Index and Smart Beta when Investors are Ambiguity Averse

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June 1, 2018

Abstract

We show that in a rational expectations equilibrium model, investors who are ambiguity averse about the market volatility give up index investing strategy. We then design a new index that takes stock idiosyncratic volatilities into the weight. If a smart beta fund offering such a new index is publicly available, ambiguity averse investors hold the fund and an information-based portfolio, and thus participate in all asset markets, directly or indirectly. This result follows from a new separation theorem, which states that an investor’s equilibrium portfolio can be decomposed into components, each matching the optimal portfolio based on only one information source (price versus private signal). Asset risk premia satisfy the CAPM with the fund as the pricing portfolio.
1 Introduction

Index investing has been recommended publicly by legendary investors. For example, Mark Cuban in an interview suggested that “If you don’t know too much about markets, the best way to invest your money is to put it in a cheap S&P 500 SPX fund.” Warren Buffett agreed and argued that consistently buying an S&P 500 low-cost index fund makes the most sense practically all of the time. These turned out to be great suggestions during the strong upward trend of U.S. equity market in the last ten years. According to Morningstar’s commentary, net flows into actively managed U.S. equity funds fell $211 billion for the 12 months ended January 2018; in contrast, net flows into passive U.S. equity funds rose by $233.5 billion in the same period.

However, it seems that when the financial market becomes intangible, investors give up the index investing strategy. For example, Morningstar estimated that when the U.S. stock market behaved like a roller-coaster in February 2018, investors pulled $8.4 billion out of US equity passive funds.

Such a huge outflow is attributed to investor ambiguity aversion about the market volatility. In such an argument, investors do not know perfectly the variance of the distribution of stock returns—they face model uncertainty. Then, in making investment decisions, ambiguity averse investors place heavy weight upon worst-case scenarios (which is the extremely volatile financial market) and thus run away from the stock market to avoid unbounded risks.

In this paper, we first provide a rational expectations equilibrium model to analyze the index investing strategy when investors are ambiguity averse about assets’ idiosyncratic volatilities. Consistent with the argument above, we show that index investing strategy becomes unattractive to ambiguity averse investors, especially when the market seesaws dramatically. We then ask whether it is possible to design a new index, such that if a passive fund commits to offering it, even extremely ambiguity averse investors are willing to hold the fund and indirectly participate in the financial market.

The new index design is indeed the smart beta investing strategy, which is gaining increasing popularity.\footnote{Smart beta strategies seek to passively follow indices, while also taking into account alternative weighting schemes such as volatility to the traditional market capitalization.} However, ambiguity averse investor do not have the information needed to implement the smart beta strategy themselves. So the question of how investors who do not know the ambiguous parameters form optimal portfolios that in-
clude the smart beta fund and other assets is economically meaningful and determinate. Further interesting questions are how the investor’s degree of knowledge about the ambiguous parameter of the economy, or the investor’s private information, affects the optimal holdings in the smart beta fund; and how the existence of the smart beta fund affects asset risk premia.

The newly designed index is different from the value-weighted market portfolio in that it includes less shares of more volatile stocks. Hence, it is indeed related to the defensive (low-volatility) investing strategy. We show that with the smart beta fund, all investors hold identical positions in the fund as a common component of their portfolios and hence participate in all assets’ markets. In consequence, a well-designed smart beta strategy can prevent ambiguity averse investors from fleeing the financial market and thus promote the market efficiency. In developing our analysis, we also provide a new separation theorem for optimal security holdings under asymmetric information, and a version of the CAPM that holds under ambiguity aversion and asymmetric information.

Specifically, in our setting there is a continuum of investors with strictly positive endowments of all risky assets. Investors hold a common uninformative prior about risky asset payoffs, so based upon the prior, holding non-zero positions of the assets is infinitely risky. Security endowments are subject to random supply shocks. Prices are set to clear the markets for all assets. For each asset, investors are divided into two groups. Members of one group receive conditionally independent private signals about the asset

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2 This separation theorem is the reason that delegation to a fund induces participation in our model. In general an individual will be willing to delegate to an agent who credibly commits to a contingent strategy (based on the agent’s knowledge) of doing whatever the individual himself would have done based on that knowledge. But it does not follow that the nonparticipation problem is cured by delegating to a fund whose manager knows the model parameters. The problem is that different potential investors in the fund have different information sets (as well as different degrees of ignorance about model parameters). So in general (without imposing the equilibrium condition, and without our portfolio information separation theorem) there is no guarantee that a single fund (or fund of funds) can persuade different investors that the fund is offering part or all of what each investor would have chosen herself (knowing what the fund knows). (In the model, in equilibrium the fund does not offer each investor the overall portfolio that she would want to hold, only a key component of such a portfolio.) It is only in equilibrium, and by virtue of our new separation theorem, that we can conclude that there is a single fund (which we characterize) which offers a key common component of the portfolios that the diverse investors themselves would have chosen if they knew what the fund knows.

It might further be suggested that even without imposing the equilibrium condition, the problem of nonparticipation could be solved by offering a separate tailored fund to each investor consisting of the portfolio that this investor would have chosen if the investor knew the ambiguous model parameters. However, even if this were a relevant option in practice, this approach does not solve the problem because it is not clear exactly what portfolios such funds should offer.
payoff and know the precision of the supply shock. Members of the other group neither receive any private signals about the asset payoff, nor know the precision of the supply shock. In particular, the uninformed investors’ subjective belief about the precision of the supply shock includes the possibility of precisions that are arbitrarily close to zero. As a result, since the uninformed investors cannot extract information about the asset payoff from its price when the precision of the supply shock is arbitrarily close to zero, they may perceive the assets to be extremely risky.

Ambiguity averse investors choose optimal portfolio to maximize expected utility under worst-case assumptions for the values of the supply volatility parameters that they are uncertain about. So for any portfolio contemplated by an investor, expected utility is calculated contingent on the unknown parameters having values that minimize traditional CARA expected utility (following the max-min expected utility proposed by Gilboa and Schmeidler (1989)).

There is a passive fund whose manager knows the supply shock precisions of all assets. The assumption that fund managers observe parameters that individual investors do not is based on the idea that fund managers are professionals who understand financial markets. The fund offers all investors a single portfolio which is a deterministic function of the exogenous parameters, including the supply volatilities of all the assets. Though investors who face model uncertainty do not know the exact weights of the portfolio, the function used for constructing the portfolio is common knowledge. We refer to the portfolio offered as ‘the fund.’

