commitment leads her to screen through types infinitely fast, which pushes her payoffs down to her outside option. The Coasean Force also applies in my model, and it typically drives the seller’s payoffs down to an outside option, but this “outside option” is a subtle object. I show below that the seller’s outside option at any point in time is given by what she would obtain from a counterfactual “stalling” strategy in which, having reached that point, she starts making unreasonable offers forever, ceasing to trade with the incumbent buyer during normal negotiations and waiting indefinitely for entry. This off-path stalling strategy has different effects when offers are public and when they are private, and the payoff ranking consists of understanding the effects of the seller’s stalling in the two regimes.

Panels (b) and (c) in Figure 1 illustrate how the Coasean Force operates in my model. Start with the public offers case. $\tilde{P}(v)$ still denotes a possible “demand curve” that traces the reservation prices of different buyer types. In my model, the seller has no cost of production, but she might still be unwilling to sell at some price $\tilde{P}(v)$ because she has an opportunity cost $C(v)$ of selling. The analysis below will make precise what this opportunity cost is. For now, to give an intuition for this cost, consider that the seller could always choose to stall for a while by making unreasonable offers, rather than make reasonable offers that lead to trade with the buyer. While the seller is stalling, entry would happen with strictly positive probability and generate strictly positive profits.
Since the seller always has this opportunity available, she must be compensated for it if she is willing to trade with the incumbent buyer. And since both the profits from that bidding war and the rate at which entrants arrive depend on $v$, this opportunity cost will depend on $v$.

Reasoning by analogy with the simple production cost example, the Coasean Force once again rules out a candidate demand curve $\tilde{P}(v)$, such as the one drawn with a thin solid curve in Panel (b), that lies above “opportunity cost” $C(v)$, drawn with a thick dashed curve. The seller cannot stop herself from trying to collect the area in between $\tilde{P}$ and $C$ as fast as possible, so that the buyer’s conjectures about future prices which led to $\tilde{P}$ would be inconsistent with the seller’s future behavior. Hence, the true equilibrium demand curve must lie flat against the equilibrium opportunity cost curve, and the seller’s payoffs must be pushed down to her outside option.

Now consider the operation of the Coasean Force with private offers. When offers are public, an entrant who arrives at some point during the game observes all the offers made up to that point, so he knows the current cutoff and conditions his entry on that Markov state. When offers are private, entrants know there must exist a cutoff type (since higher types dislike delay more), but since they cannot observe the history of offers, they do not know what the cutoff is. They must form conjectures about that cutoff. Since the only thing entrants know when they arrive is the calendar time $t$ since bargaining began, those conjectures can depend only on $t$. Therefore, entry rates depend on calendar time, and “Markov strategies” for the buyer and the seller condition both on the cutoff (which they know in common, since they know the history of offers) and on the calendar time $t$.

Panel (c) in Figure 1 draws the “inverse demand” plot for a particular calendar time $t$ of the private offers game. The seller now considers a demand curve $P(v, t)$ and an opportunity cost curve $C(v, t)$. The reasoning for applying the Coasean Force here is more subtle: since $P$ and $C$ depend on $t$, the diagram changes as time advances, and one can no longer read “time” in the diagram by moving from left to right on the x axis. However, the linearity of the seller’s objective in the screening speed, which is the essence of the Coasean Force, still holds. Hence, one can rule out demand curves like $\tilde{P}$, drawn with a thin solid curve, that lie above $C$, drawn with a thick dashed curve, and the seller’s payoffs will be driven down to her outside option.

As alluded to above, whether offers are public or private, the seller’s outside option is given by the ex ante value of stalling—the expected discounted profits at the time $t$.

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10 As I explain in Section 5, formally this follows from the linearity of the seller’s payoffs in the screening speed. Consider, for example, the seller’s flow profits from trading per unit time in a discrete time model. If the real time between periods is $\Delta > 0$, and the seller is screening through a mass of types proportional to $\Delta$ in each period, then for small $\Delta > 0$, by a first-order Taylor expansion, she obtains flow profits

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\frac{1}{\Delta} \left( \frac{F_B(K_t) - F_B(K_{t+\Delta})}{F_B(K_t)} \right) P(K_{t+\Delta}) \approx \frac{f_B(K_t)}{F_B(K_t)} P(K_t) \left( - \frac{dK_t}{dt} \right),
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where $-dK_t/dt$ is the instantaneous rate at which she screens through types.
of entry, where the expectation is over all remaining buyer types and the time of entry, assuming that the seller makes unreasonable offers forever\textsuperscript{11}. To see why this might be the case, note that, whenever $P$ lies flat against $C$, so that there is no area between the two, the seller becomes indifferent over all screening speeds\textsuperscript{12}. This includes the speed of 0, i.e., making unreasonable offers that get rejected for sure. Therefore, the seller’s payoff at that state must equal what she would have gotten from making unreasonable offers at that moment. The formal proof shows something stronger: even though the seller is making reasonable offers in equilibrium, she gets exactly what she would have gotten from making unreasonable offers forever.

Therefore, to compare the seller’s payoffs with public and private offers, it suffices to know her ex ante value of stalling at the beginning of the game in each case. That value consists of the profits upon entry, times the expected discount to entry when the seller is unreasonable. The eventual profits are the same whether offers are public or private, but the expected discount differs. When offers are public, stalling freezes the entry rate: if the seller started stalling, then entrants would “see” the seller’s unreasonable offers and would stop becoming optimistic about their possible competition. On the other hand, when offers are private, if the seller started making unreasonable offers, the entry rate would rise anyway. The entrants would never see this hypothetical deviation by the seller. Hence, they would continue to believe that offers had been reasonable (as would have been the case in equilibrium): they would conclude after seeing there was no agreement that the buyer’s value is lower than their previous estimate, and they would grow more eager to enter. Therefore, the entry rate under the stalling deviation is frozen at some level under public offers, but rises from some level under private offers. Since, at the beginning of the game, the entrants correctly believe that the seller has screened no types, the private offers entry rate must start at exactly the same level at which the public offers rate is frozen. Therefore, when offers are private, the seller discounts the eventual profits from stalling by less, which means that she is better off.

## 2 Related Literature

**Transparency and Privacy** This work relates to a recent discrete-time literature on the effects of transparency on market performance. This family of models focuses on search markets, where an informed player faces offers by a sequence of uninformed players. These models allow no recall of previous offers, so players only have short-term interactions with each other. In Hörner and Vieille (2009), short-run buyers make offers

\textsuperscript{11}In discrete time, the ex ante value of waiting is a lower bound for the seller’s payoffs, but in continuous time the bound must be tight.

\textsuperscript{12}$P$ and $C$ may fail to lie flat against each other when $C$ has non-monotonicities, which gives rise to states where trade happens in bursts. $P$ and $C$ coincide precisely on states where trade happens smoothly.
one-by-one to an informed seller. The seller knows its type, which determines both her own cost and the utility that all buyers get from the good. Kaya and Liu (2015) study a similar model with private values, where an informed buyer faces an infinite sequence of short-lived uninformed sellers, one per period. In Fuchs et al. (2016), informed sellers face a competitive market of short-lived buyers every period, for two periods. Buyers make all the offers. The authors introduce interdependent values, so buyers face a lemons problem.

In these works, making the offers private (hiding them from future uninformed players) speeds up trade and can help efficiency; in fact, Fuchs et al. (2016) show that private offers can be Pareto superior to public offers. Hörner and Vieille (2009) show that when offers are public, equilibrium ends at an impasse, after which no more trade happens, while with private offers there is always eventual agreement. Both Kaya and Liu (2015) and Fuchs et al. (2016) show that privacy makes prices more favorable to the informed player (prices are higher when the seller is informed, lower when the buyer is informed).

My paper shares the basic concerns of this literature, but the model is fundamentally different. Rather than a search market, I model a bargaining interaction where both the uninformed side making the offers and the informed side receiving them are long-lived and interact repeatedly over time. I discuss the contrast between my results and those in this literature, as well as the economic intuitions behind that contrast, in Remark 6.

My paper also relates to a broader literature on information design and the consequences of disclosure. That literature is too large to summarize here. I note, however, a recent paper that is most closely related to the problem I study. Dworczak (2017) analyzes a static mechanism design problem where the designer can control both the trading protocol and the information that gets released to an “after market” after the protocol is run, but she cannot control the after market. I discuss the contrast between this approach and mine in the conclusion.

Dynamic (Possibly Endogenous) Populations An earlier literature on static mechanism design literature considered how the need to induce the endogenous entry of agents could completely change the auction format a seller should use (McAfee and McMillan, 1987; Levin and Smith, 1994; Ye, 2007).

A number of recent papers in mechanism design consider the incentive issues that arise when populations are dynamic, i.e., agents in the mechanism arrive and depart over time (Chaves Villamizar, 2017; Garrett, 2017; Mierendorff, 2016; Gershkov et al., 2015, 2017). When agents have private information about their time on the market, any mechanism must give them incentives to arrive and depart at the right time. Since the designer could potentially excludes certain arrival or departure “types,” there is a sense in which the population is endogenous in these models. Garrett (2017), for instance, shows that, when agents can delay their arrival times, an optimal selling mechanism will give rents even to the lowest types, and it penalizes late arrivals.
focus on efficiency, and they identify arrival processes for which dynamically efficient outcomes can be implemented when agents have private information about arrival times.

To my knowledge, outside the mechanism design paradigm, very few works have studied dynamic models with both private information and endogenous entry. A notable exception is Zryumov (2017). In the model, there are good and bad entrepreneurs, and bad entrepreneurs can strategically choose when to enter the market for funding. The uninformed side of the market is competitive and makes all the offers (so the offers are pinned down by a zero-profit condition). In particular, Zryumov (2017) shows that trade flows can respond discontinuously to changes in market conditions.

**Bargaining with Frequent Offers: Coase Conjecture** This paper uses a canonical dynamic bargaining setup (Bulow [1982]; Stokey [1981]): there is one-sided incomplete information, and the uninformed party (usually a seller) makes repeated offers in discrete time with exponential discounting. I build on the literature, started by Fudenberg et al. (1985) and Gul et al. (1986), that studies what happens as the seller loses all commitment power not to renegotiate price offers. In particular, the literature has focused on whether a “Coase Conjecture” (Coase [1972]) holds: in the frequent-offers limit, does trade happen instantly, at a price that gives the seller no profit? This literature considers equilibria where continuation play only depends on a sufficient statistic for the seller’s beliefs. I follow that approach here and study only Markovian strategies with the belief cutoff as state variable.

Daley and Green (2018), which I describe in more detail below, define and prove a modified Coase Conjecture result for richer environments: because of her perfect lack of commitment, the uninformed party derives no benefit from the ability to set prices. Remark 5 explains to what extent this modified Coase conjecture holds (with private offers), or fails to hold (with public offers) in my model.

**Bargaining in Changing Environments** A growing literature studies frequent-offer bargaining where the underlying environment changes as players bargain in ways that affect their bargaining strength. The environment might evolve because of different kinds of arrivals—arrivals either of news (Daley and Green [2018]) or of other players (Inderst [2008]; Fuchs and Skrzypacz [2010])—or because the seller’s cost changes over time (Ortner [2017]). So long as the environment changes in an exogenous way, there is no role for privacy, so these papers have a very different focus from my own.

The “stage game” I use borrows from Fuchs and Skrzypacz (2010). They study a discrete-time bargaining game where some event can arrive at a constant, exogenous rate. When the arrival happens, both players get reduced form payoffs that are a function of the function of the buyer’s type. They find that in the limit as the time between offers

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13But see Ausubel and Deneckere (1989a,b) for notable exceptions.
shrinks, there is still delay in equilibrium, since the seller has an incentive to wait for entry that pushes her away from the classic Coase Conjecture result. In Inderst (2008), the seller can only interact with newly arriving buyers sequentially. Unlike Fuchs and Skrzypacz (2010), Inderst (2008) finds trading dynamics that converge to instantaneous efficient trade with the first buyer as the interval between offers shrinks.

Daley and Green (2018) focus on the exogenous arrival of news about the informed player. They study a model in which the informed party is either high or low, values are interdependent so there is a lemons problem, and both parties observe Brownian news that gradually reveals the informed party’s type. I look at a different environment and study a different issue, but I share their approach to formulating the canonical dynamic bargaining model directly in continuous time.

The equilibrium dynamics in my public offers game share a striking feature with that of Daley and Green (2018): whenever trade happens smoothly (at a rate proportional to time) the uninformed party suffers a kind of “ex post regret,” i.e., loses money relative to its outside option on every accepted offer. I explain how my notion of ex post regret is different in Remark 3 below.

3 Model and Equilibrium Notions

3.1 Stage Game and Entry Dynamics

There is a long-lived seller $S$ and a long-lived buyer $B$ bargaining over an indivisible asset owned by $S$. They face a sequence of possible short-lived second buyers ($E$, for “entrant”) who observe the course of negotiations and can choose to enter and trigger a bidding war. $B$’s willingness to pay for the asset is a privately known type $v_B \sim F_B[0,1]$. Each entrant is characterized by a private cost of entry $c \sim G[0, \bar{c}]$. By paying $c$, an entrant can privately observe his value for the asset $v_E \sim F_E[0,1]$. Assume $\bar{c} < \mathbb{E}[(v_E - v_B)_+]$, which ensures that the rate of entry is always positive. The $c$’s and $v_E$’s are independent across entrants, and they are independent of $v_B$. $F_B$ and $G$ are absolutely continuous, with full-support, strictly positive continuous densities $f_B$ and $g$, respectively. I impose the mild technical condition that $f_B(0) > 0$.

Time is continuous, with an infinite horizon. A heuristic description of the “stage game” happening within each “period” $dt$ is as follows. At each $t$ before the arrival of an entrant, the seller offers a price to $B$, who can accept or reject. If $B$ accepts, the game ends, but if $B$ rejects, one of three things can happen. First, an entrant $E$ can arrive. If no entrant arrives, the game moves on to the next “period,” i.e., bargaining between $S$ and $B$ continues. If an entrant $E$ arrives and chooses to pay his entry cost $c$, he then learns his value $v_E$ and immediately triggers bidding war against $B$, which ends the game (formally, there is an ascending auction with no reserve price). If $E$ does pay
c, he leaves, and the game moves to the next “period,” i.e., and bargaining between S and B continues as though E had never arrived. Figure 2 illustrates this heuristic “stage game” being played within each infinitesimal period of time dt.

\[ S \text{ makes offer to } B \quad B \text{ rejects} \quad E \text{ arrives, learns } c \quad E \text{ leaves, next “period”} \]

\[ S \text{ makes offer to } B \quad B \text{ accepts, game ends} \quad E \text{ pays } c, \text{ learns } v_E \quad E \text{ pays } c, \text{ learns } v_E, \text{ triggers auction, game ends} \]

\[ B \text{ accepts, game ends} \quad \text{No } E \text{ arrives, next “period”} \]

\[ dt \]

Figure 2: Heuristic “Stage Game”

If S and B agree at time \( t \) on a price \( p \) before entry occurs, they receive payoffs \( e^{-rt}p \) and \( e^{-rt}(v_B - p) \), while any entrants that may have arrived but not entered receive 0. Let \( \pi_S(v_B) = \int_0^1 \min\{x, v_B\} dF_E(x) \) and \( \pi_B(v_B) = \int_0^1 (v_B - x) dF_E(x) \) be S and B’s expected payoffs from the bidding war (i.e., the ascending auction) when B has type \( v_B \). If there is entry at time \( t \) and a bidding war is triggered, S receives \( e^{-rt}\pi_S(v_B) \) (in expectation over \( v_E \)), B receives \( e^{-rt}\pi_B(v_B) \) (also in expectation over \( v_E \)), and E receives \( (v_E - v_B)_+ - c \). For future reference, denote by \( \Pi_S(k) = \mathbb{E}[\pi_S(v_B)|v_B \leq k] \) the seller’s expected profit from the bidding war if she knows that \( v_B \leq k \), and by \( \Pi_E(k) = \mathbb{E}[(v_E - v_B)_+|v_B \leq k] \) an entrant’s expected surplus from the bidding war (gross of information acquisition costs) he believes \( v_B \leq k \).

Potential entrants arrive to the market according to a Poisson process with constant rate \( \lambda_0 \), which is independent of \( v_B \), and of all draws of \( c \) and \( v_E \). Let \( \mathcal{A} = (\mathcal{A}_t)_{t \geq 0} \) be the filtration for this arrival process on some sufficiently rich probability space. By standard results on sampling of Poisson processes, if at time \( t \) entrants interrupt the bargaining with probability \( q_t \) independent of \( \mathcal{A} \), then \( \sigma \), the stopping time of entry, is distributed according to a Poisson process with rate \( \lambda_0 q_t \). For future reference, let \( \Lambda(k) \equiv \lambda_0 G(\Pi_E(k)) \), the instantaneous rate when \( E \) enters iff \( c \leq \Pi_E(k) \).

Finally, let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space satisfying the “usual conditions” \(^{15}\) (Harrison, 2013, p. 172), with \( \mathcal{F} \) independent of all other random elements in the model \((v_B, \mathcal{A}_t)_{t \geq 0}\).

\(^{14}\) In continuous time, the particular ordering of these actions will matter whenever beliefs or behaviors “jump,” as can happen in my model. What captures my particular extensive form “stage game” is the assumption that, when his beliefs jump at \( t \) after a rejection, the entrant best-responds according to his beliefs after the jump.

\(^{15}\) The space is complete and \( \mathcal{F} \) is complete and right-continuous.
c’s and vE’s, and the Poisson process governing arrivals). Here \( \mathcal{F}_t \) is a public correlation device introduced purely for technical convenience, to be able to define continuous-time strategies consistently.

**Remark 1.** Given the assumption that entrants are short-lived, endogenous entry affects the negotiation only through the reduced-form functions \( \Lambda, \pi_S, \) and \( \pi_B. \) (Note that these depend only on model primitives). Aside from continuous differentiability, the only properties that matter for most of the analysis are

1. \( \Lambda(v) \) is strictly decreasing (if entrants believe \( v_B \) is higher, they are less likely to enter).
2. \( \Lambda(v) + \pi_B(v) + \pi_S(v) < v, \forall v \) (immediate agreement is bilaterally efficient for \( B \) and \( S \); if \( B \) and \( S \) knew \( v_B = v \) in common, they would prefer to agree immediately rather than wait for an entrant who only knows that \( v_B \leq v \)).
3. \( v - \pi_B(v) \) is increasing (\( B \)'s payoff is supermodular in type and discounting until agreement, so higher types agree sooner, i.e., there is “skimming”).
4. \( \pi_S(v) \) is non-negative and strictly increasing (upon entry, \( S \) prefers higher \( v_B \)’s).

I have motivated \( \Lambda, \pi_B \) and \( \pi_S \) with a private values bidding war against an entrant who pays a sunk cost to learn his value. However, as in Fuchs and Skrzypacz (2010), different specifications of \( (\Lambda, \pi_B, \pi_S) \) that satisfy 1-4 can model different entry games in reduced form. In fact, in Section 8 I provide entry games in which \( \Lambda(v) \) is strictly increasing, and I show how to extend the analysis to that case.

### 3.2 Equilibrium Concept

Making formal the heuristic stage game I described above requires an equilibrium concept that is tailored to my particular model, in order to avoid well-known pitfalls in continuous time game-theoretic modeling (Simon and Stinchcombe, 1989). My equilibrium notion modifies Daley and Green (2018) for the current setting. In essence, the equilibrium notion captures the following basic notions from the typical discrete-time formulation in the Coase-Conjecture literature:

- Buyers solve an optimal stopping problem: conditional on no entry, when should they accept the offer and stop the bargaining process?
- The equilibrium satisfies a “skimming property”: Higher types of \( B \) accept earlier than lower types, so \( S \) and \( E \)'s posterior beliefs about \( v_B \) after any history are a truncation of the prior.
Given the above, the current truncation (hereafter, the “cutoff type”) forms a natural state variable for the game, so the focus is on equilibria where acceptance strategies and price offers depend only on the current cutoff.

Since any given price history induces a history of realized cutoff types, along the equilibrium path the seller can be thought of as choosing cutoffs, i.e., how quickly to screen through types, rather than choosing prices.

A key difference between a discrete time formulation (even one that considers the limit as the delay between periods goes to zero) and a continuous time formulation is that, in the latter, the seller solves an “impulse control problem” ([Harrison] 2013). The impulse-control formulation lets me identify two qualitatively different dynamics: periods (instants) during which the probability mass of trade is infinitesimal, so that $S$ screens through types of $B$ one-by-one, and periods during which there are atoms of trade, so that $S$ screens through a positive mass of $v_B$’s in an instant. Meanwhile, in discrete time, in every period with trade, the seller transacts with a positive mass of $B$ types. While these masses may differ in size across periods, there is no qualitative difference between them.

Modeling the game in continuous time also dramatically simplifies the analysis. In continuous time, whenever the seller is screening buyer types one by one, prices, seller payoffs, and often trading speeds are all pinned down to a “smooth trading locus” that depends only on fundamentals. This “smooth trading locus” then guides the rest of the analysis. Even when the seller is screening buyers in chunks of positive mass, her incentives are driven by the shape of that locus, since the size of those chunks depends on the counterfactual of eventually returning to the locus.

I develop the equilibrium definition for the public offers game first, later clarifying how to adapt it for the case of private offers.

**General Buyer’s Problem and Definition of Markov State**

To define my notion of equilibrium, it is easiest to consider first a general feature of the buyer’s best-response problem. Let $\sigma$ be any $(\mathcal{F} \otimes \mathcal{A})_t$-adapted stopping time at which entry happens, and $\{P_t, t \geq 0\}$ denote any $\mathcal{F}_t$ adapted sequence of price offers. Then the buyer $v_B$ solves an optimal stopping problem (when to accept):

$$\sup_{\tau \in T} \mathbb{E} \left[ 1_{\sigma > \tau} e^{-r\tau} (v_B - P_\tau) + 1_{\sigma \leq \tau} e^{-r\sigma} \pi_B(v_B) \right].$$

(1)

where the sup is over $\mathcal{F}_t$-adapted stopping times and $\mathbb{E}[\cdot]$ is the expectation of sample paths of $P_t$ and $\sigma$.

When $v_B - \pi_B(v_B)$ is weakly increasing, the buyer’s objective satisfies increasing differences in $(-\tau, v_B)$, and any selection of maximizers of the above problem will be
decreasing in \( v_B \). Therefore, higher types will accept sooner, and at any time \( t \), for any history of prices offers and entry time with \( \sigma > t \), there will exist some cutoff type \( K_t \) such that all \( v_B \geq K_t \) would have accepted weakly before \( t \). In particular, if either the seller or the potential entrants were to observe \( B \) respond to any \((\sigma, \{P_t, t \geq 0\})\) according to [1], then their beliefs about \( v_B \) at time \( t \) conditional on no acceptance would be right-truncations of \( F_B \) at \( K_t \).

I study equilibria that are Markov in \( K_t \), the belief cutoff (or simply “cutoff” for short) for \( v_B \) at any point in time. \( k \) denotes a generic value of \( K_t \). The equilibrium objects in the public offers game consist of a triple

\[
\left( \{K_t, t \geq 0\}_{k \in [0,1]}, P(\cdot), \mathcal{L}(\cdot) \right)
\]

where each element roughly corresponds to the strategy of a particular player:

1. \( \{K_t, t \geq 0\}_{k \in [0,1]} \) is a collection of non-increasing càdlàg stochastic processes adapted to \( \mathcal{F}_t \), representing possible belief cutoff paths, one for each initial cutoff value. The seller chooses these cutoffs by solving an impulse control problem for any possible initial cutoff. This is analogous to working in “quantity space” in monopoly theory: rather than choose prices that induce rejections leading to belief cutoffs, the seller can choose cutoffs, evaluating them according to the prices that would be consistent with those cutoffs.

2. For any \( k \), \( P(k) \) is simultaneously a) \( v_B = k \)’s reservation price strategy; and b) the price offered at state \( K_t = k \). Given the interpretation of \( K_t \) as a “quantity,” \( P(\cdot) \) has a corresponding interpretation as an (endogenous) inverse demand curve faced by the seller.

3. \( \mathcal{L}(\cdot) \) is the instantaneous rate of entry as a function of the state, chosen by the entrants, so that over an infinitesimal interval \([t, t+dt]\) entry occurs with probability \( \mathcal{L}(K_t)dt \).

Buyer’s Problem, Markov Case, Public Offers For arbitrary \( P(\cdot) \) and \( \mathcal{L}(\cdot) \), \( v_B \) takes \( P(\cdot), \mathcal{L}(\cdot) \), and the law of motion for \( K_t \) as given, and solves

\[
\sup_{\tau \in \mathcal{T}} \mathbb{E}_k \left[ 1_{\sigma > \tau} e^{-r\tau} (v_B - P(K_\tau)) + 1_{\sigma \leq \tau} e^{-r\sigma} \pi_B(v_B) \right]
\]  

where

- \( \sigma \) is a modulated Poisson process, adapted to \((\mathcal{F} \otimes \mathcal{A})_t\), with rate \( \mathcal{L}(K_t) \), and
- \( \mathcal{T} \) is the set of \( \mathcal{F}_t \)-adapted stopping times;
The expectation operator over $K_t$'s and $\sigma$ induced by $\{K_t, t \leq 0\}$ on $K_0^- = k$. What will pin down $P(k)$ is a recursive condition: $P(k)$ is an optimal reservation price strategy for the marginal type $v_B = k$, when future prices are given by the “inverse demand curve” $P(k'), k' \leq k$ and the law of future $K_t$'s.

**Seller’s Problem, Markov Case, Public Offers** $S$ takes $P(\cdot)$ and $L(\cdot)$ as given, and solves

$$
\sup_{Q \in \Gamma} \mathbb{E}_k^Q \left[ \int_0^\sigma e^{-rt} P(Q_t) dF_B(Q_t) + e^{-r\sigma} \Pi_S(Q_\sigma) \right]
$$

where

- $\sigma$ is a modulated Poisson process, adapted to $(\mathcal{F} \otimes \mathcal{A})_t$, with rate $L(Q_t)$, and
- $\Gamma$ is the set of non-increasing processes on $[0, k]$ that are $\mathcal{F}_t$-Markov.
- $\mathbb{E}_k^Q$ is the expectation operator of $Q_t$'s and $\sigma$ induced by $\{Q_t, t \geq 0\}$ on $Q_0^- = k$.

**Entrants’ Problem, Markov Case, Public Offers** Entrants are short-lived, so regardless of $P(\cdot)$, $L(\cdot)$, or the law of $\{K_s, s \geq t\}$, an entrant who arrives at $t$ interrupts the bargaining iff $c \leq \Pi_E(K_t)$. This specification captures the extensive form in Figure 2 if the cutoff jumps at $t$, so that $K_{t-} \neq K_t$, entrants choose to interrupt according to their beliefs after the jump.

**Equilibrium Definition, Public Offers**

**Definition 1.** A triple

$$
\left( \{K_t, t \geq 0\}_{k \in [0, 1]}, P(\cdot), L(\cdot) \right)
$$

forms a **Markov Equilibrium** of the public offers game if

1. $K_t$ is $\mathcal{F}_t$-Markov and time-homogenous.

2. For all $k \in [0, 1]$, $P(k)$ is an optimal reservation price strategy for $v_B = k$, taking as given the law of motion for $K_t$, future prices given by $P(K_t)$ and time of entry given by $L(K_t)$:

$$
k - P(k) = \sup_{\tau \in T} \mathbb{E}_k \left[ 1_{\sigma > \tau} e^{-r\tau} (k - P(K_\tau)) + 1_{\sigma \leq \tau} e^{-r\sigma} \pi_B(k) \right]
$$
3. For any \(k \in [0, 1]\), \(\{K_t, t \geq 0\}_k\) is a seller-optimal impulse control for cutoffs taking as given transaction prices \(P(\cdot)\) and entry rates \(L(\cdot)\), i.e., \(\{K_t, t \geq 0\}_k\) solves

\[
\sup_{Q \in \Gamma} \mathbb{E}^Q_k \left[ \int_0^\infty e^{-rt} P(Q_t) dF_B(Q_t) + e^{-r\sigma} \Pi_S(Q_\sigma) \right]
\]

(5)

4. For all \(k\), entrants interrupt the bargaining iff \(c \leq \Pi_E(k)\), i.e., entry rates \(L(k)\) are given by \(\Lambda(k) \equiv \lambda_0 G(\Pi_E(k))\).

Note that the equilibrium condition on \(P(\cdot)\) only really specifies that the marginal type (the one whose type equals the state, \(v_B = k\)) is stopping optimally at \(K_t = k\). However, since the general buyer’s problem in (1) satisfies a single-crossing property in type and stopping time, whenever \(v_B = k\) wants to stop at \(K_t = k\), so will any type \(v_B > k\), while all \(v_B < k\) will want to continue. Therefore, all buyer types will be stopping optimally at all histories if \(P(v)\) is an optimal reservation price for every \(v_B = v\), and one can derive optimal stopping times at all states and for all types from \(P(\cdot)\).

For future reference, let \(J_S(k)\) denote the seller’s equilibrium value function at state \(k\), and let \(J_B^v(k)\) denote \(v_B = v\)’s equilibrium value function at state \(k\). Note that even though buyer and seller strategies in principle can condition on (are adapted to) the public correlation device \(F_t\), their equilibrium payoffs will depend on \(F_t\) only through \(K_t\). Conditional on no entry, \(S\)’s payoffs going forward depend on \(F_t\) only through \(\{K_t + s, t \geq 0\}\), which she chooses taking \(K_t = k\) as given. Similarly, her payoffs in the case of entry depend only on \(F_t\) only through \(\sigma\), but this stopping time itself depends on \(F_t\) only through \(\{K_t + s, t \geq 0\}\). Therefore, \(S\)’s value function depends on \(F_t\) only through the current state \(K_t = k\). Likewise for \(B\), at any time \(t\) \(v_B\)’s payoffs going forward depend only on \(\{K_t + s, s \geq 0\}\), and the future law of motion for \(\{K_t + s, s \geq 0\}\) depends only on the current state \(K_t = k\). Therefore, \(v_B\)’s value depends on \(F_t\) only through \(K_t = k\).

Equilibrium Definition, Private Offers

Private offers equilibria similarly consist of triples that capture the seller’s screening decision, the buyer’s optimal stopping decision, and the entrants’ entry decision.

Definition 2. A (pure) Regular Markov Equilibrium of the private offers game consists of a triple

\[
(\{K_{s+t}, t \geq 0\}_{k \in [0,1], s \in \mathbb{R}_+}, (\lambda_t^s)_{t \geq 0}, P(\cdot, \cdot))
\]

such that

1. \(t \mapsto K_{t+s}\) is non-increasing, càdlàg, and Markov with respect to \(F_{t+s}\).

2. For all \(k \in [0,1]\) and \(t \geq 0\), \(P(k, t)\) is an optimal reservation price strategy for \(v_B = k\) at time \(t\) taking as given the law of motion for \(K_{t+s}\), future prices given by
\[ P(K_{t+s}, t + s) \) and time of entry given by \( \lambda^*_t \):

\[
k - P(k, t) = \sup_{\tau \in T} \mathbb{E}^K_{k, t} \left[ 1_{\sigma > \tau + t} e^{-r(\tau + t)} (k - P(K_{\tau + t}, \tau + t)) + 1_{\sigma \leq \tau + t} e^{-r(\sigma - t)} \pi_B(k) \right]
\]

where \( \mathbb{E}^K_{k, t} \) is the expectation with respect to the law of \( \{K_{s+t}, s \geq 0\} \) and \( \sigma \), conditional on \( K_{t-} = k \) and \( \sigma > t \).

3. For any \( k \in [0, 1], t \geq 0, \{K_{t+s}, s \geq 0\}_k \) is a seller-optimal impulse control for cutoffs taking as given transaction prices \( P(\cdot, \cdot) \) and fixed, exogenous entry rates \( \lambda^* \), i.e., it solves

\[
\sup_{Q \in \Gamma} \mathbb{E}^Q_{k, t} \left[ \int_0^\sigma e^{-rs} P(Q_{t+s}, t + s) dF_B(Q_{t+s}) + e^{-r(\sigma - t)} \Pi_S(Q_\sigma) \right]
\]

where \( \Gamma \) is the set of decreasing processes on \([0, k]\) adapted to \( \mathcal{F}_{t+s} \), and \( \mathbb{E}^Q_{k, t} \) is the expectation with respect to the law of \( \sigma \) and future \( Q_{t+s} \)'s conditional on \( Q_{t-} = k \) and \( \sigma > t \).

4. \( \lambda^*_t \) satisfies

\[
\lambda^*_t = \Lambda(K_t) = \lambda_0 G(\Pi_E(K_t)),
\]

i.e., the entrants have correct conjectures about \( K_t \).

There are three main differences between this definition and Definition 1 with public offers. First, since entrants do not observe prices, they do not know \( K_t \) and need to form beliefs about it. These beliefs can only depend on how long the disagreement between \( B \) and \( S \) has lasted, since entrants observe nothing else. Therefore, the relevant state for the game is now two-dimensional, including both cutoff and calendar time. Second, in point 3., the seller now takes the law of \( \sigma \) as given. This law depends on the entrants’ beliefs about the path of cutoffs \( (K_t)_{t \geq 0} \), which the seller cannot affect since she has no way of changing what entrants observe. Suppose, for instance, the entrants believe prices leading to a particular path \( (K_t)_{t \geq 0} \) are being played. Since they cannot see the particular offers that were made, if the seller deviates to some other \( (\tilde{K}_t)_{t \geq 0} \) by making different offers, the entrants’ beliefs at \( t^* \) when the game has not ended will still be given by \( K_t \). Third, even though entrants cannot observe prices, in equilibrium, their beliefs about \( v_B \) must be consistent with \( S \) and \( B \)'s choices. Given the focus on pure strategies for the seller, this means that their entry rate must equal what it would have been had they known \( K_t \), which is exactly (6).

I focus on a subset of Markov equilibria with the property that the seller alternates between periods of sufficiently gradual trade and a few periods with bursts of trade.

**Definition 3.** A Markov Equilibrium of either the public or the private game is regular
if, for any initial \( k \), \( K_t = K_t^{ac} + K_t^J \), where \( K_t^{ac} \) is absolutely continuous in \( t \), and \( K_t^J \) is a step function with finitely many jumps.

Below I abbreviate Regular Markov Equilibria as “RME” and when convenient refer to them simply as “equilibria,” with the understanding that all equilibria I study are in this class.

**Definition 4.** Time intervals \( [\bar{t}, \tilde{t}) \) where \( K_{\bar{t}^-} - K_{\tilde{t}^-} = K_{\bar{t}^-}^{ac} - K_{\tilde{t}^-}^{ac} \) are smooth trade regions. If \(-\dot{K}\) is the (a.e.) density with respect to Lebesgue measure of \( K_{\bar{t}^-}^{ac} \) in a smooth trade region, \( \dot{K} \) is the speed of trade.

Since \( K_t \) is monotone decreasing, it has a Lebesgue decomposition. This already implies that \( K_t = K_t^{ac} + K_t^J + K_t^s \), where \( K_t^J \) is a piecewise constant jump function and \( K_t^s \) is a singular continuous function (i.e., a continuous function whose points of increase are measure zero). Therefore, the additional content in making \( K_t \) “regular” is twofold. First, regularity requires that jumps in \( K_t \) are rare. Second, it requires that the continuous part of \( K_t \) is sufficiently smooth: when the seller is driving a hard bargain, the buyer can actually see the price dropping gently over time, rather than \( K_t \) only moving in “flashes.” RME’s do rule out potential dynamics. For example, in Daley and Green (2012) where there is Brownian news and two types, singular trading dynamics correspond to certain kinds of mixing by the informed party: the latter tries to keep the uninformed player’s belief above a certain level. Belief cutoffs in my model only move downwards, so it is not clear what kind of behavior would correspond to singular cutoff dynamics.

**Remark 2 (Mixed Strategies).** In principle, Definition 1 allows for mixed strategies, using the filtration \( \mathcal{F}_t \) as a public randomization device. In practice, the scope for randomization will be limited. Since \( \Pi_E \) is strictly decreasing, entrants will have no indifference regions over which to mix. Moreover, by having the seller choose the “quantity traded” directly, I implicitly assume that the seller can break buyer indifferences in her favor. The only scope for randomization is therefore in the seller’s choice of cutoff paths, but as I discuss below, there are tight constraints on these paths, and by and large the equilibria I consider evolve deterministically along the path of play. Hence, the filtration \( \mathcal{F}_t \) mostly serves as a technical aid to avoid the ill-posedness issues that can arise in continuous-time models of with observable actions (Simon and Stinchcombe, 1989).

# 4 Seller’s Induced Preferences for Privacy

With the equilibrium definitions in hand, I can formally state the main result of the paper, i.e., the seller’s strong preference for private bargaining.

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16. When there is a gap between buyers’ values and seller cost, the dynamics in the continuous limit of Deneckere and Liang (2006), are not regular according to this definition, and they are generated by strategies that are not Markovian in the limit. See, however, Fuchs and Skrzypacz (2013), who show that the equilibria become regular Markov (using my terminology) as the gap shrinks to zero.
Theorem 1. The seller prefers every pure private offers RME to any public offers RME that starts with smooth trade.

Below I provide conditions on primitives which guarantee that every public offers RME starts with smooth trade. In fact, even when there exist multiple RME’s of the public offers game, they either all begin with smooth trade, or all begin with a jump in $K_t$ (Theorem 3). With private offers, trade happens smoothly at all states (Theorem 4), so the essential force behind Theorem 1 is that smooth trade is always worse for the seller when offers are public. To see why smooth trade is crucial for the payoff comparison, recall the discussion of the Coasean Force in the introduction. As I argued above, this force pushes the buyer’s equilibrium willingness-to-pay $P$ down towards the seller’s “opportunity cost” $C$, and it pushes the seller’s payoffs toward her outside option (the ex ante value of stalling). The condition that determines whether $P$ in fact lies flat against $C$, so that the seller is locally indifferent over all trading speeds, is exactly smooth trade, because the seller’s payoffs are \textit{linear in the speed of screening} whenever trade is smooth. Since the seller chooses $-dK_t/dt$ directly, if the seller has strict incentives to screen ($P$ lies above $C$), she will screen infinitely fast; whenever trade happens at a finite smooth speed, this speed drops out of the seller’s objective, and the seller’s payoffs can be pinned down by the ex ante value of stalling.

When the public offers game starts with a jump in $K_t$, the seller does better under public offers than her ex ante value of stalling under public offers, so the comparison between values of stalling across public and private bargaining no longer suffices to determine the seller’s induced preferences for privacy. Nevertheless, there is still a sense in which the seller has strong preferences for privacy: in a broader class of hybrid regimes that alternate between private and public bargaining, public bargaining is dominated for the seller by a regime that is private at all times but the first instant.

Indeed, consider a hybrid “starting-public” game, where the first offer is made public and all subsequent offers are private. Defining RME’s for this hybrid game is notationally cumbersome, so I relegate the formal definition to the appendix, but the idea behind it is straightforward. Since entrants observe the initial offer, the seller controls the initial level of entry. After that first moment, players face a private offers game with initial cutoff equal to the seller’s initial public choice. I show that this nearly private regime is uniformly better for the seller than public bargaining:

Theorem 2. The seller prefers every pure RME of the hybrid starting-public game to any RME of the public offers game.

Once again, the reason why fully public bargaining is dominated for the seller goes back to smooth trade and the Coasean Force. If there were some benefit from the initial

\footnote{So, for example, the seller takes entry rates at all times $t > 0$ as exogenous.}
jump of a public offers RME, the seller can still capture it in the hybrid game by inducing an initial jump with her \( t = 0 \) public offer. At the same time, when bargaining is private after the first instant, the seller can avoid avoid the harmful effects of trading smoothly in public after the initial jump. Therefore, she can capture “the best of both worlds” in the starting public game.

Crucially, the equilibrium definition for hybrid starting-public RME’s allows for the possibility that the continuation private-offers RME depends on the initial jump. This introduces a slight non-Markovian flavor into the space of strategies, but it strengthens the result that the seller would never prefer purely public bargaining: even if one could select private offers RME’s of the continuation game adversarially against the seller after the initial public offer, the seller would still unambiguously prefer the starting-public game to the public offers game.

5 Trading Dynamics Under Public Offers

In the following two subsections, I show how the trading dynamics are largely determined by the hypothetical price path that would have obtained if the equilibrium consisted only of a single smooth trade region.