The key intuition derives from a new separation theorem which applies in the setting with no model uncertainty. In this setting, there is a rational expectations equilibrium in which any investor’s equilibrium risky asset holding can be decomposed into two components. The first is a common deterministic component that plays a role in our model somewhat similar to the market portfolio in the CAPM, but is distinct from the endowed market portfolio. We call this component the Risk-Adjusted Market Portfolio

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3Most of the literature considers ambiguity aversion about asset payoffs, with the exception of Watanabe (2016) who assumes that investors are ambiguous about the mean of the asset’s random supply shock. For tractability, we similarly assume that investors are ambiguous about precisions of assets’ supply shocks. However, our main result that all investors in equilibrium hold the passive fund holds even when investors are ambiguous about other asset market parameters.

4In particular, we assume that for each asset, there is a positive measure of investors who know the precision of its random supply shock. Hence, the knowledge needed to construct an appropriate fund is available in the economy. In Section 6, we discuss how the passive fund can be implemented by a fund of funds.
(RAMP). The second is the investor’s information-based portfolio, which includes a non-zero position in an asset if and only if she receives a private signal about the asset.

This new separation theorem differs from the separation theorem derived in the literature in that any individual investor’s optimal portfolio is separated by the different parts of her information set. We therefore call it the Information Separation Theorem. Specifically, RAMP is constructed based only on the information extracted from asset prices, and is independent of the investor’s private signal. So another name for it is the learning-from-price-based portfolio. Conversely, the information-based portfolio depends only on the information derived from the private signal; it is independent of the information extracted from asset prices.

This separation derives from the model assumptions of CARA utilities and normal random variables. As is standard in such settings, any investor’s optimal position in each asset is proportional to the product of her information precision about the asset payoff and the difference between the conditional expectation of the asset payoff (given her information) and the asset price. The precision of the investor’s overall information about an asset is the sum of the precisions of the price signal and her private signal. Because of normality, her conditional expectation of the asset payoff is the average of the conditional expectations based on each of her two signals, weighted by the signal precisions. Hence, the investor’s optimal position is the sum of two components. Each is the product of the precision of one signal and the difference between the conditional expectation of the asset payoff, based on only this signal, and the asset price. RAPM is the investor’s optimal portfolio based only on the price signal, and her information-based portfolio is her optimal portfolio based only on her private signal.

The Information Separation Theorem provides new insight into how ambiguity averse investors will participate in asset markets when there is model uncertainty. Consider again a setting where investors are subject to model uncertainty and a passive fund offers RAMP. One share of the passive fund represents one unit of RAMP. Consider the following proposed strategy profile: each investor holds exactly one share of the fund, and additionally holds her investor-specific information-based portfolio (which could be a nullity). Given that all other investors behave as prescribed, no investor has an incentive to deviate.

The key insight is that the fund uses its knowledge to do precisely what each investor would choose in her non-information-based portfolio if she knew what the fund knows. Consider, for example, an investor named Lucy and a vector of precisions of
assets’ supply shocks that is possible according to her subjective belief. Given the value of this vector, she would be in a possible world without model uncertainty. In such a possible world, since all other investors are holding exactly one share of the passive fund and their own information-based portfolios, they are holding the same portfolios as they would in the rational expectations equilibrium in this world. Hence, the market clearing condition implies that the pricing function is the same as the one in the rational expectations equilibrium. Therefore, if Lucy knew the parameter values that characterize this possible world, her optimal portfolio choice would consist of $RAMP$ and her own information-based portfolio.

By holding the passive fund and her own information-based portfolio, Lucy implements exactly such an investment strategy in every possible world. In each possible world, $RAMP$, and therefore the composition of the passive fund, differ. But Lucy’s information-based portfolio does not. So even when Lucy is ambiguous about some or all assets, her optimal portfolio choice is to hold exactly one share of the passive fund together with her own information-based portfolio.

In the above argument, given other investors’ strategies, we first fix a possible world, and then calculate Lucy’s optimal investment strategy. Hence, we are implicitly assuming that Lucy has a min-max utility. However, since Lucy’s optimal investment strategy, holding one unit of the fund and her own information-based portfolio, is constant across all possible worlds in her belief support, her min-max utility is the same as her max-min utility. Put differently, a strong min-max property holds in the equilibrium, and thus holding one unit of the fund and her own information-based portfolio is also the optimal investment strategy with her max-min utility.

This argument for the optimality of investing in the fund (if all other investors behave according to the proposed equilibrium) is a powerful one, as it requires only that an ambiguity averse investor’s investment strategy be time-consistent. Imagine that Lucy could pay a fee to learn the precise values of all parameters that she does not know. This would eliminate her ambiguity aversion, so she would subsequently trade as an uninformed investor who is not subject to model uncertainty. The fund is doing just what Lucy herself would do if she were to pay the fee. So by time consistency, given equilibrium behavior of all other investors, Lucy would strictly prefer holding the fund over paying the fee, however small, to learn the parameters. In other words, delegation can replace information acquisition as a means of addressing ambiguity aversion.\footnote{Mele and Sangiorgi (2015) analyze a model in which ambiguity averse investors can acquire costly}
The presence of the fund affects asset risk premia. The fact that an investor with no private information optimally holds the passive fund implies that it is mean-variance efficient, so that the CAPM security market line holds with \( RAMP \) as the pricing portfolio. Because the pricing portfolio does not depend on the realization of the random supply shock, and the weight of each asset in the pricing portfolio are conditional on asset prices, which are publicly observable, the portfolio is potentially observed by an econometrician. This makes the model empirically testable.

Lucy’s willingness to hold the passive fund is an equilibrium outcome; the reasoning relies on her conjecture about the willingness of all other investors to hold the fund and their own information-based portfolios. Otherwise, to consider an off-equilibrium possibility, if other investors’ trading were to lead to asset prices that are almost uninformative, Lucy would not hold the passive fund, because \( RAMP \) would be extremely risky to her in this case. We consider such a scenario in detail in Subsection 4.1.

This setting endogenizes investor trust in fund managers (Gennaioli, Shleifer, and Vishny 2015). One subtle point that this approach reveals is that inducing investors to participate requires more than investor trust in the honesty and superior knowledge of fund managers about the financial market. It is crucial that investors foresee an equilibrium in which other investors also trust the fund managers and trade accordingly: as we discuss in Subsection 4.1 off the equilibrium, it is possible that an investor is not willing to hold the fund, even if she trusts the fund manager.

The equilibrium argument also highlights the fact that investors hold \( RAMP \) for risk-sharing rather than mere individual-level diversification reasons, which would apply even without analysis of market clearing. As further indication that diversification incentives are not at the heart of the result, we show that an equilibrium with ‘fund’ delegation exists even in the special case in which only one risky asset is traded in the market.\(^6\)

It may seem surprising that all investors take the same position in the passive fund, even though their beliefs about the precisions of supply shocks and thus the fund composition have different supports. Investors with different priors have different worst-case scenarios, and therefore differ in how risky they view the fund. However, owing to the Information Separation Theorem, in equilibrium all investors agree that it is good to

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\( ^6 \)We consider a multi-asset setting in order to derive cross-sectional asset pricing implications.
delegate their non-information-based investments to a fund that has access to the true values of the supply volatilities. RAMP is based upon those actual values.

Overall, these findings suggest that when an appropriate passive fund is available, investors’ ambiguity aversion alone does not explain nonparticipation. Since in fact there is limited participation, this means that our model is not an accurate description of reality. We view the primary contributions of this model as four-fold. First is providing conceptual clarification that ambiguity aversion does not, in a fairly standard setting, explain the nonparticipation puzzle unless there is also a failure of the market to offer an appropriate passive mutual fund. Second, when the fund is available, there exists an equilibrium with several strong and interesting properties. For example, a version of CAPM holds with the fund being the pricing portfolio. Third is the Information Separation Theorem. Fourth is offering normative implications, about the possible benefits of investor education and of funds offering RAMP. In particular, the index fund industry has been growing rapidly in recent decades, yet the nonparticipation puzzle remains. So the model raises the possibility that innovation in the fund industry, and in particular the introduction of funds that offer RAMP, can benefit investors.

2 A Model of Financial Markets

The model has two dates, date 0 and date 1. The economy is populated by a continuum of investors with measure one, who are indexed by \( i \) and uniformly distributed over \([0, 1]\). All investors trade at date 0 and consume at date 1.

**Assets.** Any investor \( i \) invests in a riskfree asset and \( N \geq 2 \) independent risky assets by herself. Denote by \( Q \) the set of all risky assets. The riskfree asset pays \( r \) units, and risky asset \( n \) pays \( f_n \) units of the single consumption good. In addition to trading directly, investors can also hold individual risky assets through a passive fund that commits to offering a portfolio \( X \), which is an \( N \)-dimension column vector with the \( n^{th} \) element being the shares of the \( n^{th} \) risky asset in \( X \).

Hence, if we denote by \( D_i \) the vector of shares of the risky assets held by investor \( i \) and by \( d_i \) (a scalar) the shares of the fund held by investor \( i \), investor \( i \)'s effective risky assets holdings are \( d_iX + D_i \).

**Return Information.** Let \( F = (f_1, f_2, \ldots, f_N)' \) be the vector of risky assets’ returns. We assume that all investors share a common uniform improper prior of \( F \), and so no
investor has prior information about any risky asset’s return.

Each investor $i$, however, receives a vector of private signals $S_i$ about asset returns. In particular, $S_i = F + \epsilon_i$, where $F$ and $\epsilon_i$ are independent; and $\epsilon_i$ and $\epsilon_j$ are also independent. Each $\epsilon_i$ is normally distributed, with mean zero and precision matrix $\Omega_i$. We assume that $\Omega_i$ is diagonal for all $i \in [0, 1]$, and so investor $i$’s private signal about asset $n$’s return is uninformative about asset $k$’s payoff.

We call investor $i$ an informed investor of asset $n$ if and only if the $n$th diagonal entry of $\Omega_i$ is strictly positive. Let the $n$-dimension column vector $\lambda$ summarize the measures of informed investors of each asset, with the $n$th element being the measure of the informed investors of asset $n$; we assume $\lambda_n \in (0, 1)$. Let Diag($\lambda$) be the diagonal matrix with the $n$th diagonal entry being the $n$th element of $\lambda$. We call an investor $i$ with precision matrix $\Omega_i = 0$ an uninformed investor. We assume that $\gamma \in (0, 1)$ fraction of investors are uninformed, so $\gamma \leq \min_j (1 - \lambda_j)$.

For simplicity, we assume that the private signals of all informed investors of asset $n$ have the same precision $\kappa_n > 0$. Let $\Omega$ be the $N \times N$ diagonal matrix with the $n$th diagonal entry being $\kappa_n$. Denote by $\Sigma$ the matrix of the average precision of private signals, we have

$$\Sigma = \int_0^1 \Omega_i di = \Omega \text{Diag}(\lambda)$$  \hspace{1cm} (1)

As is standard, the independence of the errors implies that in the economy as a whole signal errors average out, so that the equilibrium pricing function does not depend on the error realizations (though it does depend on their distributions).

**Random Supply.** Let $Z$ denote the random vector of supplies of all risky assets. We assume that $Z$ is independent of $F$ and of $\epsilon_i$ (for all $i \in [0, 1]$). We further assume that $Z$ is normally distributed with mean 0 and the precision matrix $U$. By independence of assets, $U$ is diagonal and positive definite, with the $n$th diagonal entry being $\tau_n$.

Investors commonly know all parameters except $U$. Specifically, we assume that only informed investors of asset $n$ know $\tau_n$; any uninformed investor $i$ of asset $n$ will have her own subjective prior belief about $\tau_n$ with the support $(\tau_{i,n}, \bar{\tau}_{i,n})$, where $\tau_{i,n} > \tau_{n} \geq 0$. Denote by $U_i$ the set of investor $i$’s belief about $U$, and by $u_i$ a typical element in $U_i$. We

\[7\] In the financial market, very few investors know all traded assets. Hence, it is reasonable to assume that an investor holds uninformative priors about the assets that they don’t even know the names. In the model, we still allow an investor to hold the assets that she does not know directly or indirectly through the fund.
allow different uninformed investors of a particular asset \( n \) to have different supports of their beliefs about \( \tau_n \).

**Ambiguity Aversion.** Taking the riskfree asset to be the numeraire, let \( P \) be the price vector of the risky assets. Also, let \( W_i = (w_{i1}, w_{i2}, \ldots, w_{iN})' \) be the endowed shareholdings of investor \( i \) (we assume that the aggregate endowments of shares of each stock is strictly positive; that is, \( W = \int_0^1 W_i\text{d}i \gg 0 \)). Then, investor \( i \)'s final wealth at date 1 is

\[
\Pi_i = r \left[ W_i' - (d_i X' + D_i') \right] P + (d_i X' + D_i') F. \tag{2}
\]

The first term in (2) is the return of investor \( i \)'s investment in the riskfree asset, and the second term is the total return from her investments in risky assets.