5.1 Smooth Trade Locus and “Regret Pricing”

Seller Incentives and Smooth Trade Prices  By assumption, an RME consists of at most finitely many jumps and no singular-continuous parts. Therefore, if the equilibrium does not consist of a single jump, there will be at least one smooth trading region of positive length.\(^{18}\) The probability of leaving the interior of such a region in the next \( dt \) is negligible, so the seller’s payoffs in that interior must evolve according to

\[
rJ_S(k) = \sup_{\dot{K} \in [0,\infty)} \left\{ \begin{array}{ll}
\text{flow payoff from accepted offers} & (P(k) - J_S(k))\frac{f_B(k)}{F_B(k)} (\dot{K}) \\
\text{arrival of entrant, game ends} & \Lambda(k)[\Pi_S(k) - J_S(k)] \\
\text{game continues with new state} & J_S'(k)(-\dot{K}) \end{array} \right. \quad (7)
\]

The seller can choose \( \dot{K} \in \mathbb{R}_+ \) freely, and this variable enters (7) linearly. Therefore, if the coefficients on \( \dot{K} \) did not add up to exactly 0, so that \( S \) were not indifferent over

\(^{18}\) Instant trade equilibria are impossible, since they would give a payoff of zero to the seller by the usual Coase Conjecture logic. The seller could do strictly better, in a time-consistent way, by keeping \( K_t \) at 1 and waiting for entry.
all possible speeds, she would choose either $\dot{K} = \infty$ (when the coefficients add up to something positive) or $\dot{K} = 0$ (when the coefficients add up to something negative). $\dot{K} = 0$ corresponds to a quiet period, i.e., a period with no trade. I rule out this case below using the buyer’s problem. For now, focus on $\dot{K} > 0$, when trade is actually taking place at a non-trivial smooth rate. Setting the coefficients on $\dot{K}$ to zero in (7), for any $k$ in the interior of a smooth trading region of any candidate equilibrium, 

$$J_S(k) = \frac{\Lambda(k)}{\Lambda(k)} \Pi_S(k)$$

Let $D(k) \equiv \frac{\Lambda(k)}{\Lambda(k) r}$; this is the expected discount until entry when entry happens with constant Poisson rate $\Lambda(k)$, and it is a crucial object for the analysis. In the public offers game, it is also the expected discount until entry when, after reaching a state $K_t = k$, the seller simply waits for entry indefinitely, i.e., makes unreasonable offers that every remaining $v_B$ would reject. Note that while $D(k)$ is a complicated function of the primitives, it is still defined entirely in terms of those primitives.

The expression for $J_S(k)$ in (8) therefore implies that, under smooth trading, the seller’s payoff is exactly the ex ante value of waiting $D(k)\Pi_S(k)$, i.e., the expected profits from bidding war if the seller screens no further, times the net present value of a dollar at the time of that bidding war if the seller screens no further. This ex ante value is the seller’s outside option or disagreement payoff at state $k$, so in that sense an aspect of the Coasean Force still applies: the seller’s payoffs under smooth trade get pushed down to her disagreement payoffs.

By analogy with $D(k)\Pi_S(k)$, one can also define $D(k)\pi_S(k)$ as the ex post value of stalling. In a region with smooth trade, only the marginal type $v_B = k$ ever accepts the equilibrium price offer $P(k)$. Therefore, conditional on $P(k)$ being accepted during smooth trading, the seller knows $v_B$ is exactly $k$. Hence, she knows that, if instead of trading at that point, she had waited for entry indefinitely by making unreasonable offers forever, she would have obtained exactly $D(k)\pi_S(k)$ in expected discounted profits. $D(k)\pi_S(k)$ is therefore the ex post (in $v_B$) value of stalling at a smooth-trading state $k$.

Plugging in $J_S(k)$ into $P(k)$ in (8) then yields the first major result:

**Lemma 1** (Regret Pricing). On a smooth trade region of any RME of the public offers game, offers prior to entry have an ex post regret property: conditional on knowing the type that gets revealed by acceptance, the seller regrets not having waited for entry:

$$P(k) = D(k)\pi_S(k) + D'(k)\Pi_S(k)F_B(k)/f_B(k) < D(k)\pi_S(k)$$

**Proof.** Use (8) and $\frac{\partial}{\partial k} (\Pi_S(k)F_B(k)) = \pi_S(k)f_B(k)$. \qed

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To understand why the seller is willing to suffer this form of regret, consider that an optimal smooth trading price at state $k$ must balance two effects on the seller’s objective: (i) the loss from the marginal type $v_B = k$, when that price is accepted; and (ii), the gain from the inframarginal types $v_B < k$ when that price is rejected. When entry is endogenous and offers are public, there is a gain from the inframarginal types because a rejection by $B$ makes entrants more optimistic about their prospects and speeds up their rate of entry. Indeed, one can rearrange the expression for smooth trade prices as follows:

$$\frac{-f_B(k)}{F_B(k)} \frac{\text{net profits from sale}}{\text{marginal buyers who accept}} \left[ P(k) - D(k)\pi_S(k) \right] = \frac{\text{profits from auction}}{\text{faster entry after rejection}} \frac{-D'(k) \, dk}{\Pi_S(k)} \text{ < 0}$$

The left hand side is the loss in (i) (since at state $k$, all buyers $v_B > k$ have been ruled out), while the right hand side is gain in (ii). The faster speed of entry affects the seller’s objective through the rate of change in the discount $D(\cdot)$. Intuitively, the seller wants to convey to entrants that entry is “safe” (competition will be weak), but she can only communicate this via low prices that get rejected—high prices that get rejected would have been rejected by so many buyer types that entrants’ would not update their beliefs or their entry rates in response.

The wedge between smooth trade prices and the seller’s ex post value of waiting will be crucial for comparing trading speeds with public and private offers:

**Definition 5** (Public Offers Regret). Let

$$\rho(k) \equiv D'(k)\Pi_S(k)F_B(k)/f_B(k)$$

be the *public offers regret* during smooth trading, so that $\hat{P}(k) = D(k)\pi_S(k) + \rho(k)$.

**Remark 3** (Related Literature, Ex Post Regret, and Belief Manipulation). The ex post regret under endogenous entry is similar to a result of Daley and Green (2018). In their continuous time bargaining model, an uninformed buyer makes repeated offers to an informed seller. The buyer’s value for the asset depends on the seller’s type, and adverse selection is severe, so that the buyer suffers from a lemons problem. There is no entry, but both parties observe a news process that slowly reveals the seller’s type. The authors show that in any smooth trade region of equilibrium, the buyer loses money on every accepted offer, i.e., suffers ex post regret from trading. In their model, the buyer offers attractive prices that get accepted only by the “bad” type of seller as a way to experiment and learn about the seller’s type: the buyer regrets having traded with the bad type at that price, but the price gets rejected often and the buyer’s belief about the seller changes by so much that the gamble is worth it.

In my model, the uninformed party (a seller) is willing to suffer ex post regret, but for
a very different reason. There is no experimentation motive. Rather, S is willing to face this regret so as to affect entrants’ inference about market conditions which she herself does not know. In that sense, the seller’s behavior bears a resemblance to the signal-jamming models of Riordan [1985] and Fudenberg and Tirole [1986]. In those models, an incumbent firm privately sets a low price in the hopes that a competitor will become pessimistic about its future profitability (for instance, the competitor may conclude that the intercept of the market demand curve is low). This is belief manipulation to forestall entry on one’s own side of the market, whereas in my model, there is belief manipulation to encourage entry on the opposite side of the market.

Buyer Incentives and Smooth Trading Speeds

Smooth trade prices are completely determined by the seller’s indifference between speeds of trade. Meanwhile, the speed of trade when trade is smooth is completely determined by the marginal buyer type’s indifference between accepting and rejecting the smooth-trade price. For a state $k$ in the interior of a smooth trade region of equilibrium, the marginal buyer’s payoffs satisfy the following HJB:

$$rJ^k_B(k) = \lambda(k)(\pi_B(k) - J^k_B(k)) + \left(J^k_B(k)\right)'(k)\left(-\dot{K}\right)$$

where $(J^k_B)'(k)$ is short-hand for $\frac{\partial}{\partial k} J^v_B(k)|_{v=k}$.Unlike the seller’s HJB in (7), the HJB for buyer $k$ at state $k$ has no term corresponding to the flow payoffs in (11). This term would consist of the difference between the stopping payoff from accepting $P(k)$ and the continuation payoff from rejecting it, but by condition (11) in the definition of equilibrium, a price of $P(k)$ makes $k$ just indifferent between accepting and rejecting.

First assume that $\dot{K} > 0$. Differentiating both sides of the equilibrium condition $J^k_B(k) = k - P(k)$ with respect to the state $k$ (as opposed to the type $v_B$ that happens to equal $k$) yields

$$(J^k_B)'(k) = -P'(k)$$

Plugging that into the buyer’s HJB in (11) and solving for $\dot{K}$, one finds that

$$\dot{K} = (\lambda(k) + r)\left[\frac{(k - P(k)) - D(k)\pi_B(k)}{P'(k)}\right]$$

must be the speed of trade in the interior of a smooth trading region of any candidate equilibrium. The numerator in the expression in square brackets represents what the marginal buyer $v_B = k$ gains from agreeing, net of losing his disagreement payoff at state $k$. The denominator represents how attractive it would be to agree an instant later and

Nonetheless, the model does have a kind of interdependence of values, since the seller’s net gain over disagreement depends on $v_B$ through $\pi_S(v_B)$. 
instead face prices $P(k) + P'(k)(-dk)$. Therefore, the trading speed in a smooth trading region balances the marginal buyer’s gain from agreeing right now with the possible benefit of rejecting and waiting for better prices.

The candidate equilibrium objects when trade happens smoothly at a positive speed form the smooth trade locus:

**Definition 6.** Let $\hat{P}$, $\hat{K}$, and $\hat{J}_S$ denote functions obtained by naively extending (8) and (12) to the entire support of $v_B$. Then $(\hat{P}, \hat{K}, \hat{J}_S)$ is the smooth trade locus.

For that locus to be meaningful, it remains to rule out $\dot{K} = 0$ in smooth trading regions. As in Daley and Green (2018), if trade were to cease, the marginal type’s implied willingness to pay would rise so much that the seller would want to trade with him, yielding a contraction:

**Lemma 2 (No Quiet Periods).** At every point on a smooth trade region of a public offers RME, $\dot{K} > 0$.

### 5.2 Characterization of Trading Dynamics

A minimal requirement for the speed of trade (12) to form part of an actual equilibrium is that it be non-negative. Plugging in the expression for smooth-trade prices derived above in (9) into the numerator of the smooth-trade speed of trade yields something proportional to $D(k)[\pi_B(k) + \pi_S(k)] - k + D'(k)\Pi_S(k)F_B(k)/f_B(k) < 0$, since $\pi_B(k) + \pi_S(k)$ is at most $k$. The expression in (12) will therefore be positive if and only if $P$ is strictly increasing at $k$.

If $\hat{P}$ were in fact strictly increasing, then (8) and (12) could be used to construct an equilibrium where trade between $B$ and $S$ always happens smoothly along the path of play. However, even in very standard cases, $\hat{P}$ can have strictly decreasing regions, which are incompatible with smooth trade. Figure 3 below plots $\hat{P}$ and $\hat{J}_S$ against $1 - k$ for the case where $F_B$, $F_E$, and $G$ are all $U[0, 1]$. I plot these curves against $1 - k$, rather than against $k$, to facilitate comparisons with the graphical intuition in the introduction. $\lambda_0$ and $r$ affect these locii only through $\gamma = \lambda_0/r$, the relative intensity of arrival. Dotted lines trace $\hat{J}_S$, and solid lines trace $\hat{P}$. The figure shows that both $\hat{P}$ and $\hat{J}_S$ for this example are strictly single-peaked at an interior point. As the relative intensity of arrival increases, $\hat{J}_S$ becomes more likely to be always decreasing (i.e., increasing in $k$), and the increasing region of $\hat{P}$ shrinks. But at the same time, $\hat{P}$ becomes steeper in that region (i.e., more decreasing in $k$). For large enough $\gamma$ (not depicted on Figure 3), the increasing region vanishes, and $\hat{P}$ becomes monotone.

The non-monotonicities in Figure 3 extend to more general primitives. Lemma 7 in the appendix gives some sufficient conditions, which can hold in very standard models, under which $\hat{P}$ is strictly non-monotone. In particular, a sufficient condition for $\hat{P}$ to be
non-monotone is that entry be very sensitive to beliefs, where sensitivity is measured by the slope of $D$: if $D$ is highly decreasing, then $J_S = D\Pi_S$ will have decreasing portions and interior local maxima, and $\hat{P}$ must have interior local maxima. Indeed, $D'(k)$ is relevant measure of sensitivity of entry as far as the seller’s incentives are concerned, since she cares not about the changes in the rate of entry per se, but about the ultimate effects on the expected discount till entry.

Figure 3: Prices, Seller Payoffs Along Smooth Trade Locus, Different Values of $\gamma \equiv \frac{\lambda_r}{r}$

Consider, then, the seller’s payoffs from causing a downwards jump in $K_t$. Since RME’s have only finitely many jumps, and by Lemma 2 they have no quiet periods, any jump must end at a state where smooth trade commences at a strictly positive speed. That is, jumps always happen onto the smooth trade locus. The price that induces a jump in $K_t$ down to $k'$ on the smooth trade locus must be precisely the price that leaves the buyer with type $v_B = k'$ indifferent between accepting and rejecting, i.e., $\hat{P}(k') = k' - J^k_B(k')$. Hence, the seller’s payoff from jumping from a state $k$ to a state $k' \leq k$ at which smooth trade commences is

$$U(k, k') = \left(1 - \frac{F_B(k')}{F_B(k)}\right) \hat{P}(k') + \frac{F_B(k')}{F_B(k)} \hat{J}_S(k').$$

$U$’s behavior significantly pares down the set of possible RME’s. First, if there is smooth trade at $k$ along the equilibrium path, it must be that $k \in \arg\max_{k \in [0,k]} U(k, \hat{k})$—otherwise, the seller would want to jump down to some $k' < k$ and trade smoothly from there instead. Second, recall that, within the class of RME’s, all jumps in $K_t$ happen onto the smooth trade locus. Therefore, if there is a jump at $k$ to $k' < k$ along the equilibrium path, it must be that $k' \in \arg\max_{k \in [0,k]} U(k, \hat{k})$ and $k' \in \arg\max_{k \in [0,k']} U(k', \hat{k})$—otherwise, either the seller would not want to jump to $k'$, or once there would want to jump again. These kinds of considerations lead to the second main result:

**Theorem 3** (Trade Dynamics). Let $\hat{P}$ be the smooth trade locus for prices from (8).
extended to all $k \in [0, 1]$. In any RME of the public offers game,

1. Trade can only happen smoothly in states $k$ where $\hat{P}(k)$ is strictly increasing. Moreover, trade must be smooth at any $k$ such that $\hat{P}$ is increasing on $[0, k]$.

2. Trade happens in bursts at any state $k$ where $\hat{P}(k)$ is strictly decreasing.

3. At any such $k$, the state jumps to a local maximum of $\hat{P}$ to the left of $k$.

Almost everywhere in the smooth trading regions, trade happens at a speed

$$\dot{K} = (\Lambda(k) + r) \left[ \frac{(k - \hat{P}(k)) - D(k)\pi_B(k)}{\hat{P}'(k)} \right].$$

Theorem 3 shows that two very different kinds of trading dynamics can co-exist in equilibrium: slow, gradual trading, interspersed with burst of trade. The proof, which I present in the appendix, relies solely on local optimality arguments, the key for which is the following expression:

$$U_2(k, k') = \left(1 - \frac{F_B(k')}{F_B(k)}\right)\hat{P}'(k'),$$

where $U_i$ denotes the partial derivative of $U$ with respect to the $i$-th argument. By (14), the local maxima and local minima of $k' \mapsto \hat{P}(k')$ and $k' \mapsto U(k, k')$ will coincide for the sub-domain $[0, k]$ so optimal jumps must end at local maxima of $\hat{P}$.

When $\hat{P}$ is single-peaked, as in the uniform-uniform example from Figure 3 in all public offer RME’s, the seller starts by causing a burst of agreement at the first instant of bargaining, after which the parties settle in for a long impasse during which agreement happens in dribs and drabs. Indeed, once the state reaches the peak of $\hat{P}$, trade will proceed smoothly until the end of the game, since there are no more decreasing regions of $\hat{P}$ to run into. To an outside observer who saw a large cross-section of bargaining interactions according to the model, the empirical CDF of agreement times would look like the one in Figure 4: a jump up at $t = 0$, followed by a gradual asymptote towards 1 as $t$ grows.

Whenever $\hat{P}$ is single-peaked, the above reasoning suffices to explicitly construct an RME:

**Proposition 1.** If $\hat{P}$ is single-peaked with an interior maximum at $\hat{k}$, and the ODE

$$\dot{k}(t) = - (\Lambda(k(t)) + r) \left[ \frac{(k(t) + \hat{P}(k(t))) - D(k(t))\pi_B(k(t))}{\hat{P}'(k(t))} \right], \quad k(0) = \hat{k}$$

For instance, if $k'$ is a local minimum of $\hat{P}$, then $\hat{P}'$ goes from negative to positive in the neighborhood around $k'$. This sign change is transferred to $U_2(k, k')$, which means $U(k, k')$ is also at a local minimum.

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has a decreasing solution on $[0, T]$ for any $T$, then there exists an RME of the public offers game. For all such RME’s,

1. on histories with no entry, for any $k > \hat{k}$ trade starts in a burst where $[\hat{k}, k]$ accept instantly, and then trade proceeds smoothly at speed $\hat{k}$.

2. 

$$P(k) = \begin{cases} \hat{P}(\hat{k}), k \geq \hat{k} \\
\hat{P}(k), k < \hat{k} \end{cases}$$

3. $\mathcal{L}(k) = \Lambda(k)$.

The argument, in the appendix, adapts the construction Daley and Green (2018) to the current setup with no news and a continuum of types. Note that this proposition includes the case of monotone increasing $\hat{P}$, in which case standard Peano theory arguments provide existence results for the ODE in (15) without additional conditions.\(^{21}\) When $\hat{P}$ has an interior local maximum, however, the existence requirement on (15) is non-trivial, since the right hand side of the ODE explodes near the initial condition. In economic terms, the equilibrium calls for smooth trade to commence at infinite speed in the first instant after the initial jump, which might create an atom of trade unless the speed declines sufficiently quickly after that first instant. Intuitively, smooth trade prices must have enough curvature around their peak to provide buyer types with an incentive to wait: in the limit, if they were indeed flat for a range of $k$’s to the left of the peak of $\hat{P}$, a positive mass of buyer types would want to accept in the instant after the first jump (and supposedly final) jump.

**Remark 4** (Mixed Strategies Revisited). Lemma 2 and Theorem 3 severely limit the regions of indifference that the seller might have over cutoff paths. At any state, in a public offers RME, the seller could only possibly randomize

\(^{21}\) The construction used here could in principle be extended to more general shapes of $\hat{P}$. The main limitation is that, for arbitrary shapes of that smooth trade locus—in particular, if $\hat{P}$ has multiple interior peaks—it becomes harder to verify the global optimality of a given jump. Which peak should the seller jump to? Nevertheless, in the case with multiple peaks, Theorem 3 would still significantly narrow down the set of candidate equilibria on which to try the construction above.
over jump sizes at a jump state $k$ (i.e., a state with $\dot{P}(k) < 0$), and only if (i) $\dot{P}$ has multiple local maximizers in $[0, k)$ and (ii) $k' \mapsto U(k, k')$ has multiple global maximizers on the same domain;

over solution paths of $\dot{P}'$ at the start of a smooth trading region, if the ODE has has multiple solutions with that state as an initial condition.

Therefore, the equilibrium path of any mixed RME of the public offers game will be a randomization over pure RME equilibrium paths, and any realized path of a mixed RME is itself a pure RME path. This differs from the private offers case, where the restriction to pure strategies is an essential part of the equilibrium definition.

The jumps in $K_t$ along the equilibrium path have an important consequence:

**Remark 5** (Failure of the Modified Coase Conjecture). Daley and Green (2018) define a modified Coase Conjecture result: because of her perfect lack of commitment, the uninformed party making the offers derives no benefit from the ability to negotiate prices. The equilibrium may involve delay of trade and positive profits, depending on the environment, but the uninformed party’s payoffs must be exactly what it would receive if it were unable to make offers at all.