All investors are risk averse, so when all model parameters are common knowledge, at date 0 their expected utility is CARA,

\[
\mathbb{E}_i u(\Pi_i) = \mathbb{E}_i \left[ -\exp \left( -\frac{\Pi_i}{\rho} \right) \right]. \tag{3}
\]

The expectation in equation (3) is taken based on investor \( i \)'s information about asset returns. Because the common prior about asset returns is uninformative, any investor \( i \)'s information consists of the equilibrium price vector and the realization of a private information signal \( S_i \) only.

However, investor \( i \) may be subject to model uncertainty about the precisions of some assets’ random supplies, and will choose an investment strategy \( (d_i, D_i) \) to maximize the infimum of her CARA utility. Formally, each investor \( i \)'s decision problem is

\[
\max_{d_i, D_i} \inf_{u_i \in \mathcal{U}_i} \mathbb{E}_i \left[ -\exp \left( -\frac{\Pi_i}{\rho} \right) \right]. \tag{4}
\]

**Equilibrium** We are interested in a rational expectations equilibrium defined as follows.

**Definition 1** A pricing vector \( P^* \) and a profile of all investors’ risky assets holdings \( \{d_i^*, D_i^*\}_{i \in [0,1]} \) constitute a rational expectations equilibrium, if

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\(^8\)An investor’s utility in this paper differs slightly from that defined in Gilboa and Schmeidler (1989). Because any investor’s subjective prior about the precision of the random supply shock has a non-compact support, the investor maximizes the infimum, rather than the minimum, of her CARA utility among all possible precisions.
1. Given $P^*$, $(d_i^*, D_i^*)$ solves investor $i$’s maximization problem in equation \((4)\), for all $i \in [0,1]$; and

2. $P^*$ clears the market, that is,

$$\int_0^1 (d_i^* X + D_i^*) \, di = W + Z, \quad \text{for any realizations of } F \text{ and } Z. \tag{5}$$

As in the literature on rational expectations models, equilibrium prices play two roles: clearing the market and partially aggregating private information. Hence, a rational expectations equilibrium differs from a Walrasian equilibrium mainly in that the asset prices convey information about the asset payoffs to investors. This is especially important in our setting with ambiguity averse investors. Since they hold neither informative priors nor private signals about the asset payoffs, observation of asset prices is what allows them to update their beliefs to have finite conditional variances, making them willing to participate in the markets for risky assets. When the precision of the supply shock of an asset is arbitrarily close to zero, the asset price becomes uninformative, and so ambiguity averse investors will hold a zero position of the asset.

3 Ambiguity Aversion and Index Investing

In this section, we discuss ambiguity averse investors’ asset holdings when the index fund is offering the value-weighted market portfolio. We first provides a model of investor unawareness to argue that when the financial market swings dramatically, news can lead investors who are subject to model uncertainties to form extreme priors about random supply precisions. We then show that such extreme priors cause the investors to flee from the market of the assets they are ambiguity averse about. Importantly, if an investor is ambiguity averse about at least one asset, he does not hold the index fund.

3.1 News and Investor Knowledge Update

In this section, we argue that media news can cause investors to update their knowledge, and such knowledge updating may lead to extreme ambiguity aversion. We provide an intuitive argument here, and we refer the readers who are interested in a formal state space model of investor unawareness to Appendix A.

Following Li (2009), we model a state of the world as the combination of a question and the affirmative and negative answers to it. Specifically, for asset $q$, the state contains
the question “Can the random supplies in the asset $q$’s market be extremely volatile?” Before realizing the question, the answer to this question is dropped in an investor $i$’s knowledge set. That is, without realizing the question $q$, investor $i$ is at her subjective state space where she does not know the possibility of extremely volatile random supplies and, more importantly, she does not know she does not know such a possibility. In such a scenario, we say that investor $i$ is unaware of the possibility that $\tau_q = 0$, and investor $i$ has a subjective belief that the asset $q$’s random supply precision is $\tilde{\tau}_p^i > 0$.

When the market swings dramatically, newspapers and TV news discuss how volatile the market could be. Take the market swing in early February 2018 as an example. Almost all media, including The Wall Street Journal, Financial Times, CNN, and CNBC, discussed the market volatility as headlines. Such discussions will make investors realize the question about the asset about which they are uncertain. Since any investor cannot exclude the possibility that $\tau_q = 0$, the investor’s answer to question $q$ will be affirmative, and her knowledge is updated to $\tau_q^i = 0$. That is, the investor will have the lower bound of the support be zero, which implies that she becomes extremely ambiguity averse about asset $q$.

We emphasize that because of the maxmin representation of investor ambiguity aversion preference, only the lower bound of an investor’s prior support of the random supply precision matters. In particular, even though the lower bound of an investor’s prior support of the random supply precision is zero, the investor need not believe that the random supply precision is very close to zero. Indeed, the investor may still have a prior that the random supply precision is relatively high; however, if the lower bound of the prior is $\tau_q = 0$, our results hold. In Appendix A we provide an example to illustrate this point.

### 3.2 Unattractive Index Investing

Investor $i$ is risk averse, so she will not hold any non-zero position of asset $n$, unless her subjective belief of asset $n$’s payoff has a finite variance, conditional on her information. When investor $i$ is uninformed about asset $n$, however, she has neither prior information nor private information about asset $n$’s payoff. Hence, she estimates the payoff based only on the price, which partially aggregates informed investors’ private information. The informativeness of price increases in the precision of the random supply. When the precision of the random supply is arbitrarily close to zero, price becomes almost uninformative.
Investor $i$ does not know the precision of asset $n$’s random supply. In particular, when the market volatility is high, investor $i$ becomes extremely ambiguity averse, as we argue in Section 3.1. That is, investor $i$’s subjective prior about $\tau_n$ has the support $(0, \bar{\tau}_n)$. Investor $i$ may extract some information about $\tau_n$ from the price of asset $n$. However, because all random variables in our model are normally distributed, the support of investor $i$’s belief about $\tau_n$ does not change. So as she considers the worst-case scenario in making the investment decision, investor $i$ focuses on the possibility that the true $\tau_n$ is very close to 0, since in such a case, asset $n$’s price is almost uninformative.

Suppose that investor $i$ holds a non-zero position of asset $n$. As the price becomes almost uninformative, the payoff variance conditional upon price diverges to infinity. So holding a non-zero position is extremely risky in the worst-case scenario. To avoid this risk, investor $i$ optimally chooses a zero position. Proposition 1 below summarizes the argument above.

**Proposition 1** If an investor $i$ is uninformed about asset $n$, and $\tau^i_n = 0$, then investor $i$ will hold a zero position of asset $n$.