An immediate consequence of Theorem 3 is that this form of the Coase Conjecture fails in the present model. If the seller were to lose the ability to make price offers at a state $k$, she could only wait for entry indefinitely, and her continuation value would be exactly $\hat{J}_S(k) = D(k)\Pi_S(k)$. In contrast, if $k$ is a jump state (one where $\dot{P}$ is decreasing), then the seller’s payoff in any RME is

$$\max_{k' \leq k} U(k, k') > U(k, k) = \hat{J}_S(k),$$

where the strict inequality follows from the fact that $k$ is not a local optimum of $k' \mapsto U(k, k')$ on $[0, k]$. Hence, at jump states the seller strictly benefits from the ability to renegotiate prices.

### 6 Trading Dynamics Under Private Offers

Note that, without the consistency requirement on entrants in beliefs (condition 4), the private offers RME definition just describes a game with an exogenous, time-varying entry-rate. Schematically,

$$\begin{pmatrix} \text{Private Offers} \\ \text{Endogenous Entry} \end{pmatrix} = \begin{pmatrix} \text{Time-varying} \\ \text{Exogenous Entry} \end{pmatrix} + \begin{pmatrix} \text{Consistent} \\ \text{Beliefs} \end{pmatrix}$$
Indeed, the key to characterizing private offers dynamics is to understand the seller’s incentives in a game of exogenous but time-varying entry. Aside from laying the groundwork for comparing public and private offers, this step highlights the new forces and incentives that are due to the endogeneity of entry itself.

6.1 Smooth Trading and No-Regret Pricing

A crucial object for the discussion below is

$$D^X(t|\lambda) \equiv \int_t^\infty \lambda_s e^{-\int_t^s (\lambda_u + r) du} ds,$$

the time-\(t\) present value of a dollar at entry when entry occurs according to a rate \((\lambda_s)_{s \geq 0}\). With this object in hand, I can state the first key result on time-varying exogenous entry:

**Proposition 2.** Let \(\lambda\) be any entry rate process such that \(D^X(t|\lambda)\) exists and for all \(t\) and is uniformly bounded away from 0. Then all RME’s of the exogenous entry game have smooth trade at all states \((k,t)\), at a speed

$$\frac{dK_t}{dt} = -(\lambda_t + r) \left[ \frac{K_t - \lambda_t}{\lambda_t + r} \left( \pi_B(K_t) + \pi_S(K_t) \right) \right].$$

Aside from the integrability condition on \(\lambda\), Proposition 2 makes no assumptions about entry. A priori it might have seemed plausible that large, discontinuous changes in \(\lambda\) could cause jumps or quiet periods in \(K_t\). If the entry rate is going to plummet tomorrow, a possible argument might have gone, the seller has discontinuous incentives to agree today. Shouldn’t she trade in a burst today in anticipation? This kind of reaction is in fact impossible when traders are forward looking, and the reasoning again follows from the Coase logic. If entry rates were going to collapse tomorrow, \(B\)’s willingness to pay today would rise in such a way that the seller had a strict incentive to trade.

In particular, Proposition 2 implies that bursts of trade are only possible with public offers. Since cutoff paths \(K_t\) are decreasing, equilibrium entry rates under pure RME’s of the private offers game must be increasing and bounded from above by \(\Lambda(0)\) and from below by \(\Lambda(1)\). Therefore the discount factors \(D^X(t|\lambda^*)\) for any candidate equilibrium entry rate \(\lambda^*\) will be well defined and bounded between \(\frac{\Lambda(1)}{\Lambda(1) + r}\) and \(\frac{\Lambda(0)}{\Lambda(0) + r}\). Moreover, since the seller chooses cutoff paths taking the equilibrium entry rate path as fixed and exogenous, Proposition 2 applies to the seller’s best response problem taking \(\lambda = \lambda^*\).

**Theorem 4.** Pure Regular Markov Equilibria of the private offers game have only smooth trade at all states \((k,t)\). If \(\lambda^*\) is the equilibrium entry rate process, trade proceeds at a
speed

\[ \frac{dK_t}{dt} = -\left( \lambda^*_t + r \right) \left[ \frac{K_t - \frac{\lambda^*_t}{\lambda^*_t + \pi_B(K_t)} (\pi_B(K_t) + \pi_S(K_t))}{D^X(t|\lambda^*) \pi^t_S(K_t)} \right]. \]

The additional complication from adding exogenously time-varying entry to the model is that the smooth trade locus for the seller’s value function becomes harder to characterize. In section 5.1, the seller’s indifference over trading speeds sufficed to pin down the seller’s payoffs along the smooth trade locus in closed form, and the smooth trade locus in turn sufficed to characterize the optimal jumps. With time-varying exogenous entry, that indifference does not suffice to solve for the seller’s payoffs. Let \( \hat{J}^X_S(k, t|\lambda) \) be the seller’s payoff at a state \((k, t)\) with exogenous entry rate \( \lambda = (\lambda_s)_{s \geq 0} \). Then as in the public offers case, the linearity of the seller’s objective in the speed of trade \( \dot{K} \) means that \( \hat{J}^X_S(k, t|\lambda) \) must be independent of the speed:

\[ r \hat{J}^X_S(k, t|\lambda) = \lambda_t [\Pi_S(k) - \hat{J}^X_S(k, t|\lambda)] + \frac{\partial}{\partial t} \hat{J}^X_S(k, t|\lambda). \] (16)

Unlike the public offers case, the smooth trade locus here pins down an ODE that this value must satisfy, but it gives no guidance about the right boundary condition.\(^{22}\) To avoid this complication, the proof of Proposition 2 proceeds by identifying the smooth trade locus and proving that the seller never wants to jump onto it a final time. Since RME’s have finitely many jumps, this implies that candidate equilibria cannot have jumps at all. The seller’s payoffs must then drift down smoothly to 0; this information acts as a boundary condition with which to solve (16).

Public bargaining therefore has trading dynamics that are impossible under private offers. But even when trading dynamics are smooth in both regimes, there are deep qualitative differences between them. To see this, consider first the ex post value of waiting when the seller cannot affect entry rates. Recall that the ex post value of waiting is the discounted profit the seller would expect to receive if instead of trading smoothly at some state, she made unreasonable offers forever and waited for entry—where the expectation is conditional on what she learns from trading smoothly. The seller’s ex post value of stalling, when entry happens at a rate \( (\lambda_s)_{s \geq 0} \) that the seller cannot affect, is \( D^X(t|\lambda) \pi_S(k) \). Compared to the public offers ex post value of waiting, the discount applied to entry is no longer \( D(k) \), but \( D^X(t|\lambda) \), since the entry rate follows its (exogenous) expected path even after the seller starts making unreasonable offers.

The seller was willing to tolerate ex post regret from her public offers because low prices, when rejected, helped lure entrants faster and caused a bidding war sooner. In economic terms, she accepted losses on the marginal buyer (relative to the ex post outside option) because this created gains from the inframarginal ones. When the seller cannot

\(^{22}\) More precisely, the possible boundary condition depends on equilibrium play after the smooth trade region ends or before it starts, but this equilibrium play is precisely what one is trying to determine.
affect the entry rate through her offers, she cannot affect the equilibrium amount she
would receive from the inframarginal types, and there is nothing to balance the loss from
marginal types if the price were lower than the ex post value of waiting. Formally,

**Lemma 3.** Let $\lambda$ be any entry rate process such that $D^X(t|\lambda)$ exists for all $t$ and is
uniformly bounded away from 0. Then at all states $(k, t)$ equilibrium prices have a **no ex post regret** property:

$$P^X(k, t|\lambda) = D^X(t|\lambda)\pi_S(k).$$

The result is derived as part of the proof of Proposition 2 in the appendix; the expression
takes into account the smooth trade region after the final jump, but in light of Proposition
2, the same result will hold for all $(k, t)$.

Crucially, when entry is endogenous and offers are private, the seller’s ex post value
of waiting must also take the path of entry rates as given. Therefore, evaluating Lemma
3 at the equilibrium entry rates $\lambda^*$ of a pure private offers RME yields

**Lemma 4** (No-Regret Pricing, Private offers). In any pure RME of the private offers
game, equilibrium prices have a **no ex post regret** property:

$$P^X(k, t|\lambda^*) = D^X(t|\lambda^*)\pi_S(k).$$

Lemma 4 makes clear that the ex post regret in Lemma 1 is due exclusively to the public
nature of the offers, and is not a result of endogenous entry on its own. Even though
private-offer trading speeds affect entry in equilibrium, the seller best responds as though
she cannot affect entry, and as a result she loses any incentive to post “regrettable” offers.

In general, saying more about the public-private contrast would require explicitly
constructing these equilibria. In turn, that would require characterizing the fixed point
of entry rates, but this fixed point in continuous time is (at least in principle) extremely
intractable, since it is characterized by a non-linear functional equation. (In particular,
with exogenous entry, the smooth trade locus $\dot{K}_t^X$ depends on all of $(\lambda_s)_{s \geq 0}$ through (26).)

However, the consistency condition, together with the smooth nature of private offer
RME’s, suffices to make a number of clear comparisons between public and private offers.
In particular, those two features make it possible to prove the existence of pure private
offer RME’s:

**Lemma 5** (Existence: Private Offers). There always exists a pure RME of the private offers game.

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23 In a model with exogenous and constant rates of entry, Fuchs and Skrzypacz (2010) had found that
in any atomless limit of their discrete-time equilibria, prices had a “no ex post regret” property; Lemma
3 generalizes this result to arbitrary time-varying exogenous rates.
The proof, in the appendix, is by construction, and it provides an algorithm to numerically compute private offers RME’s. Below, I use that numerical approach to compare equilibrium payoffs in cases where the comparison is not pinned down by the smooth trade locus.

7 Ranking Payoffs, Prices, and Delay

Equipped with the existence result of Lemma 5 I can make meaningful comparisons between the equilibrium sets of the public and private offers games. Ultimately, one wants to make statements about equilibrium payoffs and welfare. Making offers private affects payoffs through three separate channels. It changes (i) transaction prices on histories with no entry, (ii) the time of agreement between B and S on histories with no entry, and (iii) the timing of entry. I address each of these channels in turn, though notice that even signing the direction of each effect does not on its own lead to a payoff comparison.

Consider, for example, the buyer’s ex post equilibrium payoff in the simplest case, when \( F_E(v) = 1_{v \geq 1} \) so that \( \pi_B(v) = 0 \forall v \). If the buyer \( v_B = v \) trades agrees to a price \( p \) at a time \( t \) on histories with no entry, he receives \( 1_{t < \sigma} e^{-\sigma t} (v - p) \), where \( \sigma \) is the time of entry. By changing \( t \), one changes the cutoff path leading up to \( t \), which in turn changes \( \sigma \). Therefore, even changes that postpone the time of trade can benefit the buyer if they postpone the time of entry by less. Payoff comparisons will require additional arguments.

Comparing Seller Payoffs In spite of all the sources of ambiguity I mentioned above, the logic of smooth trading is sufficiently powerful to pin down a payoff comparison for the seller. The discussion in Sections 5 and 6.1 imply that, at any state \( K_t = k \) where trade proceeds smoothly under public bargaining, the seller’s public offers equilibrium payoffs equal the ex ante value of stalling with public offers \( D(k) \Pi_S(k) \), while with private bargaining those payoffs always equal the ex ante value of stalling with private offers, \( D^X(t|\lambda^*) \Pi_S(k) \).

To prove Theorem 1, it therefore suffices to rank the discount factors \( D^X(t|\lambda^*) \) and \( D(k) \) along the equilibrium path, i.e., at states \( (k, t) \) such that \( K_t = k \). Let \( t^*(k) \) denote the time at which \( v_B = k \) trades in a pure RME of the private offers game. (Since trade is smooth with private offers, \( t^*(k) \) is always a strictly decreasing, continuous function).

Lemma 6. For any pure RME of the private offers game, with corresponding equilibrium entry rates \( \lambda^* \),
\[
D^X(t^*(k)|\lambda^*) > D(k) \quad \forall k > 0
\]
Hence, applying Lemma 6 at $k = 1$,

$$D(k = 1) \Pi_S(1) < D^X(t = 0|\lambda^*) \Pi_S(1).$$

(20)

so the seller prefers any private bargaining RME to any public bargaining RME that starts with smooth trade, which proves Theorem 1.

The proof of Lemma 6, relegated to the appendix, formalizes the intuition given in the introduction: $D(k)$ is the expected discount to entry when the entry rate gets frozen at $\Lambda(k)$, whereas $D^X(t^*(k)|\lambda^*)$ is the discount to entry when the rate starts at $\Lambda(k)$ and rises. Both $D(k)$ and $D^X(t^*(k)|\lambda^*)$ are the discounts to entry in the counterfactual where the seller starts making unreasonable offers indefinitely at state $K_t = k$. When offers are public, entrants would see these unreasonable offers and cease updating their beliefs. Meanwhile, when offers are private, the entrants’ initial conjecture at $K_t = k$ was correct, so in the first instant they enter at rate $\Lambda(k)$. But they never notice this hypothetical deviation by the seller, so they become increasingly optimistic over time and enter at faster and faster rates.

The logic of Theorem 1 extends to any point in the public offers game with smooth trading:

**Corollary 1.** Conditional on reaching a cutoff state $k$ where there is smooth trading under public offers along the equilibrium path, the seller has a lower continuation value under public offers than under any pure private offers RME.

When offers are public, the seller has a lever that she lacks with private offers: she can lure entrants into a bidding war faster by offering lower prices that get rejected. One might expect that this lever could only help the seller and hurt the buyer. After all, isn’t the seller creating more competition for the buyer? Surely, if $F_E$ were skewed towards very high values, a bidding war should be far more profitable than almost any bargain she could strike with the first buyer; shouldn’t the ability to make the more profitable alternative happen sooner help her? Theorems 1 and 2 show that having this lever available can harm the seller. Her lack of commitment pushes her payoffs down to an outside option, and this outside option is usually worse with public offers than with private offers. While the additional competition is attractive for the seller, it exacerbates her commitment problem.

Note also that which of Theorems 1 and 2 to apply depends (almost) entirely on primitives. By Theorem 3, if $\hat{P}$ is everywhere increasing, every public offers RME starts with smooth trading, and Theorem 1 suffices for understanding the seller’s induced preferences for privacy. Meanwhile, if $\hat{P}'(1) < 0$, every public offers RME starts with a jump in $K_t$. For such primitives, Theorem 2 still shows that the seller has robust preferences.
for private bargaining. Since $\hat{P}$ depends only on model parameters, these condition on $\hat{P}$ can be checked for given primitives without solving for RME explicitly.

Theorems 1 and 2 still leave open the possibility that the seller could sometimes prefer strictly public offers to strictly private offers. Whether this happens depends on fine details of the model parameters $\lambda_0, r, F_B, F_E,$ and $G$, so it will require solving for private offer RME’s explicitly. Any advantages to the seller from public offers would come from (i) sales happening more often through bidding wars than in the normal course of negotiations, and (ii), trade happening faster (Theorem 6). Therefore, to see whether bursts of trade can overwhelm the advantages of private offer bargaining, I choose an example where (i) auctions are extremely profitable, i.e., entrants are unbeatable, and (ii) the seller really dislikes delay, i.e., a high discount rate. In Table 1 I compute equilibrium payoffs for an environment with $r = 0.3$ and $F_E = 1_{v_e \geq 1}$. I consider different arrival environments by changing the rate of arrivals $\lambda_0$ from fractions of $r$ to multiples of $r$, while leaving $G$ fixed at $U[0,1]$, for tractability. Likewise, I consider different informational environments by varying $F_B$ within the family of power distributions $F_B(v) = v^\beta, \beta > 0$. I vary $\beta$’s in a wide range surrounding $\beta = 1$, the uniform case. Each entry of Table 1 represents the percentage gain in the seller’s objective when moving from the lowest pure RME of the private offers game to the (unique, at least in payoff terms) RME of the public offers game. Rows represent different values of $\beta$, and columns represent different values of $\lambda_0$.

<table>
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<th>$\beta \setminus \lambda_0$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.6</th>
<th>1.0</th>
</tr>
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<td>-4.14</td>
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<tr>
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<td>-17.02</td>
<td>-10.77</td>
<td>-3.38</td>
<td>-1.55</td>
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<tr>
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<td>-20.25</td>
<td>-8.21</td>
<td>-4.42</td>
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<tr>
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<td>-35.15</td>
<td>-19.28</td>
<td>-13.33</td>
</tr>
<tr>
<td>6.0</td>
<td>-50.76</td>
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<td>-24.07</td>
<td>-17.51</td>
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<td>10.0</td>
<td>-53.01</td>
<td>-43.03</td>
<td>-25.87</td>
<td>-19.07</td>
</tr>
</tbody>
</table>

Table 1: % Gain in $J_S$, Public RME vs. (Pure) Private RME. Model Parameters: $F_B(v) = v^\beta, F_E(v) = 1_{v_e \geq 1}, G = U[0,1], r = 0.3$. Grid Parameters: $T = 4 \times 10^4, \Delta = 4 \times 10^{-3}$.

The numerical evidence in Table 1 suggests that the forces identified in Theorem 1 and Corollary 1 in favor of private offers are strong and operate robustly beyond the case of monotone $\hat{P}$. Bargaining in public as opposed to in private can decrease the seller’s payoffs by more than 50% in some cases. The higher ex ante value of waiting more than compensates the seller for the higher delay incurred during from smooth trading.

**Comparing Price Offers** It seems intuitive that, since smooth-trade public offers are “regrettable,” while private offers have a no-regret property, smooth-trade equilibrium prices under public offers should be lower than equilibrium prices under private offers. However, this argument is insufficient because private offers have a no-regret property
with respect to stalling under private offers and public offers have a regret property with respect to stalling under public offers. Both cases consider the same counterfactual seller strategy of making unreasonable offers until the auction happens, but this strategy has very different effects in the two cases, so the ex post values of stalling will differ. Nevertheless, as I show in the following theorem, it is still possible to rank equilibrium prices across privacy regimes, uniformly for all buyer types and regardless of whether they trade smoothly or in a burst:

**Theorem 5** (Privacy and Price Ranking). Fix any two pure RME’s under endogenous entry, one where offers are public and one where offers are private. Let $p^{\text{pub}}(k)$ and $p^{\text{priv}}(k)$ be the prices that type $v_B = k$ pays when it buys the good from from $S$ before entry. Then $p^{\text{pub}}(k) < p^{\text{priv}}(k), \forall k > 0$.

The proof is in the appendix. By Lemma 6, the discount factors are ranked across public and private offers in a way that reinforces the regret effect. This is still not enough to rank equilibrium prices for all types, since there may exist types who never trade smoothly with public offers. Nevertheless, the argument goes through by considering that, if a type trades in a public offers jump, he pays exactly what the endpoint of that jump would have paid in smooth trading.

**Remark 6** (Related Literature on Transparency). Theorem 5 contrasts with previous studies on the effects of transparency on the division of surplus (Fuchs et al., 2016; Kaya and Liu, 2015). These papers have typically found that making offers private (i) can be a Pareto improvement and will generally benefit the informed party, and (ii) makes prices more favorable to the informed party. As I mentioned in the introduction, this comparison is subject to the caveat that in this paper “making offers private” means something different. In those models there is a sequence of uninformed players but no entrants, whereas in mine there are entrants but only one uninformed party. In the current setting, the offers are being hidden from (potentially) informed entrants, while in these papers the offers are hidden from future uninformed players.

A further caveat is that the uniform price ranking result I obtain is of a very different kind from, say, the one in Kaya and Liu (2015). Their ranking results refer to the prices offered period-by-period, whereas mine refer to prices paid type-by-type (and even then, only prices paid outside of bidding wars). Also, unlike Kaya and Liu (2015), one cannot conclude from the uniform price ranking I obtain that $B$ is unambiguously worse off under private offers without knowing the effect on (i) trading times in the absence of entry, and (ii) on the probability of entry.

That caveat notwithstanding, one can summarize the difference between my price ranking result and those in the previous literature is as follows. In Kaya and Liu (2015)’s

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25 However, by focusing on realized prices type-by-type, rather than period-by-period, I can obtain a uniform price ranking without any additional assumptions on $F_B$. 