Proposition 1 emphasizes that if investor $i$ is extremely ambiguity averse about asset $n$, she will hold a zero position of such an asset, including the positions held through the index fund. Indeed, the value-weighted market portfolio in our model is equivalent to the portfolio $W$, which includes a deterministic positive number of shares of each asset. Hence, when the index fund offers the value-weighted market portfolio, holding a non-zero position of the index fund means holding a non-zero position of all assets (unless the investor intentionally hold the opposite positions of some assets directly to offset the holdings through the index fund). Therefore, if investor $i$ is extremely ambiguity averse about at least one asset, holding a non-zero position of the index fund will bring her infinitely high risks. As a result, investor $i$ will avoid the index fund, if she is extremely ambiguity averse about at least one asset.

**Corollary 1** When the index fund offers the value-weighted market portfolio $W$, an investor $i$ will not hold it if $\tau^i_n = 0$ for at least one $n \in Q$.

Corollary 1 helps explain why the stock-market tumble in February 2018 sent investors fleeing index funds and ETFs. At the beginning of the market swing, media news reported huge volatilities of the market, making investors review the question how volatile the market of each stock could be. This leads to their extreme ambiguity
aversion. Investors then avoided the stocks about which they are ambiguity averse and index funds or ETFs that follow the value-weighted market portfolio.

Corollary 1 also identifies a shortcoming of the traditional index, the value-weighted market portfolio. In particular, since the stocks in the value-weighted market portfolio are weighted by their market capitalizations only, many other important factors, such as volatility, are not included. Therefore, when investors are ambiguous about the volatilities of stocks, the index fund that offers the value-weighted market portfolio becomes less attractive, and so it cannot help encourage investors to participate in the financial market through it and thus increase the financial market efficiency.

4 Smart-beta Design and Full Participation

Since traditional index investing is less attractive when investors are ambiguity averse, it becomes interesting and important to design a new index that could attract ambiguity averse investors. Such a new design is one of smart beta strategies, which receive increasing popularity recently. We design an index as

\[ X = \left[ I + \frac{1}{\rho^2} (\Sigma U)^{-1} \right]^{-1} W \]  

and name it the “risk-adjusted market portfolio” (RAMP). We assume that a fund that offers RAMP is publicly available to all investors. Ambiguity averse investors do not know \( U \) and hence do not know the exact composition of \( X \). However, all investors commonly know \( X \) as a function of \( U \) that is specified in equation (6). This is consistent with the practice in the smart beta fund industry.

We show that in equilibrium, such a smart beta fund induces all investors to participate in all asset markets. We also show how investors will allocate their initial wealth among the fund, their direct holdings of risky assets, and the riskfree asset, even when investors do not know the exact composition of the fund. We then argue that the full participation with the fund follows from the Information Separation Theorem that applies in financial markets without a fund and without ambiguity aversion.

Before the equilibrium analysis, we want to emphasize that RAMP echoes the “defensive investment strategy” in practice. Specifically, RAMP differs from the ex-ante

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9One evidence is that by the end of December 2017, smart beta funds have hit the $1 trillion of assets milestone.
endowed market portfolio \( W \) in that it is also influenced by the assets’ idiosyncratic volatilities \( (\Sigma U)^{-1} \). Investigating equation (6) carefully, we find that assets’ idiosyncratic volatilities enter \( \text{RAMP} \) negatively. That is, \( \text{RAMP} \) includes less shares of more volatile assets. Therefore, offering \( \text{RAMP} \) will be a passive asset management strategy that can largely reduce the risk of loss.

4.1 An Equilibrium with Full Participation

In the model with a fund that offers the portfolio \( X \), any investor \( i \)’s investment strategy \((d_i, D_i)\) leads to effective asset holdings \( d_iX + D_i \). Proposition 2 below then shows that in an equilibrium, all investors, including those who are extremely ambiguity averse, hold exactly one share of the fund and thus participate in all asset markets.

**Proposition 2** In the model with a passive fund that commits to offering the portfolio \( X \) specified in equation (6), there is an equilibrium in which

1. All investors will buy one share of the passive fund, and so \( d_i^* = 1 \) for all \( i \in [0, 1] \);
2. Any investor \( i \) will hold an extra portfolio \( \rho \Omega_i (S_i - rP) \); and
3. For any given \( F \) and \( Z \), the equilibrium price is

\[
P = \frac{1}{r} \left[ F - \frac{1}{\rho} \left( \Sigma + \rho^2 \Sigma U \Sigma \right)^{-1} W - \frac{1}{\rho} \Sigma^{-1} Z \right].
\] (7)

The intuition of Proposition 2 arises from a new separation theorem that applies in the setting without ambiguity aversion and the passive fund. Since such an intuition is not straightforward, we discuss it in detail in Section 4.2. In the rest of this subsection, we discuss some properties of the equilibrium characterized in Proposition 2.

First, in equilibrium, uninformed investors are indifferent between holding the passive fund and not participating in asset markets. When contemplating a position in the passive fund, uninformed investors believe that the fund’s holdings of all assets are very close to zero, when the precisions of all assets’ supply shocks are almost zero. Hence, by holding the fund, the infima of the uninformed investors’ utilities are the same as the utility from not participating.

Nevertheless, the only reasonable conclusion is that an uninformed investor who is ambiguity averse and otherwise-rational holds the passive fund (when other investors follow equilibrium behavior). In particular, since any uninformed investor \( i \)’s subjective
prior about the precision of any asset $n$’s supply shock is $(0, \tau_n^i)$, she knows for sure that $\tau_n > 0$. For any given $\tau_n$, holding the passive fund is strictly better than not participating. So while investor $i$ has the same (infimum) utility ex ante, once $\tau_n$ is realized, she knows she will be strictly better off holding the passive fund. So only holding the passive fund is time-consistent.

Given this, it is not surprising that there are ways to express preferences that capture formally the fact that an investor is not indifferent, even ex ante, as to whether to invest in the fund. This can be done by considering perturbations of the model. Consider a sequence of perturbed models in which all uninformed investors’ priors about $U$ have strictly positive lower bounds. When the perturbed lower bounds converge to zero, the perturbed models converge to our original model. In any of these perturbed models, strictly positive lower bounds of investors’ priors about $U$ imply that holding the passive fund is investor $i$’s unique best response to other investors’ strategies in an equilibrium, as shown in the proof of Proposition 2. Hence, when investor $i$’s prior knowledge about model parameters switches a little bit, investor $i$ strictly prefers to hold the passive fund; then, by the revealed preference, investors would like to choose the passive fund in the original model, given all other investors’ strategies. Therefore, the equilibrium characterized in Proposition 2 is *near strict*, an equilibrium refinement concept defined by Fudenberg, Kreps, and Levine (1988).