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model, making offers private lowers prices by removing the buyer’s incentives to reject prices in order to signal their value to future sellers. Here, making offers private raises prices by removing the seller’s ability to jump-start entry and expose buyers to more competition.\footnote{In fact, the price ranking in Theorem\,\ref{thm:price} only uses the seller’s incentive conditions and the consistency of entrants’ beliefs.}

**Comparing Trading Speeds** Under an additional condition on the regret component of the public offers smooth locus, one also obtains a uniform ranking of trading times, to go along with the uniform ranking of prices in Theorem\,\ref{thm:price}:

**Theorem 6 (Privacy and Cutoff Ranking).** Let $K_{\text{pub}}^t = \{K_{\text{pub}}^t\}_{t \geq 0}$ and $K_{\text{priv}}^t = \{K_{\text{priv}}^t\}_{t \geq 0}$ be any equilibrium sample paths for cutoffs under some public offers and private offers, respectively. If the public offers regret $\rho(k)$ is decreasing in $k$, then $K_{\text{pub}}^t$ is uniformly smaller than $K_{\text{priv}}^t$:

$$K_{\text{pub}}^t < K_{\text{priv}}^t \text{ for all } t > 0.$$  

In words, Theorem\,\ref{thm:privacy} states that, if the seller’s belief-manipulation motive decreases over time, then on histories with no entry all but the highest buyer type trade later when offers are private.\footnote{Note that $K_{\text{pub}}^0 = K_{\text{priv}}^0$, but $K_{\text{pub}}$ may have a jump at $t = 0$. So either $K_{\text{pub}}^0 = K_{\text{priv}}^0 = 1$, and the highest type trades at the same time in both regimes, or $K_{\text{pub}}^0 < K_{\text{priv}}^0 = K_{\text{priv}}^0$, and the highest type trades strictly earlier with public offers.} The proof is in the appendix. The argument relies on comparing the smooth trading speeds at any point at which $K_{\text{priv}}$ and $K_{\text{pub}}$ cross. I show that at any such point, the speed of trade under public offers is faster, so $K_{\text{pub}}$ must be crossing $K_{\text{priv}}$ from above. Since $K_{\text{pub}}$ and $K_{\text{priv}}$ start at the same point, and $K_{\text{pub}}$ declines either in jumps or at a faster smooth speed than $K_{\text{priv}}$, then $K_{\text{pub}}$ must lie uniformly below $K_{\text{priv}}$.

$\rho(\cdot)$ is decreasing, for example, for $F_B = F_E = G = U[0, 1]$, or whenever $D(\cdot)$ is concave and $F_B$ is log-concave. The monotonicity condition on $\rho(\cdot)$ is sufficient for a uniform ranking of $K_{\text{pub}}$ and $K_{\text{priv}}$, but it is probably much stronger than necessary. Even when $\rho$ is increasing, there are many reasons to expect that $K_{\text{pub}}$ will be uniformly smaller than $K_{\text{priv}}$. First, $K_{\text{pub}}$ has downward jumps, whereas $K_{\text{priv}}$ always decreases smoothly. Second, in smooth trade regions, the speed at which cutoffs fall is given by the buyer’s indifference between waiting for a price change and losing the chance to trade when an entry occurs. A close look at the proof of Theorem\,\ref{thm:privacy} shows that by and large, making offers private affects this indifference condition in ways that slow down trade:

1. Because of the entrant-luring incentives pushing down prices under public offers, the gain in utility from accepting right now rather than stalling is always higher with public offers.
2. At the same time, the ex-post component of prices is always steeper in \( k \) with private offers: under public offers, the present value of a dollar at entry drops as \( k \) rises, but it stays constant under private offers.

Force 1 implies that trading right now is more attractive under public offers, which pushes up the rate of change in \( K_t \): in order to equalize rates of change in \( K_t \) across the two regimes, prices would have to be dropping much faster in \( k \) under public offers. However, Force 2 implies that, net of changes in \( \rho \), prices are dropping slower in \( k \) under public offers, which pushes the rate of change under public offers even higher. Altogether, \( \rho \) would have to increase extremely quickly to counteract 1 and 2 and generate a slower rate of change under public offers.

**Comparing Buyer Payoffs** Theorem 5 and Theorem 6 together imply

**Corollary 2.** On histories without entry, all buyer types are worse off under private offers.

Indeed, in the private offers game, they trade later at higher prices. However, this kind of argument will not help with payoff comparisons more generally. When trade slows down (\( K_t \) rises), entrants become more reluctant to enter, and the time of entry rises. Therefore, as one moves from public to private offers, the time of trade on histories with no entry and the time of entry move in the same direction. So it is not clear whether (discounted) probability mass moves from histories with entry to histories without, or the reverse. Moreover, even in a simple static setting (e.g., facing a take-it-or-leave-it offer versus an auction), being more exposed to competition in the form of an auction can be good or bad for a buyer, depending on his type and on the distribution of his competitor’s type. Even if one knew in which direction discounted probability mass was shifting, it would be hard to determine whether the buyer was becoming better or worse off with that shift. However, high enough types face entry so rarely that their payoffs are driven mostly by what happens before entry, and for them, the ranking of prices directly implies a ranking of payoffs:

**Proposition 3.** There exists \( k^* < 1 \) such that all initial buyer types in \((k^*, 1]\) strictly prefer any RME of the public offers game to any pure RME of the private offers game.

Given Theorem 5, Proposition 3 follows from an appropriate envelope theorem. I relegate that proof to the appendix. Note that, since the seller does not value the good, high value buyers represent high surplus deals. Proposition 3 therefore suggests a margin on which making negotiations private harms efficiency at the top.

**Remark 7** (Conflicts of Interest, Interim vs Ex Ante). Moreover, Proposition 3 taken together with Theorem 1 implies that, at least at the interim stage, there exists a conflict
of interest between the seller and the buyer as to whether to bargain in public or in private. Nevertheless, for the case when entry causes an ascending auction with no reserve price, there is a sense in which the seller and the buyer may want to agree on public bargaining ex ante. With this model of entry, their bilateral surplus upon entry as a function of the buyer’s type is \( \pi_S(v) + \pi_B(v) = v \), which exactly matches their bilateral surplus when they agree before entry. Therefore, since Theorem 6 implies that the bilateral surplus is delayed more with private offers, and that surplus is the same whether entry happens or not when the game ends, the buyer and the seller’s ex ante discounted bilateral surplus shrinks when they move to private bargaining. The buyer could potentially, at the ex ante stage, offer the seller a fixed fee in order to bargain in public, e.g., to waive an NDA.

8 Extensions

8.1 Common Values

Common values between bidders are a prominent feature of the mergers and acquisitions markets that I used to motivate the model. The reader should then wonder to what extent the results are special and rely on the private values assumption. Indeed, a common element in valuations introduces a new consideration for the entrants. When values were fully private, as the seller skims through buyer types, an entrant became more optimistic about the prices he would have to pay in a bidding war, while the value he expected to get upon winning remained constant. Altogether, delays in agreement between \( B \) and \( S \) were strictly good news for entrants, such that in equilibrium they triggered bidding wars faster over time as the disagreement dragged on. If instead values are common, more skimming still reduces the prices an entrant expects to pay if he wins, but it reduces the value he expects to obtain. Altogether, delays in agreement may become bad news for entrants, so that they trigger bidding wars more slowly as the disagreement drags on. I show in this subsection that what matters for the shape of equilibria and the effects of privacy is not really the commonality of values, but whether skimming encourages or discourages entry.

Disagreement is Good News for Entrant, Regardless of Commonality With the right normalization of a sum-of-signals model, my setup can span the range from fully common to fully private values without changing the analysis or the direction of the public-private comparison. Suppose that instead of observing their values, initial buyers and entrants now observe estimates (signals) that are indicative of their common value. \( B \) observes a signal \( \varepsilon_B \sim F \), entrants who pay \( c \sim G \) observe \( \varepsilon_E \sim F \), and as before, \( \varepsilon_B \), all \( \varepsilon_E \)’s and \( c \)’s, \( F \) and the Poisson process for arrival are mutually independent. Unlike
the setup in Section 3, $i \in \{B, E\}$ has a value for the asset

$$v_i = \alpha \bar{\varepsilon} + (1 - \alpha)\varepsilon_i,$$

where $\bar{\varepsilon}$ is the average signal, with the average taken over bidders currently in the market. Therefore, from $B$'s perspective, $\bar{\varepsilon} = \varepsilon_B$ before entry by $E$, and $\bar{\varepsilon} = (\varepsilon_B + \varepsilon_E)/2$ after entry by $E$. Meanwhile, when some entrant is considering entering, he knows he will face an average signal $\bar{\varepsilon} = \varepsilon_B$ before entry by $E$, and $\bar{\varepsilon} = (\varepsilon_B + \varepsilon_E)/2$ after entry.

Modeling the post-entry bidding war as a full-information ascending auction, it is standard to check (Milgrom and Weber, 1982) that $i$ drops out at (bids) $\varepsilon_i$, so the player with the highest signal wins and pays the losing player’s signal. Hence, buyer type $\varepsilon_B$ expects a surplus in the bidding war equal to

$$\mathbb{E}_{\varepsilon_B \sim F}[(\alpha \bar{\varepsilon} + (1 - \alpha)\varepsilon_B - \varepsilon_E) 1_{\varepsilon_B \geq \varepsilon_E}] = \left(1 - \frac{\alpha}{2}\right) \mathbb{E}_{\varepsilon_B \sim F}[(\varepsilon_B - \varepsilon_E) 1_{\varepsilon_B \geq \varepsilon_E}]$$

$$= \left(1 - \frac{\alpha}{2}\right) \int_0^1 (\varepsilon_B - \varepsilon_E) dF(\varepsilon_E).$$

Therefore, $\varepsilon_B$’s surplus for a given $\alpha$ is $\pi_B^\alpha(\varepsilon_B) = \left(1 - \frac{\alpha}{2}\right) \pi_B(\varepsilon_B)$, with $\pi_B$ defined exactly as in Section 3. Since $B$’s value when agreeing with $S$ before entry is $\varepsilon_B$, $\varepsilon_B - \pi_B^\alpha(\varepsilon_B)$ is strictly increasing in $\varepsilon_B$ for any $\alpha$, and the skimming property continues to hold in this common-values model. Beliefs about $\varepsilon_B$ for $S$ and $E$ are therefore truncations of $F$, just as before. Let $k$ now denote the truncation points for $\varepsilon_B$. By arguments identical to the ones in the derivation for $\pi_B^\alpha$, entrants now pay to enter iff $c \leq \Pi_E^\alpha(k)$, where $\Pi_E^\alpha(k) = \left(1 - \frac{\alpha}{2}\right) \Pi_E(k)$, while the seller expect profits of $\Pi_S(k) = \mathbb{E}[\pi_S(\varepsilon_B)|\varepsilon \leq k]$, with $\Pi_E$ and $\pi_S$ defined as in Section 3.

Altogether, modeling common values in this way alters the original model only by rescaling the reduced forms $\pi_S, \pi_B$ and $\Pi_E$. In particular, $\Lambda(k) = G(\Pi_E^\alpha(k))$ will still be strictly decreasing, so all the properties in Remark 3 continue to hold, and the qualitative analysis of equilibrium is unchanged. Note especially that, with a sum-of-signals model, the basic shape of equilibria, and by extension all the conclusions about the effects of privacy, will be the same regardless of how private or common values are.

**Disagreement is Bad News for Entrant** Consider instead a different parametrization of common values. Suppose the buyer has a value $v_B$, which he observes perfectly, while any entrant has a value $v_E = \varphi v_B$ for $\varphi > 1$. Entrants again have a private cost

\[28\] Note that while bidders here have asymmetric beliefs, the equilibrium of the symmetric full-information ascending auction model in Milgrom and Weber (1982) is an ex post Perfect Bayesian Nash Equilibrium, so it will remain an equilibrium for any beliefs of the bidders.
c ∼ G to learn \( v_E \) and enter a bidding war. Entrants will always win the auction if they enter at a price \( v_B \), so for all \( v \), (i) \( \pi_B(v) = 0 \) and \( \pi_S(v) = v \). \( v - \pi_B(v) \) is therefore trivially strictly increasing, so that the buyer’s best response functions always feature skimming from the top and posterior beliefs are always right-truncations. Entrants who believe \( v_B \leq k \) enter at a rate \( \Lambda(k) = G((\varphi - 1)E[v_B|v_B \leq k]) \), which is now strictly increasing in \( k \). Nonetheless, one can define RME’s for the public and private offers game as before.

The analysis goes through almost exactly as before with the comparison between public and private offers exactly reversed because of the sign-reversal on \( \Lambda(\cdot) \).

- With public offers, the seller ex post “rejoices” after an offer is accepted: she does strictly better than if she had waited for entry.
- If \( D'(k)\Pi_S(k)F_B(k)/f_B(k) \) is increasing, trade is slower with public offers.
- With private offers, equilibria are always smooth.
- Prices on histories with no entry are higher with public offers.
- High value buyers are rse off with public offers and would prefer to bargain in private.
- The seller always prefers to bargain in public. Indeed, adapting the arguments in Lemma \[\text{[a]}\] yields that \( D^X(t^*(k)|\lambda^*) < D(k) \) when \( \Pi_E(k) \) is strictly increasing. Therefore, \( D(k = 1)\Pi_S(k = 1) > D^X(t = 0|\lambda^*)\Pi_S(k = 1) \), and the ex ante value of waiting is higher with public offers. Therefore, when the public offers RME starts with smooth trade, the equilibrium payoffs in both regimes equal the respective ex ante values of waiting, and the seller must prefer public bargaining. If public bargaining starts with a burst of agreement, then as per Remark \[\text{[b]}\] the seller obtains strictly more than her ex ante value of waiting with public offers, while she obtains exactly the private-offers ex ante value of waiting when bargaining is private.

Note that, since \( D \) is now increasing in \( k \), \( \hat{P} \) is more likely to be everywhere increasing, and bursts of trade with public offers become less less likely.

### 8.2 Firm-Union-Regulator Example

I motivated the endogenous entry model by thinking of a buyer and a seller, with possible entry of another buyer, but the analysis above gives insights on the effects of privacy in a range of other bargaining problems with endogenous entry. For example, consider a labor union bargaining with a firm over a new wage contract. Aside from showing the
flexibility of my model, this example will emphasize the main point of the common values discussion: the crucial feature driving the effects of privacy is whether delay is good or bad news for entrants.

The firm has private information about $v_B \sim F_B$, its profitability gross of wages (so it will replace the “buyer” in the original model), and the union has no private information and wants to negotiate the highest possible wage (so it replaces the “seller” in the original model). They discount the future at the same rate, and the union makes all the offers.

While they bargain, they are subject to the risk that a negative shock will wipe out the firm’s value entirely, say, because their disagreement spooks downstream corporate clients that fear interruptions in their supply chains, or because the consumer market makes adverse inferences about the product’s quality. For example, Krueger and Mas (2004) shows that Firestone’s labor dispute caused large losses when quality declined in striking factories. The shock arrives with a constant Poisson rate $\lambda_0$.

Their bargaining happens in the shadow of regulator who cares about preserving total value but does not know $v_B$. The regulator can intervene at a cost $c \sim G$ and force a fixed surplus split between the firm and the union. The regulator is inattentive, so it only decides whether to intervene and save the firm when the shock hits. It only gets credit for the value it “saves” when a crisis hits (say, because only during a crisis does it get enough publicity to garner political capital). Let $\eta$ be the share of surplus that the union receives after an intervention. Then the firm’s payoff when bargaining is interrupted and the regulator intervenes is $\pi_B(v_B) = (1 - \eta)v_B$, while the union’s payoff is $\pi_S(v_B) = \eta v_B$.

If a crisis hits and the regulator does not intervene, all players receive a payoff of 0.

The analysis above implies that the union and the regulator will have truncated beliefs about the firm’s gross profitability after any history. Focus on Regular Markov Equilibria with the truncation $k$ as the state. The regulator therefore intervenes at state $K_t = k$ if $c < \mathbb{E}[v_B | v_B \leq k]$. Let $\Pi_S(k) = \mathbb{E}[\pi_S(v_B) | v_B \leq k]$ as before, and let $I_E(k) = G(\mathbb{E}[v_B | v_B \leq k])$ denote the probability of intervention after a shock.

Whenever there is smooth trading with public offers, the union’s HJB is

$$r J_S(k) = \frac{d}{d k} \left[ f_B(k) \left( P(k) - J_S(k) \right) \right] + \lambda_0 \left[ I_E(k) \Pi_S(k) - J_S(k) \right] + J'_S(k)(-K)$$

so the above reasoning pins down the smooth trade locus. Let

$$D(k) = \frac{\lambda_0 I_E(k)}{\lambda_0 + r}.$$

Then on that smooth locus,

$$J_S(k) = \frac{\lambda_0 I_E(k)}{\lambda_0 + r} \Pi_S(k) \text{ and } \dot{P}(k) = D(k) \pi_S(k) + D'(k) \Pi_S(k) \frac{F_B(k)}{f_B(k)}.$$

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Therefore, on a smooth trade region, \( D(k) \) is the “expected discount to intervention” when the union makes unreasonable offers after state \( k \), and the union gets exactly the ex ante value of waiting for an intervention. Note that \( D(\cdot) \) is now increasing (since \( I_E(\cdot) \) is), so \( \hat{P}(k) > D(k)\pi_S(k) \).

Meanwhile, with private offers, a regulator who pays attention after a shock at time \( t \) has beliefs \( I_t \), which in a pure RME must equal \( I_t^* \equiv I_E(K_t) \). Similar steps as in Section 6 yield an equilibrium “discount to intervention”

\[
D^X(t|I^*) = \frac{\lambda_0}{\lambda_0 + r} \int_t^\infty I_s^* ds.
\]

In a private offers RME, at state \( K_t = k \) the intervention probability starts at \( I_t^* = I_E(k) \) and falls, since the regulator becomes increasingly pessimistic about the value of the firm and finds it less worthwhile to intervene. Therefore, an analogous result to Lemma 6 applies:

\[
D^X(t^*(k)|I^*) < D(k).
\]

That suffices to conclude (assuming RME’s exist for both public and private offers):

- With public offers, the union ex post “rejoices” after an offer is accepted: it does strictly better than if it had waited to settle with that firm type during a regulatory crisis intervention.
- If \( D'(k)\Pi_S(k)F_B(k)/f_B(k) \) is increasing, trade is slower with public offers.
- With private offers, equilibria are always smooth.
- Equilibrium wages (conditional on survival and no intervention) are higher with public offers.
- Highly profitable firms are worse off with public offers and would prefer to bargain in private.
- By analogous arguments to the ones in the \( v_E = \varphi v_B \) common values example above, the union always prefers to bargain in public.

As in the second common values example, since \( D \) is increasing, \( \hat{P} \) is more likely to be everywhere increasing, and bursts of trade with public offers game are less likely.

9 Conclusion

I have shown that privacy is a first order concern in bargaining. There are differences in kind between bargaining in public and bargaining in private. When offers are public, equilibria alternate between phases when the seller and initial buyer reach agreement in
bursts (atoms of of trade in an instant), and phases where they reach agreement only in dribs and drabs (when the seller screens the buyers “smoothly,” type by type). If in addition entrants become encouraged by the skimming of buyer types, then when offers are public the seller suffers a kind of ex post regret: she makes offers so low that, conditional on the information revealed by their being accepted, she wishes she had waited for entry instead. When offers are private, the seller’s offers never have this “regret” property—in fact, they have the opposite feature: a no ex post regret property—nor are there any bursts of agreement.

There are also marked differences in degree between bargaining in public and in private. When entrants become encouraged by the skimming of buyer types, hiding the offers from entrants always raises the prices paid by every buyer type who transacts in the normal course of bargaining. Moreover, it delays agreement for all types of the buyer. Sellers often prefer private negotiations while high value buyers dislike them, even though going public provides the seller with the ability to lure in competitors that can outbid the original buyer. Moreover, I show that the seller never wants to bargain only in public. Indeed, whenever public bargaining starts with a burst in a way that improves on private bargaining, the seller can do even better by combining parts of the public and private offers RME’s. That is, the seller prefers to start bargaining with a public offer that creates a jump in $K_t$, and then switch to bargaining in private for the smooth trading phase, which is always more profitable than smooth trading in public offers.

My analysis suggests two main lessons about the effect of privacy or publicity in bargaining with endogenous entry—two lessons that plausibly apply regardless of whether the parties involved are buyers and sellers, unions and firms, or political parties, whether they fear interruption by other buyers triggering bidding wars, by regulators forcing them to agree, or by journalists reporting on them. First, negotiating in public matters because it gives the player making the offers a new trade-off. When offers are public, the entrants learn about market conditions not only from the amount of delay, but also from the particular offers that caused that delay. This allows the party making an offer to affect the profits she obtains from inframarginal types, by speeding up or slowing down the eventual interruption of negotiations. Second, the direction in which privacy affects bargaining depends on whether “skimming” of the informed type encourages or discourages entry. Assume that, upon entry, higher types of the informed player are better for the uninformed player. Then, if skimming encourages entry, publicizing offers will tend to hurt the uninformed player, lower prices, and speed up trade. If it discourages entry, publicizing offers will tend to help the uninformed player, raise prices, and slow down trade.

Moving forward, the approach here suggests new avenues for research in information design. In particular, my paper complements recent work in mechanism design that studies the optimal revelation of information acquired in a mechanism to interested third
parties. In Dworczak (2017) the designer can control both the trading protocol and the information that gets released to the market after the protocol is run. This is the right approach for many financial markets, such as the US market for corporate bonds, in which regulators can dictate legal requirements for centralized trading platforms that want to market-make.

In many markets, however, the designer cannot control the trading protocol at all, but it can mandate certain kinds of disclosure. In mergers and acquisitions, for example, regulators may lack the legal tools to prevent parties from renegotiating *ad infinitum*, so long as the final decision respects shareholders’ best interests. (And for privately owned companies, the regulator may not even be able to enforce that much.) Given those constraints, how should the regulator think about the disclosure of materially relevant buyout offers? Currently, the SEC requires companies to disclose expressions of interest by buyers and the terms of firm offers they receive, whereas (aside from committing fraud), there are far fewer limits on the behavior of privately held companies. Part of the rationale for these disclosure rules is that they act as a check against moral hazard by the management. My research ignores the moral hazard dimension in mergers and acquisitions, but it points to an unexpected trade-off: a target’s shareholders may be harmed by forcing its management to bargain with an acquirer in full public view.

In principle, the set of disclosure policies is much larger than two extremes I have discussed—having to disclose materially all relevant information (as with publicly traded companies) and not having to disclose anything (as with privately held companies). The set of possible policies includes not only hybrids in between the two, but also options more extreme than either. Should companies only have to disclose initial offers, but not subsequent revisions, as in the conjecture above? Should there be a real-time public record of all communications between acquirers and targets? Should companies only have to disclose tentative acquisitions terms that are above a certain premium on their share price? Who would benefit or be harmed by these policies?

Incorporating the idea that the designer cannot control the trading protocol also leads to a different space of policies than those considered by the Bayesian Persuasion literature. If the designer cannot control the trading protocol, then it becomes problematic to argue that she can choose arbitrary garblings of the offer history. A regulator can mandate disclosure in a way that forces bargaining to be public, but how could it enforce private bargaining on parties that would like to bargain in public? This places a one-sided constraint on the level of publicity that a designer can implement. My results are a first step for understanding both the relevant trade-offs and the relevant constraints on information design in this setting.
10 Appendix A: Omitted Proofs

10.1 Proofs for Section 5

Proof of Lemma 3. Assume otherwise. Since all remaining types are rejecting when \( \dot{K} = 0 \), as in Daley and Green (2018), it is without loss to make the marginal type indifferent between accepting and rejecting \( P \), i.e., \( J_S(k) = k - P(k) \). Then (11) holds with \( \dot{K} = 0 \), and \( J_B^k(k) = D(k)\pi_B(k) \). Hence, \( P(k) = k - D(k)\pi_B(k) \). Since \( J_S(k) = D(k)\Pi_S(k) \), \( \dot{K} \) is smooth in a trade region, then using \( \frac{\partial}{\partial k}(\Pi_S(k)F_B(k)) = \pi(k)f_B(k) \), the coefficients on \( \dot{K} \) in the seller’s HJB must equal

\[
(k - D(k)\pi_B(k) - J_S(k))\frac{f_B(k)}{F_B(k)} - J_S'(k) = (k - D(k)(\pi_B(k) + \pi_S(k)))\frac{f_B(k)}{F_B(k)} - D'(k)\Pi_S(k),
\]

By Remark 1.2, the first term on the right hand side is strictly positive. The second is strictly positive as well, since \( \Lambda'(k) < 0 \), so \( \dot{K} = 0 \) cannot be optimal for the seller. □

Lemma 7.

1. If \( \dot{J}_S \) is non-monotone, \( \dot{P} \) is non-monotone. It is strictly increasing in a neighborhood of \( k = 0 \), and strictly decreasing in a neighborhood of every local maximizer of \( \dot{J}_S \).

2. Let \( \mathbb{E}[v] \) and \( \mathbb{E}[v^{2:2}] \) be the mean and expected second order statistic of two independent samples from \( F_E \). If \( r \) is sufficiently large, \( F_E = F_B \), \( 2\mathbb{E}[v^{2:2}] > \mathbb{E}[v] \), and \( G = U[0,1] \), then \( \dot{J}_S \) is strictly decreasing in a neighborhood of \( k = 1 \).

Proof. Since \( \dot{J}_S(k) = D(k)\Pi_S(k) \) is continuously differentiable and non-monotone, it has at least one strict interior local maximum. Let \( k^* \) denote one such local maximum. Then

\[
\dot{P}'(k^*) = \dot{J}_S'(k^*) + \dot{J}_S''(k^*)\frac{d}{dk} \left( \frac{F_B(k)}{f_B(k)} \right) \bigg|_{k=k^*} + \dot{J}_S''(k^*)\frac{F_B(k^*)}{f_B(k^*)}. \tag{21}
\]

Since \( k^* \) is a strict local maximum of \( \dot{J}_S \), \( \dot{J}_S'(k^*) = 0 \) and \( \dot{J}_S''(k^*) < 0 \). Therefore, plugging into (21), \( \dot{P}'(k^*) < 0 \), and \( \dot{P} \) must have decreasing regions.

\( \dot{P} \) must also have increasing regions. First note that \( \Pi_S(0) = 0 \) and \( \Pi'_S(0) > 0 \), so \( \dot{J}_S(0) = D'(0)\Pi_S(0) + D(0)\Pi'_S(0) > 0 \) (where \( \Pi'_S(0) \), etc. refers to the right derivative at the corner). Moreover,

\[
\frac{d}{dk} \left( \frac{F_B}{f_B} \right) \bigg|_{k=0} = \frac{f_B^2 - F_B'f_B}{f_B^2} \bigg|_{k=0} = 1
\]

so again plugging into (21), \( \dot{P}'(0) = 2\dot{J}_S(0) > 0 \). Since \( \dot{P} \) is continuously differentiable, it must therefore have at least one increasing region near \( k = 0 \).
To prove part 2 of the lemma, using $G(x) = x$, first re-write $D(k)$ as \( \frac{\Pi_E(k)}{\Pi_E(k)+1/\gamma} \), where \( \gamma = \frac{d}{r} \). Then by straight-forward calculus, and omitting the dependence on \( k \) to simplify notation,

\[
\hat{J}_S = \frac{\Pi'_{E} \Pi_{S}}{\gamma (\Pi_{E} + 1/\gamma)^2} + \frac{\Pi_{E}}{\Pi_{E} + 1/\gamma} \left[ -\frac{f_{B}}{F_{B}} \Pi_{S} + \frac{1}{F_{B}} \pi f_{B} \right]
\]  

(22)

Using \( \Pi_{E}(k) = \int_{0}^{1} \int_{0}^{k}(v-p)_{+}d(F_{B}(p)/F_{B}(k))dF_{B}(v) \), it is easy to verify that \( \Pi'_{E}(1) = -f_{B}(1)\Pi_{E}(1) \). Plugging that back into (22) evaluated at \( k = 1 \), and using the fact that \( \Pi_{E}(1)f_{B}(1)/(\Pi_{E}(1) + 1/\gamma) > 0 \), conclude that \( \hat{J}'_{S}(1) < 0 \) if and only if

\[
- \frac{\Pi_{S}(1)}{\gamma \Pi_{E}(1) + 1} - (\pi_{S}(1) - \Pi_{S}(1)) < 0
\]

Note that by the definition of \( \pi_{S} \), \( \pi_{S}(1) = \mathbb{E}[v] \) (since \( B \) always wins the auction and pays \( v_{E} \sim F_{B} \) when \( v_{B} = 1 \)) and \( \Pi_{S}(1) = \mathbb{E}[v^{(2:2)}] \) (since at \( k = 1 \), revenue from the auction is that of the symmetric IPV case with distribution \( F_{B} \)). Therefore, \( \hat{J}'_{S}(1) < 0 \) if and only if

\[
\mathbb{E}[v] - \mathbb{E}[v^{(2:2)}] \left( 1 + \frac{1}{\gamma} \right) < 0.
\]

If, as was assumed, \( \mathbb{E}[v] < 2\mathbb{E}[v^{(2:2)}] \), there exists \( r^{*} \) large enough that for any \( r > r^{*} \), \( \hat{J}'_{S}(1) < 0 \), which proves the result. \( \square \)

**Proof of Theorem 3.** First, I derive (14)—the remainder of the proof relies on local optimality arguments from that expression. Let \( A(k,k') = 1 - \frac{F_{B}(k')}{F_{B}(k)} \) be the probability of immediate acceptance by \( B \) in a jump from \( k \) to \( k' \leq k \). Then differentiating jump payoffs in (13) with respect to \( k' \),

\[
U_{2}(k,k') = A_{2}(k,k') \left( \hat{P}(k') - \hat{J}_{S}(k') \right) + (1 - A(k,k')) \hat{J}'_{S}(k') + A(k,k') \hat{P}'(k')
\]

(23)

The result then follows from plugging the expression for \( \hat{P} \) in (8) in the above, and using the identity

\[
A_{2}(k,k') = -\frac{f_{B}(k')}{F_{B}(k')} \frac{F_{B}(k')}{F_{B}(k')} = -\frac{f_{B}(k')}{F_{B}(k')} (1 - A(k,k'))
\]

With (14) in hand, I verify points 1-3 of the theorem.

For 1 and 2, note that, since \( k \) is at a corner, trade can be smooth at \( k \) only if the \( k \) satisfies the first and second order conditions for local optimality, i.e., \( U_{2}(k,k^-) \leq 0 \), and for some \( \varepsilon > 0 \), \( k' \mapsto U_{2}(k,k') \) is decreasing for all \( k' \in [k - \varepsilon, k] \). The former is true for any \( k \): using (14) and the continuity of \( F \), \( U_{2}(k,k^-) = 0 \). The latter also follows from (14). Since \( \hat{P} \) is strictly increasing at \( k \) by assumption, there exists some \( \varepsilon > 0 \) such that \( U_{2}(k,k') = \left( 1 - \frac{F_{B}(k')}{F_{B}(k)} \right) \hat{P}'(k') > 0 \) for all \( k' \in (k - \varepsilon, k) \). And since \( U_{2}(k,k^-) = 0 \), one can find a further \( \varepsilon' \in (0, \varepsilon) \) such that \( k' \mapsto U_{2}(k,k') \) is strictly decreasing in \( k' \) for all \( k' \in (k - \varepsilon', k] \). In particular, when \( \hat{P} \) is strictly increasing on \( [0,k] \), the above argument
implies that \( k \) is a \emph{global} optimum of \( k' \mapsto U(k, k') \) on the subdomain \( k' \in [0, k] \). Hence, trade \emph{must} be smooth at any such \( k \) and \( \hat{P} \).

A symmetric argument to the one in the previous paragraph shows that \( k \) is a local minimum of \( k' \mapsto U(k, k') \) when \( \hat{P} \) is decreasing at \( k^- \), which means that there must be a jump at any such \( k \).

Next I verify point 3. Since jumps must end on the smooth trade locus, and \( U(k, k') \) gives the payoff from jumping onto the smooth trade locus at \( k' \) from \( k \), then \( k' \in \arg\max_{k \in [0, k]} U(k, k') \) is a necessary condition for an equilibrium jump from \( k \) to \( k' \). As remarked in the main text, expression (14) implies that \( \hat{P}(k') \) and \( U(k, k') \) have the same local maxima in \( k' \in [0, k) \). This suggests local maxima of \( \hat{P} \) on \( [0, k) \) as candidate endpoints for the jump, and they will be valid candidates so long as smooth trade can be supported at those endpoints. Since \( \hat{P} \) is left-differentiable everywhere and \( k' \) is a local maximum of \( \hat{P} \), it follows that, for some \( \epsilon > 0 \), \( \hat{P}'(k') > 0 \) for all \( (k' - \epsilon, k') \). Therefore, smooth trade can be supported at \( k' \), by the argument in previous paragraph.

\( \square \)

\textbf{Proof of Proposition 1.} The proof follows the construction in Daley and Green (2018), which uses a verification approach (Harrison, 2013, chapter 7).

\textbf{Seller Optimality} To verify seller optimality, consider any \( k < \hat{k} \), such that on the domain \([0, k]\), \( \hat{P} \) is strictly increasing and equal to \( P \) in point 2 of the theorem statement. Then, the verification argument in (Harrison, 2013) shows that smooth trading until the end of the game is globally optimal for the seller among all impulse controls. First, quiet periods are ruled out by Lemma 2. Second, there are no local maxima of \( \hat{P} \) to the left of \( k \) (so by Theorem 3 it is optimal not to jump when facing prices \( P = \hat{P} \)). Third, payoff from smooth trading, \( D(k)\Pi_S(k) \), is \( C^2 \) and satisfies the HJB in (7). Therefore \( J_S(k) = D(k)\Pi_S(k) \) is indeed the value function for the seller’s problem on \([0, k]\), and the candidate strategy is seller-optimal for \( k < \hat{k} \).

To complete the verification proof for seller optimality in (Harrison, 2013, chapter 7), it remains to show that (i) \( J_S(k) = \max_{k' \leq k} A(k, k')P(k') + (1 - A(k, k'))J_S(k') \) for all \( k \geq \hat{k} \), and (ii) \( J_S(k) \) defined by the candidate policy is indeed \( C^1 \). Point (i) follows directly from point 3 in Theorem 3. To check point (ii), note that \( J_S(k) = A(k, \hat{k})P(\hat{k}) + \ldots \)

\( \text{30} \) Proposition 7.1 in (Harrison, 2013, chapter 7), which is required for the verification argument, can be adapted to the present setting by noting that in the present setting as follows. Given the assumption that (15) has a solution, the implied state process

\[
K_t = K_0 - \int_0^t \dot{K}(s) ds - (K_{0^-} - \hat{k}),
\]

where \( \dot{K}(k) \) is given by the right hand side of (12), is bounded and has bounded variation, and its drift \( \dot{K}(k) \) is integrable. These integrability and variation conditions on the underlying process allow the proof of Proposition 7.1 of (Harrison, 2013) to go through essentially unmodified.
(1−A(k, ⌣k))J_S(⣣k) for any k > ⣣k, while \( J_S(k) = D(k)\Pi_S(k) \) for k < ⣣k. \( J_S \) is therefore \( C^1 \) on [0, ⣣k] and on (�k, 1], so it remains to check that \( J_S \) is continuous at ⣣k and \( J'_S(�k) = J'_S(�k^+) \). For continuity, note that since there is zero buyer mass between ⣣k + and ⣣k −, \( J_S(�k^+) \) cannot be strictly greater than \( J_S(�k^-) \). Moreover, \( J_S(�k^+) \geq D(�k)\Pi_S(�k) \) (since the seller could always credibly stall), and \( J_S(�k^-) = D(�k)\Pi_S(�k) \) (by the seller’s indifference over smooth trading speeds), so it must be that \( J_S(�k^+) = J_S(�k^-) = D(�k)\Pi_S(�k) \).

To check that \( J_S \) is \( C^1 \) at ⣣k, on the one hand, calculate that

\[
J'_S(�k^+) = \frac{F_B(�k)}{F_B(�k^+)^2}f_B(�k^+) \left[ P(�k) - D(�k)\Pi_S(�k) \right].
\]

By the continuity of \( f_B \), the above simplifies to

\[
J'_S(�k^+) = \frac{f_B(�k)}{F_B(�k)} \left[ P(�k) - D(�k)\Pi_S(�k) \right].
\]

On the other hand, \( J'_S(�k^-) = \frac{f_B(�k)}{F_B(�k)} \left[ \hat{P}(�k) - J_S(�k) \right] \), so \( J'_S(�k^-) = J'_S(�k^+) \), and \( J_S \) is indeed \( C^1 \).

**Entrant Optimality**  This follows immediately from \( L(k) = \Lambda(k) \).

**Buyer Optimality**  For buyer optimality, note that, by construction with the marginal buyer’s HJB, on any k with smooth trading, \( P(k) = \hat{P}(k) \) is an optimal reserve price given continuation play by the usual verification argument for optimal stopping. \( \hat{K} \) in \( \Pi \) is set such that \( J^k_B(k) = k - P(k) \) (value-matching for \( v_B = k \) at state k), and \( (J^k_B)'(k) = -P'(k) \) (smooth pasting for the marginal type \( v_B = k \) at state k). Hence, for types that trade in smooth regions, the usual verification argument for optimal stopping holds. Optimal stopping for all types \( v^B > k \) and \( v^B < k \) follows by the single-crossing property of buyers’ preferences. By construction of \( \hat{P}(k) \) and the single-crossing preferences of buyers, \( P(k) = \hat{P}(k) \) is an optimal reservation price for \( v^B > ˆk \) at any jump state k, while all \( v^B < ˆk \) find it optimal to wait.

\[\Box\]

**10.2 Proofs for Section 6.1**

**Proof of Proposition 2.** Suppose, by way of contradiction, that an RME under exogenous entry features jumps. Since RME’s have only finitely many jumps, there must be a final one. Say this last jump happens at time \( ℓ \), and goes from ⣣k to \( k \in (0, k) \). (The endpoint \( k \) must be strictly greater than 0, since \( v_B = 0 \) would only accept a price of 0 and the seller can do better than that by trading smoothly.) Since the equilibrium is regular and
there are no jumps after arriving at \((k, \tilde{t})\), there has to be smooth trade until the end of the game.

By identical arguments to the ones in section 5.1, the seller’s payoff must satisfy the HJB below in this final smooth-trading region:

\[
\min_{\dot{K} \in (0, \infty)} \left\{ \left( P(k, t|\lambda) - J_S^X(k, t|\lambda) \right) \frac{f_B(k)}{F_B(k)} \left( \dot{K} \right) + \lambda_t[\Pi_S(k) - J_S^X(k, t|\lambda)] + \frac{\partial}{\partial k} J_S^X(k, t|\lambda)(-\dot{K}) + \frac{\partial}{\partial t} J_S^X(k, t|\lambda) \right\}
\]

\[\text{exogenous arrival of entrant, game ends,}\]
\[\text{game continues with new cutoff/time state}\]
\[\forall k \leq k, t \geq \tilde{t}. \quad (24)\]

Continuing as in section 5.1, all terms with \(\dot{K}\) must vanish from (24), so \(J_S^X(k, t|\lambda)\) must satisfy the differential equation

\[
rJ_S^X(k, t|\lambda) = \lambda_t[\Pi_S(k) - J_S^X(k, t|\lambda)] + \frac{\partial}{\partial t} J_S^X(k, t|\lambda) \quad (25)
\]

(with boundary conditions to be determined).

In particular, \(S\) is equally well off as if she chose \(\dot{K} = 0\) for all \(k \leq k, t \geq \tilde{t}\). So instead of solving (25) directly, one can use the fact that, with \(\dot{K} = 0\), \(S\) is not trading, and her payoff must equal the outside option of waiting for an entrant:

\[
J_S^X(k, t|\lambda) = \hat{J}_S^X(k, t|\lambda) = \left[ \int_{t}^{\infty} \lambda_s e^{-\int_{t}^{s}(\lambda_u+r)du} ds \right] \Pi_S(k), k \leq k, t \geq \tilde{t} \quad (26)
\]

This expression suffices to determine the smooth-trade locus and derive the contradiction. First, I claim that the final jump cannot end in a “quiet period.” Let \(P^X(k, \tilde{t}|\lambda)\) be the equilibrium “demand curve.” Then, the seller’s payoffs from jumping to \(k < \bar{k}\) from \(\bar{k}\) at time \(\tilde{t}\) equal

\[
U^X(\bar{k}, k, \tilde{t}|\lambda) = A(\bar{k}, k) P^X(k, \tilde{t}|\lambda) + (1 - A(\bar{k}, k)) \hat{J}_S^X(k, \tilde{t}|\lambda),
\]

where \(A(\bar{k}, k) = (1 - F_B(k)/F_B(\bar{k}))\) is the mass of buyer types in the atom of trade. Assume first that \(P^X\) is differentiable in its first argument at \(k\). A necessary condition for the seller to find it optimal to jump from \(\bar{k}\) to \(k\) at time \(\tilde{t}\), is

\[
\liminf_{h \downarrow 0} \left[ \frac{U(\bar{k}, k + h, \tilde{t}) - U(\bar{k}, k, \tilde{t})}{h} \right] \leq 0.
\]

Otherwise, there would exist a strictly smaller jump that makes the seller strictly better off than the jump to \(k\). By the properties of \(\lim\inf\)’s of sums, the above “first order
condition” reduces to

$$- (1 - A(\bar{k}, k)) \left( (P_X(k, \bar{t}|\lambda) - \hat{J}_S^X(k, \bar{t}|\lambda)) \frac{f_B(k)}{F_B(k)} - \frac{\partial}{\partial k} \hat{J}_S^X(k, \bar{t}|\lambda) \right) + A(\bar{k}, k) \lim inf_{h \downarrow 0} \left[ \frac{P_X(k + h, \bar{t}|\lambda) - P_X(k, \bar{t}|\lambda)}{h} \right] \leq 0. \quad (27)$$

where I use $A_2(\bar{k}, k) = -(1 - A(\bar{k}, k)) \frac{f_B(k)}{F_B(k)}$.