Second, while investors have heterogeneous priors about $U$ and thus different beliefs about the fund’s composition, they all hold exactly one share of the fund. Take two investors, Lucy and Martin, for an example. Lucy is uninformed and believes that $\tau_n$ (for any $n$) could be arbitrarily close to 0; Martin does not receive private information about asset payoffs either, but he knows the true precisions of all supply shocks. According to Proposition 2, both Lucy and Martin will hold one share of the passive fund, but neither Lucy nor Martin holds any extra positions because they don’t have any private information about assets’ payoffs. Hence, Lucy and Martin are effectively holding the same portfolio. So differences in investors’ holdings arise only from differences in their information signals, not from differences in their model uncertainty or ambiguity aversion.

Third, Proposition 2 shows the importance of risk sharing among investors in their

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10 Formally, a strategy profile $\sigma$ is *near strict* in a game $\Gamma$ if there exists a sequence of games $\{\Gamma^n\}$ and a sequence of strategy profiles $\{\sigma^n\}$, such that (i) $\lim_n \Gamma^n = \Gamma$; (ii) for each $n$, $\sigma^n$ is a strict equilibrium of $\Gamma^n$; and (iii) $\lim_n \sigma^n = \sigma$. Here, a *strict equilibrium* is an equilibrium in which any investor’s strategy is her unique best response to all other investors’ strategies in the equilibrium.
optimal portfolio choices. Specifically, consider an investor who faces model uncertainty about a subset of traded assets, and views the return distributions as exogenous. Even if she can indirectly trade those assets through a passive fund, it may not be optimal for her to do so, because she cannot calculate the fund’s expected return and risk. Therefore, arguments based on the incentive of individuals to diversify do not, under radical ignorance, justify holding of the fund. In contrast, in our equilibrium setting, an investor optimally holds the fund, given her belief that other investors will also do so (together with their direct portfolios). Hence, she is willing to hold the fund too, which achieves the benefit of optimally sharing risk with other investors.

The fact that equilibrium rather than just diversification considerations are crucial for the full participation result can be seen more concretely by considering the off-equilibrium possibility that other investors trade in a fashion that causes asset prices to be almost uninformative. In such a scenario, an ambiguity averse investor (Lucy) would not hold the passive fund, because RAMP would be perceived as extremely risky. Specifically, suppose that the off-equilibrium trading strategy profile of other investors leads asset price informativeness to converge to zero as the random supply shock precisions go to zero. This convergence could be even faster than the convergence of the asset positions in RAMP to zero. Hence, taking any non-zero position of the passive fund will give Lucy infinite risks in the worst-case scenario, since she believes that random supply shock precisions could be extremely close to zero. Therefore, Lucy will not hold the fund. In contrast, in such a case, an uninformed investor who knows the supply shock precisions may still hold asset positions that are bounded away from zero, because the investor can extract asset payoff information from asset prices, resulting in finite risk.

Proposition 2 more broadly suggests that the reason why actual investors often fail to diversify goes beyond investor ambiguity aversion. In particular, for an investor to hold the fund, all other investors need to behave according to the prescribed equilibrium strategy profile. If imperfectly rational investors reason about possible portfolios based solely on partial equilibrium risk and return arguments, portfolios containing assets that investors are ambiguous about might seem extremely risky (or in the limiting case, infinitely risky). Proposition 2 shows that, owing to equilibrium considerations, even ambiguity averse investors, if otherwise rational, will hold such assets. But actual investors may not understand the equilibrium reasoning which underlies this result.
4.2 The Information Separation Theorem

Proposition 2 is a surprising result. It is true that investors are willing to hold the fund because the fund knows the precisions of all assets’ supply shocks. However, the result is not driven by any overall informational superiority of the fund over investors. Off equilibrium, if few informed investors trade, an ambiguity averse investors will not hold the fund, because the prices are still rather uninformative.

Nor is the fund offering investors great safety. For the strategy profile described in Proposition 2 to be an equilibrium, the fund has to offer the portfolio \( X \), specified in equation (6). We verify that if the fund offers another portfolio

\[
X' = \left[ I + \frac{1}{\rho^2} \left( \Sigma U^2 \right)^{-1} \right]^{-1} W,
\]

which is also a function of \( U \) and will converge to 0 as \( U \) converges to zero, ambiguity averse investors will not hold the fund and will refrain from participating in the markets of the assets they are uncertain about. This is because when \( U \) converges to 0, \( X' \) converges to 0 much slower than \( U \). But the conditional variance of holding any non-zero positions diverges to infinity at the same speed as \( U \) converges to 0. Hence, the risk of holding \( X' \) diverges to infinity as \( U \) converges to 0, implying that holding the passive fund is extremely risky for uninformed investors in the worst-case scenarios.

Hence, the intuition of Proposition 2 must go beyond the passive fund’s superior knowledge about the financial markets. We now provide greater insight into this result based upon a new separation theorem for financial markets with asymmetric information, but without model uncertainty or funds.

We now modify the model described in Section 2 by assuming that \( U \) is common knowledge among all investors and that there is no passive fund. Then the model is a traditional rational expectations equilibrium model with multiple risky assets, analyzed by Admati (1985). Proposition 3 characterizes a linear rational expectations equilibrium model with multiple risky assets, analyzed by Admati (1985). Proposition 3 characterizes a linear rational expectations equilibrium model with multiple risky assets, analyzed by Admati (1985).
and shows investors’ optimal risky assets holding when all parameters are common knowledge.

**Proposition 3** In the model whose parameters are all common knowledge among investors, there exists an equilibrium with the pricing function

\[ P = B^{-1} [F - A - CZ], \]  

where

\[ A = \frac{1}{\rho} \left[ \rho^2 (\Sigma U \Sigma) + \Sigma \right]^{-1} W \]  

\[ B = rI \]  

\[ C = \frac{1}{\rho} \Sigma^{-1}. \]

Any investor i’s risky asset holding is

\[ D_i = \left[ I + \frac{1}{\rho^2} (\Sigma U)^{-1} \right]^{-1} W + \rho \Omega_i (S_i - rP). \]  

Owing to supply shocks, asset prices are not fully revealing, so information asymmetry persists in equilibrium and different investors have different asset holdings. An investor’s asset holding is the sum of two components. The first term in equation (12),  

\[ \left[ I + \frac{1}{\rho^2} (\Sigma U)^{-1} \right]^{-1} W \]  

is just RAMP, which is deterministic.

The second component of any investor’s risky asset holding, the second term in (12), is what we call information-based portfolio. This position, \( \rho \Omega_i (S_i - rP) \), consists of extra holdings in the securities about which the investor has information. Investor i holds such an extra position of an asset n if and only if the \( n \)th diagonal entry of \( \Omega_i \) is \( \kappa_n > 0 \). This suggests that any investor i holds direct positions of a risky asset because possessing an informative signal about such an asset reduces its conditional volatility (independent of the signal realization). Investor i’s direct positions of a risky asset also come from her speculation, which is taken to exploit superior information. Different investors, even if they are informed about asset n, hold different speculative portfolios, because they receive heterogeneous private signals.
A critical feature of any investor’s equilibrium asset holdings in equation (12) is that its two components are influenced differently by investors’ information sets. The first component, \( RAMP \), is formed based only on the information that the investor gleans from asset prices; it is independent of the investor’s private information. In contrast, the second component, the information-based portfolio, can be formed based only on the investor’s own private information; it is independent of the information content of the market price. Since the supply shock precisions do not affect the distributions of the private signals, it follows that the information-based portfolio is independent of the supply shock precisions. The reason for this independence is that private signals and the random supply shocks are normally distributed, so that the conditional expectation of the asset payoffs is linear in private signals and the price signal\(^{12}\).

This independence implies a new separation theorem under asymmetric information.

**Theorem 1 (The Information Separation Theorem)** When the characteristics of all assets are common knowledge, equilibrium portfolios have three components: a deterministic risk-adjusted market portfolio (\( RAMP \)); an information-based portfolio based upon private information and equilibrium prices but no extraction of information from prices; and the riskfree asset.

This separation turns out to be important for understanding market participation and asset prices when ambiguity averse investor face model uncertainty, and can hold the risky assets through a passive fund, as analyzed in Subsection \([4.1]\). In particular, information separation implies in that setting that the unknown model parameter, the noise supply shock precision, does not affect the information-based portfolio, so that investor holdings outside their fund holdings can be analyzed simply.

Theorem\(^{14}\) indicates that investors can form an optimal portfolios in separate steps: (1) buy one share of \( RAMP \); (2) buy the information-based portfolio using only private information, not the information extracted from price; and (3) put any left-over funds into the riskfree asset. This separation theorem derives from market equilibrium as well as optimization considerations. This differs from those (non-informational) separation theorems in the literature that are based solely on individual optimization arguments\(^{13}\).

\(^{12}\)Vives (2008) derives investors’ equilibrium asset holdings in a single-asset environment with a normal prior and zero aggregate endowment. Therefore, his result cannot be directly used in our analysis when investors are ambiguity averse about some assets.

\(^{13}\)It may seem puzzling that none of the three portfolio components depend on the information that an investor extracts from price. How then does this information enter into the investor’s portfolio decision?
In our model, the fund can provide \textit{RAMP} because it knows all the model parameters, and \textit{RAMP} does not include any investor’s private information. Meanwhile, the information-based portfolio is exactly the same as the direct holdings of the risky assets in Proposition\textsuperscript{2}. To form the information-based portfolio, an investor does not need to extract information from the equilibrium price: she can treat the equilibrium prices as given parameters, and solve for the information-based portfolio from her CARA utility maximization problem as in a partial equilibrium model.

The Information Separation Theorem provides the intuition of investors’ equilibrium investment strategies in the setting with model uncertainties. Consider the model in which investors are uncertain about the precisions of some assets’ supply shocks. For each possible world $\mathbf{U}_i \in \mathbf{U}$, investor $i$ can solve her optimal risky assets holdings, assuming that the equilibrium pricing function is the one in equation\textsuperscript{7} with $\mathbf{U}$ being $\mathbf{U}_i$. Importantly, because all other investors are holding one share of the fund and their own direct information-based portfolio, they are effectively holding the risky assets as in the world with $\mathbf{U}_i$ being common knowledge. Therefore, in the possible world $\mathbf{U}_i$, the market clearing condition implies that the pricing function is the one specified in equation\textsuperscript{7} with $\mathbf{U}$ being $\mathbf{U}_i$. That is, investor $i$’s belief about the pricing function is correct. So, she would like to hold the risky assets as in the world $\mathbf{U}_i$. Such risky assets holdings can be implemented by holding one share of the passive fund and her information-based portfolio, so investor $i$ would like to use the investment strategy in Proposition\textsuperscript{2}.

Furthermore, investor $i$ is still uncertain about $\mathbf{U}$, so holding the risk-adjusted market portfolio through holding one share of the fund is strictly preferred.

In the above, investor $i$ chooses the investment strategy to maximize her expected CARA utility for any fixed possible world (given that all other investors trade according to the prescribed strategy profile). Here, we indeed implicitly assume that investor $i$ has a min-max utility. However, because investor $i$’s optimal investment strategy, holding one share of the fund and her own information-based portfolio, is a constant across all possible worlds, her max-min utility is the same as her min-max utility. That is, a strong max-min property holds in the equilibrium, and hence, in our model with investor $i$ having max-min utilities, holding one share of the fund and the information-based portfolio is also investor $i$’s optimal investment strategy.

\textsuperscript{2}The answer is that \textit{RAMP} is optimal precisely because of the ability of investors to extract information from price. As mentioned before, \textit{RAMP} is deterministic; it does not depend on the private signals. But the fact that \textit{RAMP} is an optimal choice is true only because investors update their beliefs based on price. So the optimal portfolio choice is indeed influenced by such information extraction.
The argument above shows how the Information Separation Theorem helps understand the full participation of ambiguity averse investors in an equilibrium. Indeed, the same argument can also be applied when investors are unaware of some assets or when investors have heterogeneous risk tolerances. In the online appendix, we extend our model to allow for investor unawareness (defined as a diffuse prior over the parameter values that characterize the capital market) or for heterogeneous risk tolerances. We find that in each of these extensions, there exists an equilibrium with full participation.

5 CAPM Pricing with a RAMP Fund

Propositions 2 and 3 indicate that the model with ambiguity aversion and a fund that offers RAMP has an equilibrium in which investors’ effective risky assets holdings are exactly the same as in the rational expectations equilibrium in the setting without model uncertainties. Therefore, the fund induces full participation even with ambiguity aversion. It can also reduce asset risk premia, because in the equilibrium, uninformed investors are sharing risks with informed ones.