I show by contradiction that this first order condition cannot hold. Let

$$\delta \equiv \lim inf_{h \downarrow 0} \left[ \frac{P_X(k + h, \bar{t}|\lambda) - P_X(k, \bar{t}|\lambda)}{h} \right].$$

First, suppose $\delta < 0$. Given the buyer’s single-crossing preferences, this is impossible, as it mean that lower types have higher reservation prices and accept sooner. Therefore $\delta \geq 0$. If $\delta = +\infty$, (27) cannot hold, so focus on the case where it is finite. If $\delta > 0$, then by the first-order condition in (27),

$$\left( P_X(\tilde{k}, \bar{t}|\lambda) - \hat{J}_S^X(\tilde{k}, \bar{t}|\lambda) \right) \frac{f_B(\tilde{k})}{F_B(\tilde{k})} - \frac{\partial}{\partial \tilde{k}} \hat{J}_S^X(\tilde{k}, \bar{t}|\lambda) > 0.$$ 

But this would mean the coefficients on $\dot{K}$ in the seller’s HJB would be strictly positive at $(k, \bar{t})$, and the seller would deviate from the zero speed $\dot{K} = 0$ at $(k, \bar{t})$.

Next, consider the case where $\delta$ is exactly 0. If the first order condition (27) holds with equality, then the left hand side of (28) would also equal 0. Solving for $P_X$ one obtains

$$P_X(k, \bar{t}|\lambda) = D_X(\bar{t}|\lambda)\pi_S(k)$$

using the same manipulations as in (9), where $D_X(\bar{t}|\lambda)$ is the coefficient on $\Pi_S(k)$ in (26). This expression is differentiable and strictly increasing in $k$, which again yields contradicts $\delta = 0$. If the instead first order condition is a strict inequality, then once again (28) would hold, the coefficients on $\dot{K}$ in the seller’s objective would add up to a positive quantity, and the seller would want to deviate from the zero trading speed.

Therefore, it is safe to assume that the final jump is to a state with a positive speed of smooth trade, i.e., onto the “smooth trade locus.” The same manipulations as in (9) imply a smooth trade locus satisfying (superscript “X” for “eXogenous”):

$$\hat{P}_X(k, t|\lambda) = \hat{J}_S^X(k, t|\lambda) + \left[ \frac{\partial}{\partial t} \hat{J}_S^X(k, t|\lambda) \right] \frac{F_B(k)}{f_B(k)} = D_X(t|\lambda)\pi_S(k). \quad (28)$$
and corresponding jump payoffs:

\[ U^X(k, k', t|\lambda) = \left(1 - \frac{F_B(k')}{F_B(k)}\right) \hat{P}^X(k', t|\lambda) + \frac{F_B(k')}{F_B(k)} \hat{J}^X_S(k', t|\lambda) \]

Instead of jumping from from \( \bar{k} \) to \( k \), the seller could have jumped either closer \( (k' > k) \) or farther \( (k' < k) \) before resuming smooth trade. A necessary condition for this final jump to have been optimal is therefore

\[ U_2^X(\bar{k}, k, t|\lambda) = \left(1 - \frac{F_B(k)}{F_B(\bar{k})}\right) \hat{P}_1^X(k, t|\lambda) = 0, \]

where the simplified expression for \( U_2^X \) follows by identical arguments to those in the proof of Theorem 3. However, since \( \pi_S(k) \) is strictly increasing, it is clear from (28) that \( \hat{P}^X(k, t|\lambda) \) must be strictly increasing in \( k \), and \( U_2^X(\bar{k}, k, t|\lambda) > 0 \). A final jump to \( k \) cannot be optimal for any \( k < \bar{k} \), and the result follows.

To find the speed of trade, again I consider the equilibrium payoffs of the marginal buyer. From the HJB for buyer \( v^B = k \) at state \( (k, t) \), by arguments identical to those behind equation (12),

\[ \hat{K} = (\lambda_t + r) \left[ \frac{\lambda_t}{\lambda_t + r} \pi_B(k) - k + \hat{P}^X(k, t|\lambda) - (\lambda_t + r)^{-1} \frac{\partial}{\partial t} \hat{P}(k, t|\lambda) \right] \]

\[ = (\lambda_t + r) \left[ \frac{\lambda_t}{\lambda_t + r} \pi_B(k) - k + D^X(t|\lambda)\pi_S(k) - (\lambda_t + r)^{-1} \frac{\partial}{\partial t} D^X(t|\lambda)\pi_S(k) \right] \]

where the second equality uses the smooth trade prices in (28).

To simplify further, apply Leibniz’s rule to find that

\[ \frac{\partial}{\partial t} D^X(t|\lambda) = -\lambda_t + (\lambda_t + r)D^X(t|\lambda), \]

so

\[ \hat{P}^X(k, t|\lambda) - (\lambda_t + r)^{-1} \frac{\partial}{\partial t} \hat{P}^X(t|\lambda) = \frac{\lambda_t}{\lambda_t + r} \pi_S(k). \]

Therefore, the numerator in square brackets in (29) is

\[ k - \frac{\lambda_t}{\lambda_t + r} (\pi_B(k) + \pi_S(k)) > 0, \]

where the inequality follows from Remark 1 (immediate agreement is efficient).

The proof of Lemma 5 is the only result that explicitly uses \( \pi_S(k) + \pi_B(k) = k \).
Proof of Lemma 5. Let $\mathcal{R}$ denote the set of increasing functions from $\mathbb{R}_+$ into
$$[\Lambda(1), \Lambda(0)] = [\lambda_0 G(\Pi_E(1)), \lambda_0 G(\Pi_E(0))],$$
and $\mathcal{K}$ be the set of decreasing functions from $\mathbb{R}_+$ into $[0, 1]$. Let $Z_E : \mathcal{K} \to \mathcal{R}$ be the entrants’ best response to a conjectured cutoff path:
$$Z_E(K)(t) = \Lambda(K_t). \tag{30}$$
Also, let $Z_S : \mathcal{R} \rightrightarrows \mathcal{K}$ denote the correspondence
$$Z_S(\lambda) = \left\{ K \in \mathcal{K} : K_t \text{ solves } K_0 = 1, \frac{dK_t}{dt} = -\frac{rK_t}{D^X(t|\lambda)\pi'_{S}(K_t)} \right\}. \tag{31}$$
$Z_S$ maps conjectures of entry rates by $B$ and $S$ into smooth-trade cutoff paths that make the marginal buyer type indifferent between accepting and rejecting at every point.\footnote{Note that, since $\bar{c} < E[(v_E - v_B)_+]$, $G(\Pi_E(1)) > 1$ and $D^X(t|\lambda)$ is uniformly bounded away from zero, for any $t$ and any $\lambda$ above the constant function $\Lambda(1)$.}
However, it is more intuitive to think of $Z_S$ as a pseudo “best response” of the seller to a particular entry rate path (hence the subscript). From Proposition 2 and Theorem 4, the seller must best-respond to any entry rate with smooth trading. During smooth trading, she is indifferent over speeds and the speed is set by the marginal buyer’s indifference condition, but any cutoff path in the last display is consistent with some best response to $\lambda$, depending on the reservation prices of $B$. For future reference, let $K[\lambda]$ denote a typical element of $Z_S(\lambda)$, and let $Z = (Z_S, Z_E)$ denote the self-correspondence on $\mathcal{K} \times \mathcal{R}$.

$Z$ is not a true best-response correspondence, in that it only explicitly includes a local indifference condition for the marginal buyer type (the first line) and a best response condition for the entrants (the second line), but it is mute on (i) non-marginal buyer incentives, and (ii) the seller’s global incentives. Nonetheless, if there exist $(\lambda^*, K^*)$ such that $(\Lambda^*, K^*) \in Z(\lambda^*, K^*)$, then one can construct a pure private offers RME that respects (i) and (ii) by setting prices $P(k, t) = P^X(k, t|\lambda^*)$ and on-path cutoffs according to $K^*_t$. Then, by the single-crossing property of buyer payoffs, $P(\cdot, \cdot)$ will satisfy buyer optimality (and optimal stopping for all types given $(K^*_t, t \geq 0)$ and $\lambda^*$). Moreover, with that inverse demand curve, the seller will be locally indifferent over smooth trading speeds, and taking $\lambda^*$ as given, will have no incentives to jump. Therefore, $K^*_t$ will be a best response for the seller, and fixed points of $Z$ suffice to construct pure RME’s of the private offers game.

With that in mind, I construct a fixed point of $Z$ by successive approximations. Define a sequence of entry rates $\lambda_i, = 0, 1, 2, \ldots$ and cutoff paths $K^i, i = 1, 2, \ldots$ as follows:

1. $\lambda^0 = \lambda_0 \Lambda(1)$ \quad $\forall t$;
2. $K^i \in Z_S(\lambda^{i-1})$;
3. \( \lambda^{i+1} = Z_E(K^i) \);

and so on. I explain below how to select from the (possibly multi-valued) \( Z_S \) in Step 2.

For any \( \lambda, \bar{\lambda} \in \mathcal{R} \), say \( \bar{\lambda} >_E \lambda \) if \( \lambda^0 \) is strictly higher everywhere but \( t = 0 \), and \( \bar{\lambda}_0 = \lambda_0 = \lambda_0 \). Similarly, for any \( \bar{K}, K \in \mathcal{K} \), say \( \bar{K} >_S K \) if \( K^1 \) is strictly higher everywhere but \( t = 0 \), and \( \bar{K}_0 = K_0 = 1 \).

I claim that, for all \( i \geq 2 \),

\[
\begin{align*}
\lambda^{i-1} >_E \lambda^i >_E \lambda^{i-2}, & \quad i \text{ odd} \\
\lambda^{i-2} >_E \lambda^i >_E \lambda^{i-1}, & \quad i \text{ even}
\end{align*}
\]

(32)

while for all \( i \geq 3 \),

\[
\begin{align*}
K^{i-1} >_S K^i >_S K^{i-2}, & \quad i \text{ even} \\
K^{i-2} >_S K^i >_S K^{i-1}, & \quad i \text{ odd}.
\end{align*}
\]

(33)

Therefore, each term in the sequences \( \{K^i\} \) and \( \{\lambda^i\} \) is sandwiched (pointwise) between the two previous terms, “bisecting” the space between them.

The proof is by induction, using the monotonicity properties of \( Z_E \) and \( Z_S \). First, since \( \Lambda(k) \) is strictly decreasing in \( k \), \( Z_E \) is a strictly decreasing function on \( \mathcal{K} \). Second, note that \( Z_S \) is a strictly increasing correspondence (in the strong set order with respect to \( >_E \) on the domain and \( >_S \) on the range). Indeed, by the argument at the beginning of Lemma 6, if \( \lambda^1 >_E \lambda^2 \), \( D^X(t|\lambda^1) > D^X(t|\lambda^2) \) for all \( t \). Therefore, any cutoff path \( \bar{K} \in Z_S(\bar{\lambda}) \) must be pointwise strictly higher (and equal at \( t = 0 \)), than any cutoff path \( K \in Z_S(\lambda) \): both paths start at the same point with the same (possibly infinite) slope, and if they were ever to cross, since \( 0 > d\bar{K}_i/dt > dK_i/dt \) at the intersection, \( \bar{K} \) would be cutting across \( K \) from below. In particular, \( Z_S \) always has a strictly \( >_E \)-increasing selection. Denote this selection by \( \hat{Z}_S \). Below, I always assume \( K^i \) is chosen from \( Z_S(\lambda^{i-1}) \) according to \( \hat{Z}_S \), i.e., \( K^i = \hat{Z}_S(\lambda^{i-1}) \).

With that in mind, consider the base case for induction. Since \( K^1 = \hat{Z}_S(\lambda^0) \) is strictly decreasing and starts at \( K^1_0 = 1 \), and \( \lambda^1 = \Lambda(k = 1) \forall t, \lambda^1 = Z_E(K^1) >_E \lambda_0 \). Since \( Z_S \) is strictly increasing, \( K^2 = \hat{Z}_S(\lambda^1) \) must satisfy \( K^2 >_S K_1 \). And again, since \( Z_E \) is strictly decreasing

\[ \lambda^1 = Z_E(K^1) >_E Z_E(K^2) = \lambda^2. \]

But \( K^2 \) is strictly decreasing and starts at \( K^2_0 = 1 \), so \( Z_E(K^2) >_E \lambda^0 \). Altogether, \( \lambda^1 >_E \lambda^2 >_E \lambda_0 \), the base case for \( \lambda^i \)'s for even \( i \). Meanwhile for \( K^i \)'s, one more iteration produces \( K^3 = \hat{Z}_S(\lambda^2) \), which, using \( \lambda^1 >_E \lambda^2 > \lambda^0 \) and \( K^1 = \hat{Z}_S(\lambda^0), K^2 = \hat{Z}_S(\lambda^1) \), implies that \( K^2 >_S K^3 >_S K^1 \), the base case for \( K \) with odd \( i \). The odd base case for \( \lambda^i \)'s and even base case for \( K^i \)'s follow from identical reasoning. Since the induction step would exactly replicate the steps above starting at a generic \( \lambda^{i-1} \in \mathcal{R} \), I omit it.

The inequalities \((32)\) and \((33)\) imply that \( \{K^i\} \) and \( \{\lambda^i\} \) are both Cauchy in the sup
norm, respectively in $\mathcal{K}$ and $\mathcal{R}$. To see this, assume for instance that $i$ is odd. Then re-write $K^{i-2} >_S K^i$ from (33) as $K^{i-2} - K^{i-1} >_S K^i - K^{i-1}$. Also, from (33), the last inequality is exactly

$$|K_t^{i-2} - K_t^{i-1}| > |K_t^{i-1} - K_t^i| \quad \forall t > 0$$

(with equality at $t = 0$). $\{K^i\}$ is therefore pointwise Cauchy. To prove it is uniformly so, note that, for all $i \geq 2$, $\lim_{t \to \infty} K_t^i = 0$ (since $dK_t^i/dt < 0$ so long as $K_t^i > 0$). Since, in addition, $K_0^i = 1$, and for each $i$, $K^i$ is strictly decreasing, it follows that $\sup_{t \in \mathbb{R}_+} |K_t^{i+1} - K_t^{i-1}| < 1$, and the supremum always occurs at an interior $t \in (0, \infty)$. Therefore, since (34) implies that the distance between successive terms shrinks in $i$ at every interior point, $\sup_{t \in \mathbb{R}_+} |K_t^{i+1} - K_t^{i-1}|$ must be strictly decreasing in $i$. A similar argument shows that $\lambda^i$ is uniformly Cauchy: using $\lim_{t \to \infty} K_t^i = 0$, one concludes that $\lim_{i \to \infty} \lambda_i^i = \Lambda(0)$, and the following steps are identical.

$\{K^i\}$ and $\{\lambda^i\}$ will then converge to a well-defined limit in $\mathcal{K}$ and $\mathcal{R}$ because these are complete spaces. Take $\mathcal{K}$ first. The set $B(\mathbb{R}_+)$ of bounded functions on $\mathbb{R}_+$ is complete under the sup norm. The subset of $B(\mathbb{R}_+)$ that is bounded between $[0, 1]$ and (weakly) decreasing is a closed with that same norm, so it is also complete. Therefore, $\{K^i\}$ and $\{\lambda^i\}$ must converge to some $(\lambda^*, K^*) \in \mathcal{R} \times \mathcal{K}$. For future reference, I note in particular $\lim_{i \to \infty} K_i^* = 0$ and $\lim_{i \to \infty} \lambda_i^* = \Lambda(0)$. It remains to verify that $(\lambda^*, K^*) \in Z((\lambda^*, K^*))$.

This step follows from the “sandwiching” inequalities (32) and (33). By construction, $\lambda^i$ satisfies the following inequalities.

$$\lambda_i^{i-1} >_E \lambda^* \geq_1 \lambda_i^{i-2}, \quad \text{i odd}$$

$$\lambda_i^{i-2} >_E \lambda^* \geq_1 \lambda_i^{i-1}, \quad \text{i even}$$

In particular, using $\lambda_i^i = Z_E(K_i^i),

$$Z_E(K_i^{i-2}) >_E \lambda^* \geq_1 Z_E(K_i^{i-3}), \quad \text{i odd}$$

$$Z_E(K_i^{i-3}) >_E \lambda^* \geq_1 Z_E(K_i^{i-2}), \quad \text{i even}.$$

So in particular, using the fact that for each $i$, $\lambda^*$ and $Z_E(K^i)$ are

- $\lambda^*$ is monotone increasing,
- start at $\Lambda(1)$, and
- converge to $\Lambda(0)$ as $t \to \infty$,

the maximum distance between $Z_E(K^i)$ and $\lambda^*$ must occur at an interior point. Hence the last display implies that $\| \cdot \|,$ $\| Z_E(K^i) - \lambda^* \|$ $\to 0$, where $\| \cdot \|$ denote the supremum norm. Then by the triangle inequality,

$$\| \lambda^* - Z_E(K^*) \| \leq \| \lambda^* - \lambda^m \| + \| \lambda^m - Z_E(K^*) \| \to 0,$$
which proves $\lambda = Z_E(K^*)$.

To show $K^* = \hat{Z}_S(\lambda^*)$ (so in particular $K^* \in Z_S(\lambda^*)$), note once more that

$$K^* \succ_S K^{i-1}, \quad i \text{ even} \quad \text{and} \quad K^* \succ_S K^{i-2}, \quad i \text{ odd}.$$ 

by construction, so that

$$\hat{Z}_S(\lambda^{i-2}) \succ_S K^* \succ_S \hat{Z}_S(\lambda^{i-3}), \quad i \text{ even} \quad \text{and} \quad \hat{Z}_S(\lambda^{i-3}) \succ_S K^* \succ_S \hat{Z}_S(\lambda^{i-2}), \quad i \text{ odd}.$$ 

Once more leverage the idea that for all $i$, $K^*$ and $\hat{Z}_S(\lambda^i)$ both

- are strictly decreasing,
- start at 1, and
- converge to 0 as $t \to \infty$.

to conclude that the maximum distance between them occurs at an interior point, which implies that $\|\hat{Z}_S(\lambda^*) - K^m\| \to 0$. Therefore, by the triangle inequality

$$\|K^* - \hat{Z}_S(\lambda^*)\| \leq \|K^* - K^m\| + \|K^m - \hat{Z}_S(K^*)\| \to 0,$$

which concludes the proof.

\[\square\]

10.3 Proofs for Section 7

Proof of Lemma 6. I claim that, for any two entry rate processes $\lambda^1 = (\lambda^{1}_t)_{t \geq 0}$ and $\lambda^2 = (\lambda^{2}_t)_{t \geq 0}$ where $\lambda^1$ is (almost everywhere) strictly greater after $s$, $D(s|\lambda^1) > D(s|\lambda^2)$. If $\sigma_1$ is a Poisson stopping time with rate $\lambda^1$ and $\sigma_2$ is a Poisson stopping time with rate $\lambda^2$, then for any $t > s$,

$$P(\sigma_1 < t | \sigma_1 > s) = 1 - e^{-\int_s^t \lambda^1 \nu}.$$ 

Therefore $\sigma_1 | \sigma_1 > s$ strictly first-order-stochastically dominates $\sigma_2 | \sigma_2 > s$ when $\lambda^1$ is almost surely strictly greater from $s$ onwards. Since $e^{-rt}$ is strictly decreasing in $t$, and for $i = 1, 2$, $D^X(s|\lambda^i) = E[e^{-r\sigma_i}|\sigma_i > s]$, it follows that $D^X(s|\lambda^1) > D^X(s|\lambda^2)$, as required. (Intuitively, the present value under the former entry rate is higher, since entry happens faster at each instant going forward).

Now let $\tilde{\lambda}$ be any entry rate process that satisfies $\tilde{\lambda}_s = \Lambda(k)$ for all $s \geq t^*(k)$. Then, since $D(k)$ is the present value of a dollar at entry when the seller keeps the state at $k$
permanently, $D(k) = D^X(t^*(k)|\tilde{\lambda})$. Since

$$\lambda^*_{t^*(k)} = \Lambda(K^*_{t^*(k)}[\lambda^*]) = \tilde{\lambda}_{t^*(k)}$$

by definition of a private offers RME, and $\lambda^*$ strictly increases from $t$ onwards while $\tilde{\lambda}$ stays constant, it follows that $D^X(t^*(k)|\lambda^*) > D(k)$ by setting $\lambda^1 = \lambda^*$ and $\lambda^2 = \tilde{\lambda}$ in the claim at the beginning of the proof.

With Lemma 6 in hand, I can provide the proof for Theorem 2. First, I define RME’s of the hybrid starting-public game. I use the notation $[k_0]$ on functions and sequences to denote that they condition on an initial jump to $k_0$.

**Definition 7.** A (pure) Regular Markov Equilibrium of the starting-public game consists of a triple

$$\{(K_{s+t}, t \geq 0)_{k \in [0,1], s \in \mathbb{R}^+, k_0 \geq k}, (\lambda^*_t[\cdot])_{t \geq 0}, P(\cdot, \cdot)[\cdot]\}$$

such that

1. $t \mapsto K_{t+s}$ is non-increasing, càdlàg, and Markov with respect to $\mathcal{F}_{t+s}$.

2. For all $k \in [0, k_0]$ and $t \geq 0$, $P(k, t)[k_0]$ is an optimal reservation price strategy for $v_B = k$ at time $t$ taking as given the law of motion for $K_{t+s}$, future prices given by $P(K_{t+s}, t+s)[k_0]$ and time of entry given by $\lambda^*_t[k_0]$

$$k - P(k, t)[k_0] = \sup_{\tau \in \mathcal{T}} E^K_{k,t,k_0} \left[ 1_{\sigma > \tau + t} e^{-r(\tau + t)} (k - P(K_{\tau+t}, \tau + t)[k_0]) + 1_{\sigma \leq \tau + t} e^{-r(\tau-t)} \pi_B(k) \right]$$

where $E^K_{k,t,k_0}$ is the expectation with respect to the law of $\{K_{s+t}, s \geq 0\}$ and $\sigma$, conditional on $K_{t-} = k$ and $\sigma > t$.

3. (a) At $t = 0$, $\{K_t, t \geq 0\}_{k=1,s=0,k_0=1}$ is a seller-optimal impulse control, i.e., it solves

$$\sup_{Q \in \Gamma} \left[ (1 - F_B(Q_0)) P(Q_0, 0) + \int_0^\sigma e^{-rs} P(Q_t, t)[Q_0] dF_B(Q_t) + e^{-r\sigma} \Pi_S(Q_\sigma) \right]$$

where $\Gamma$ is the set of decreasing processes on $[0,1]$ adapted to $\mathcal{F}_t$, and $E^Q$ is the expectation with respect to the law of $\sigma$ and future $Q_t$’s.

(b) For any $t > 0$, $k \in [0, k_0]$, $\{K_{t+s}, s \geq 0\}_k$ is a seller-optimal impulse control for cutoffs taking as given transaction prices $P(\cdot, \cdot)[k_0]$ and fixed, exogenous entry.
rates \( \lambda^*[k_0] \), i.e., it solves

\[
\sup_{Q \in \Gamma_{k,t,k_0}} \mathbb{E}_{k,t,k_0}^Q \left[ \int_0^\sigma e^{-rs} P(Q_{t+s}, t+s) [k_0] dF_B(Q_{t+s}) + e^{-r(\sigma-t)} \Pi_S(Q_\sigma) \right]
\]

where \( \Gamma \) is the set of decreasing processes on \([0, k_0]\) adapted to \( \mathcal{F}_{t+s} \), and \( \mathbb{E}_{k,t,k_0}^Q \) is the expectation with respect to the law of \( \sigma \) and future \( Q_\sigma \)'s conditional on \( Q_{t-} = k \) and \( \sigma > t \).

4. For all \( k \leq k_0, t \geq 0 \), \( \lambda^*_t[k_0] \) satisfies

\[
\lambda^*_t[k_0] = \Lambda(K_t) = \lambda_0 G(\Pi_E(K_t)), \tag{34}
\]

i.e., the entrants have correct conjectures about \( K_t \).

With the formal definition in hand, I can establish the result.

**Proof of Theorem 3** Let \( \hat{k} \) be the initial jump of some public offers RME. Since that jump ends on the smooth trade locus, the seller’s value at \( k = 1 \) under public offers is

\[
U(1, \hat{k}) = \left( 1 - F_B(\hat{k}) \right) \hat{P}(\hat{k}) + F_B(\hat{k}) D(\hat{k}) \Pi_S(\hat{k}).
\]

I derive a lower bound on the seller’s payoff in any RME of the starting-public game and prove that this bound exceeds \( U(1, \hat{k}) \). When the game starts with a public offer, the seller has the possibility of inducing an initial burst of trade. By identical arguments to those in Theorem 3, a necessary condition for the initial public offer of that starting-public game is

\[
K_0 \in \arg \max_{k_0 \in [0,1]} \left\{ (1 - F_B(k_0)) P(k_0, t = 0)[k_0] + F_B(k_0) D^X(t = 0|\lambda^*[k_0]) \Pi_S(k_0) \right\}.
\]

Indeed, taking as given \( P(\cdot, \cdot)[\cdot] \) and \( \lambda^*[\cdot] \), the seller could always choose an impulse control for \( K_t \) with a different size of initial jump, and after the initial jump the seller faces a continuation game that looks like a private offers game with initial state \( K_0- = k_0 \). In particular, this means that the seller’s ex ante value in the starting-public is at least

\[
(1 - F_B(\hat{k})) P(\hat{k}, t = 0)[\hat{k}] + F_B(\hat{k}) D^X(t = 0|\lambda^*[\hat{k}]) \Pi_S(\hat{k})
\]

Consistency of entrant beliefs requires that, for \( k_0 = \hat{k} \), \( \lambda^*_0[\hat{k}] = \Lambda(\hat{k}) \). Then by identical arguments as in the proof of Lemma 4, \( D^X(t = 0|\lambda^*[\hat{k}]) > D(\hat{k}) \). Adapting the arguments in 4 and Lemma 4 to a private offers game with initial state \( K_0- = \hat{k} \), it follows that

\[
P(\hat{k}, t = 0)[\hat{k}] = D^X(t = 0|\lambda^*[\hat{k}]) \Pi_S(\hat{k}) > \hat{P}(\hat{k}).
\]

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Altogether, comparing term by term,

\[
(1 - F_B(\hat{k})) P(\hat{k}, t = 0)[\hat{k}] + F_B(\hat{k})D^X(t = 0|\lambda^*[\hat{k}])\Pi_S(\hat{k}) > U(1, \hat{k}),
\]

which proves the result.

\[\square\]

**Proof of Theorem 5.** Let \(\lambda^* = (\lambda_t^*)_{t \geq 0}\) be the fixed point of entry rates in the private offers equilibrium. By Lemma 4,

\[
p^{\text{priv}}(k) = D^X(t^*(k)|\lambda^*)\pi_S(k) \tag{35}
\]

\((k's\ trading\ price\ under\ private\ offers\ is\ the\ smooth\ trade\ price\ when\ he\ is\ marginal)\)

while, for any \(v_B = k\) that trades smoothly under public offers, Lemma 1 says that

\[
p^{\text{pub}}(k) < D(k)\pi_S(k). \tag{36}
\]

Then Lemma 6, together with (35) and (36) yield \(p^{\text{priv}}(k) > p^{\text{pub}}(k)\) for any \(k\) that trades smoothly under public offers.

For any \(k\) that trades in a burst, by Theorem 3 there exists some \(\hat{k} < k\) (the left endpoint of the jump that \(k\) is part of) such that \(\hat{k}\) trades smoothly at a price \(p^{\text{pub}}(\hat{k}) = \hat{P}(k)\), and \(k\) pays exactly \(p^{\text{pub}}(k)\). Hence \(p^{\text{pub}}(k) = p^{\text{pub}}(\hat{k})\). By Theorem 4, pure RME’s of the private offers game have trade smooth trade at a strictly positive speed for all states \((k, t)\). Therefore, along the equilibrium path, \(t^*(k)\), the time at which \(v_B = k\) trades, is strictly decreasing in \(k\). So if I can show that equilibrium private-offer prices have a negative total derivative with respect to time,\(^{32}\) the result will follow, since \(\hat{k}\) trading strictly later would imply \(p^{\text{priv}}(k) > p^{\text{priv}}(\hat{k})\), and therefore

\[
p^{\text{pub}}(k) = p^{\text{pub}}(\hat{k}) < p^{\text{priv}}(\hat{k}) < p^{\text{priv}}(k).
\]

Consider then the marginal buyer’s HJB along the equilibrium path of private offers RME:

\[
rJ_B^K_t(K_t, t|\lambda^*) = \lambda_t^* \left[ \pi_B(K_t) - J_B^K_t(K_t, t|\lambda^*) \right] + \frac{d}{dt}J_B^K_t(K_t, t|\lambda^*)
\]

where \(J_B^v(k, t|\lambda^*)\) is the continuation value of type \(v^R = v\) at state \((k, t)\). Rearranging,

\[
(\lambda_t^* + r) \left[ J_B^K_t(K_t, t|\lambda^*) - \frac{\lambda_t^*}{\lambda_t^* + r} \pi_B(K_t) \right] = \frac{d}{dt}J_B^K_t(K_t, t|\lambda^*). \tag{37}
\]

Since \(K_t\) is marginal at time \(t\), \(J_B^K_t(K_t, t|\lambda^*) = K_t - \hat{P}^X(K_t, t|\lambda^*)\), so using the expression

\(^{32}\) Equilibrium prices change over time because they track the smooth trade locus as a function of \((k, t)\), and as time moves forward, \(t\) increases while \(k\) decreases. The total derivative considers both effects.
for $\hat{P}^X(\cdot, \cdot|\lambda^*)$ in $[18]$, the right hand side of (37) is proportional to

$$K_t - D^X(t|\lambda^*)\pi_S(K_t) - \frac{\lambda_t^*}{\lambda_t^* + r}\pi_B(K_t)$$

Since in addition $D^X(t|\lambda^*) > D(K_t) = \frac{\lambda_t^*}{\lambda_t^* + r}$ by Lemma 6, the left hand side of (37) is bounded below by something proportional to

$$K_t - D^X(t|\lambda^*)(\pi_S(K_t) + \pi_B(K_t)) > 0,$$

where the inequality follows from the assumption that immediate trade is bilaterally efficient for $B$ and $S$.

Meanwhile, using again the fact that $K_t$ is marginal,

$$\frac{d}{dt}J^K_{B_t}(K_t, t|\lambda^*) = -\frac{d}{dt}\hat{P}^X(K_t, t|\lambda^*).$$

(38)

where I hold the first term in $J^K_{B_t}(K_t, t)$ constant since it is the buyer’s type and here I am taking the total derivative with respect to calendar time as it affects the state $(k, t)$.

Altogether, the bound on the RHS of (37) and (38) imply that equilibrium prices with private offers strictly decrease in time, and the result follows.

$\Box$

Proof of Theorem 6. I show that, whenever $K^\text{pub}_t = K^\text{priv}_t$ and both are dropping smoothly, $K^\text{pub}$ is dropping faster. Since $K^\text{pub}_0 = K^\text{priv}_0 = 1$, and $K^\text{pub}_t$ has downward jumps but $K^\text{priv}_t$ always decreases smoothly, the result will follow.

Let $t^*$ be the time at which $K^\text{pub}_t$ and $K^\text{priv}_t$ meet, and let $k^*$ denote their common value at that time. Let $\lambda^*$ be the fixed point of beliefs in the private offers equilibrium, so that $\lambda_t = \lambda_0 G(\Pi_E(K^\text{priv}_t)) = \Lambda(K^\text{priv}_t)$. Since $K^\text{pub}_t = K^\text{priv}_t \equiv k^*$ and the public offers equilibrium is smooth at $t^*$, then at time $t^*$ entry under public offers is also happening at rate $\lambda^*_t = \Lambda(k^*)$. In particular, note that $D(k^*) = \frac{\lambda_t^*}{\lambda_t^* + r}$, which I will use repeatedly in the rest of the proof.

Under private offers, the seller best-responds to entrants’ conjectures as though their entry were exogenous, so by $[29]$, $K^\text{priv}$ is decreasing at speed

$$-\frac{dK^\text{priv}_t}{dt} = (\lambda_t^* + r) \left[ k^* - \frac{\lambda_t^*}{\lambda_t^* + r} \left( \pi_B(k^*) + \pi_S(k^*) \right) \frac{D^X(t|\lambda^*)\pi'_S(k^*)}{\pi_S(k^*)} \right]$$

(39)

while (by $[12]$) $K^\text{pub}$ is decreasing at speed

$$-\frac{dK^\text{pub}_t}{dt} = (\lambda_t^* + r) \left[ k^* - \hat{P}(k^*) - \frac{\lambda_t^*}{\lambda_t^* + r} \pi_B(k^*) \right]$$

(40)
Let \( NUM_i, i \in \{\text{pub, priv}\} \) be the numerator of the term in square brackets in (39) and (40), while \( DEN_i \) denotes the denominator of those same terms. I claim \( 0 < NUM_{\text{priv}} < NUM_{\text{pub}} \) and \( DEN_{\text{priv}} > DEN_{\text{pub}} > 0 \), so that \( 0 < -dK_t^{\text{priv}}/dt < -dK_t^{\text{pub}}/dt \), and \( K^{\text{pub}} \) is decreasing faster.

Indeed, plugging in \( \hat{P} \) from Proposition 1 into \( NUM_{\text{pub}} \) gives

\[
NUM_{\text{pub}} = k^* - \frac{\lambda_*}{\lambda_* + r} (\pi_B(k^*) + \pi_S(k^*)) - D'(k^*) \Pi_S(k^*) \frac{F_B(k^*)}{f_B(k^*)} > NUM_{\text{priv}},
\]

since \( D'(\cdot) < 0 \).

To compare denominators, use Proposition 1 again:

\[
DEN_{\text{pub}} = \hat{P}'(k^*) = D(k^*)\pi'_S(k^*) + D'(k^*)\pi_S(k^*) + \rho'(k^*)
\]

\[
DEN_{\text{priv}} = \frac{\partial}{\partial k} \hat{P}^X(t^*|\lambda^*) = D^X(t^*|\lambda^*) \pi'_S(k^*).
\]

As in the proof of Theorem 5, \( D(k^*) < D^X(t^*|\lambda^*) \), since these terms are present values at entry, and future entry rates are frozen at \( \Lambda(k^*) \) in the former but start at \( \Lambda(k^*) \) and strictly increase in the latter. Therefore, since \( \pi_S(\cdot) \) is strictly increasing and \( D(\cdot) \) strictly decreasing,

\[
D^X(t^*|\lambda^*) \pi'_S(k^*) > D(k^*)\pi'_S(k^*) + D'(k^*)\pi_S(k^*)
\]

When, in addition, regret \( \rho \) is decreasing, the right hand side in the last display is smaller than \( \hat{P}'(k^*) \), and \( DEN_{\text{priv}} > DEN_{\text{pub}} \). By Theorem 3, \( \hat{P} \) is strictly increasing at \( k^* \), so both denominators are positive, and

\[
-dK_t^{\text{priv}}/dt < -dK_t^{\text{pub}}/dt
\]

whenever \( K_t^{\text{priv}} = K_t^{\text{pub}} \).

Proof of Proposition 3: Since \( v_B = 1 \) trades in the first instant under both public and private offers, it gets \( 1 - p^{\text{pub}}(1) \) in the former case, which by Theorem 5 is strictly greater than \( 1 - p^{\text{priv}}(1) \), what it obtains in the latter case. The result will follow by continuity if one can show that the buyer’s problem is sufficiently well-behaved that indirect utilities are continuous in type.

Let \( V_{\text{pub}}(v_B) \) and \( V_{\text{priv}}(v_B) \) denote \( v_B \)'s indirect utility in an arbitrary public offers RME and an arbitrary pure private offers RME, respectively. Consider the function

\[
h(\tau, v_B) = \mathbb{E}_0^{v_B} \left[ \mathbf{1}_{\sigma > \tau} e^{-r\tau} (v_B - P(K_\tau)) + \mathbf{1}_{\sigma \leq \tau} e^{-r\sigma} \pi_B(v_B) \right]
\]

for any fixed offers \( P \), almost surely finite \( \mathcal{F} \otimes \mathcal{A} \)- stopping time \( \sigma \), and decreasing \( \mathcal{F} \)-
Markov process $K_t$. By a standard envelope theorem argument, $w'(v_B) = F(v_B)$, so
\[ h_2(\tau, v_B) = E_0 \left[ 1_{\sigma > \tau} e^{-r\tau} + 1_{\sigma \leq \tau} e^{-r\sigma} F(v_B) \right] \in (0, 1) \quad (42) \]
for all $(\tau, \sigma, (K_t)_{t \geq 0})$. Theorem 2 of Milgrom and Segal (2002) therefore implies that $V_{pub}$ and $V_{priv}$ are absolutely continuous, since they are upper envelopes of $h$ in the first argument, and $h$ has a uniformly bounded derivative in the second argument. Since $V_{pub}^1 > V_{priv}^1$, there exists a $k^*$ that satisfies the conditions of the theorem. 

11 Appendix B: Details for Numerical Computation of Private Offers RME’s

Recall from the discussion of Lemma 5 that even though $Z$ given by (30) and (31) is not really a best response correspondence, its fixed points suffice for constructing pure RMEs of the private offer game. Moreover, since the proof of Lemma 5 is constructive, it provides an algorithm to find a fixed point efficiently. To compute $Z$, which maps between spaces of functions with unbounded domain, I discretize a long time horizon into many “periods” $T$ and a small “period length” $\Delta$.

I discretize a long time horizon into many “periods” $T$ and a small “period length” $\Delta$. In that way I calculate the integral for $D^X(t|\lambda)$ for each $t$ on the grid numerically and I solve the differential equation in $Z$ by iterating it forward as a difference equation in discrete “steps,” each of length $\Delta$. (Note that in general there might be multiple solutions to the differential equation, making $Z$ multi-valued, but as described in the proof, one finds a fixed point constructively using any selection from $Z$. Therefore it is without loss to treat $Z$ and its discretizations as operators). I choose a fine time grid with $T = 4 \times 10^4$ periods and a period length $\Delta = 4 \times 10^{-3}$, for a total time horizon of $T \times \Delta = 160$. I chose these tuning parameters to make sure that the associated $\lambda$’s fully converged to $\Lambda(0)$ by end of the time horizon and the resulting $K_t$ and $\lambda^*_t$ and $D^X(t|\lambda^*)$ paths were sufficiently smooth. Call $Z^\dagger$ the discretization of the operator obtained by “chaining” the two best responses in $Z$, i.e., $Z^\dagger$ takes entry rates as inputs, computes pseudo-best-response cutoffs, and outputs best response entry rates to these cutoffs.

Let $t_n = n \times \Delta$. The algorithm for finding the fixed point of $Z^\dagger$ is as follows:

- Initialize $\lambda^{(old)}$ with the constant function $\Lambda(1)$.
- While $\max_{n \in \{0, 1, \ldots, T\}} \left| \frac{\lambda_n^{(new)} - \lambda_n^{(old)}}{\lambda_n^{(old)}} \right| < \epsilon$

Since the path of $\lambda$ gets truncated at the end end of the (finite) time horizon, I approximate $D^X(t|\lambda)$ by assuming that after time $T \times \Delta$, $\lambda_t$ stays at the constant level $\Lambda_{T \times \Delta}$. For a large enough horizon, this “end-game” scenario will have a minuscule effect on $D^X$. I verify that the time horizon is long enough that all relevant quantities converge to their respective limits by $t = T \times \Delta$. 

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1. Set $\lambda^{(\text{new})} = Z^\dagger(\lambda^{(\text{old})})$.

2. Check convergence condition.
   - If satisfied, break, return $\lambda^* = \lambda^{(\text{new})}$.
   - If no convergence, set $\lambda^{(\text{old})} = \lambda^{(\text{new})}$, repeat.

References