In order to analyze the effect of the fund on risk premia, we first return to the setting without model uncertainty. In such a model, the supply shocks make the asset prices in equilibrium imperfectly revealing, and so in the equilibrium, there are information asymmetries among investors and different risky asset holdings. Hence, the setting is very different from the classic CAPM setting, which assumes identical beliefs and has the implication that all investors hold the same risky asset portfolio.

Since holding the market is equivalent to the CAPM pricing relation, it might seem that in our setting there would not be a way to identify a portfolio that prices all assets and is identifiable ex ante based upon publicly available information. Nevertheless, even with information asymmetry, we identify an efficient portfolio in the model and therefore an implementable version of the CAPM pricing relationship.

From Proposition 3, we know that in the setting without model uncertainties, investors hold RAMP as a common component of their holdings. Therefore, it is natural to consider RAMP as a candidate for CAPM pricing. From equation (8), the equilibrium pricing function is

\[ P = \frac{1}{r} \left[ F - A - \frac{1}{\rho} \Sigma^{-1} Z \right], \]

where \( A = \frac{1}{\rho} \rho^2 (\Sigma U \Sigma) + \Sigma^{-1} W. \)
Given any realized equilibrium price $P$, the volatility of asset payoffs derives from the supply shock only. Let $\text{diag}(P)$ be an $N \times N$ diagonal matrix, whose off-diagonal elements are all zero and whose $n^{th}$ diagonal element is just the $n^{th}$ element of the vector $P$. Generically, as no asset has a zero price, $\text{diag}(P)$ is invertible. Then, by the definition of $\text{diag}(P)$,

$$\text{diag}(P)^{-1}P = 1,$$  \hspace{1cm} (14)

where $1 = (1, 1, \ldots, 1)'$. From the equilibrium pricing (equation (13)), we have

$$\text{diag}(P)^{-1}E(F) - r1 = \text{diag}(P)^{-1}A.$$  \hspace{1cm} (15)

Here, $E(F)$ is the expected payoff conditional on the equilibrium price. The LHS of equation (15) is just the vector of the risky assets’ equilibrium risk premia.

Given a realized equilibrium price, the risk-adjusted market portfolio $X$ has value $P'X$. Then the vector of the weights of risky assets in the risk-adjusted market portfolio is

$$\omega = \frac{1}{P'X} \text{diag}(P)X.$$  

Hence, conditional on the price $P$, the difference between the expected return of $\text{RAMP}$ and the riskfree rate is

$$E(R_X) - r = \omega' \text{diag}(P)^{-1}E(F) - r$$

$$= \frac{1}{P'X} X' \text{diag}(P) \text{diag}(P)^{-1} (A + rP) - r$$

$$= \frac{1}{P'X} X' A,$$  \hspace{1cm} (16)

where the expectations are all conditional on the equilibrium price.

The variance of $\text{RAMP}$ is

$$\text{V}(R_X) = \mathbb{E} \left[ \left( \omega' \text{diag}(P)^{-1} CZ \right) \left( \omega' \text{diag}(P)^{-1} CZ \right)' \right] = \left( \frac{1}{P'X} \right)^2 X' CU^{-1} CX,$$  \hspace{1cm} (17)

and the covariance between all risky assets and $\text{RAMP}$ is

$$\text{Cov}(R, R_X) = \frac{1}{P'X} \text{diag}(P)^{-1} CU^{-1} CX.$$  \hspace{1cm} (18)

Let $\alpha$ be the CAPM alpha. From equations (15)-(18), and since $X = \rho (CU^{-1} C)^{-1} A$, we have the following proposition.
Proposition 4 (Risk Premia with Supply Shocks) In the model with all parameters being common knowledge, asset risk premia satisfy the CAPM where the relevant market portfolio for pricing is the risk-adjusted market portfolio.

This result may seem surprising, since investors have heterogeneous asset holdings, and since the portfolios held by informed investors are not mean-variance efficient with respect to the public information set. Nevertheless, in equilibrium, there are no extra risk premia incremental to those predicted by the CAPM using RAMP.

The CAPM pricing relation using RAMP is equivalent to the assertion that RAMP is mean-variance efficient conditional only on asset prices. This efficiency can be seen from the utility maximization problem of an investor who is uninformed about all assets. Such an investor balances the expected returns and the risks of her holdings, and her information consists of the equilibrium price only. In equilibrium, such an investor holds RAMP, implying that RAMP is mean-variance efficient conditional only on equilibrium prices.

Privately informed investors also hold RAMP as a component of their portfolios; this is the piece that does not depend upon their private signals (except to the extent that their signals are incorporated into the publicly observable market price). In addition, they have other asset holdings taking advantage of the greater safety of assets they have more information about, and for speculative reasons based upon their private information. RAMP is not mean-variance efficient with respect to their private information sets, but it is efficient with respect to the information set that contains only publicly available information.

A very different version of the CAPM has been derived in somewhat similar model setups (see, for example, Easley and O’Hara (2004), Biais, Bossaerts, and Spatt (2010), and the online appendix of Van Nieuwerburgh and Veldkamp (2010)). In these models, the market portfolio for CAPM pricing is the ex-post total supply of the risky assets, the sum of the endowed risky assets and the random supply of risky assets ($W + Z$ in our model). This market portfolio is mean-variance efficient conditional on the average investor’s information set, and so the CAPM return-covariance relation holds from the perspective of the average investor. The version of the CAPM presented in Proposition 4 differs in that the pricing portfolio is determined ex ante (prior to the realization of the random supply shocks) and that risk premia are conditional only upon the public information set (market prices). This makes the market portfolio more directly observable to an econometrician.
In the model whose parameters are all common knowledge among investors, RAMP is a natural candidate for the CAPM pricing portfolio, because it is the common component in all investors’ risky asset holdings. We show that RAMP is mean-variance efficient unconditional on any investor’s private information. Therefore, the CAPM security market line relation holds without conditioning on private information, with respect to RAMP. One of the further contributions here is to establish that increasing information asymmetry, and its effect on investor participation, does not clearly predict whether there will be an increase versus decrease in risk premium.

We are now in a position to see how a passive fund that offers RAMP affects asset risk premia in the setting with model uncertainty. Proposition 2 shows that in the model where investors are uncertain about the precisions of asset supply shocks, they all hold one share of the passive fund. Therefore, the passive fund makes assets’ risk premia satisfy the CAPM, even if investors have heterogeneous information and are uncertain about different model parameters. Corollary 2 presents this even more surprising result.

**Corollary 2** In the model where investors are uncertain about the precisions of some assets’ supply shocks, and a passive fund offering portfolio X specified in equation (6), asset risk premia satisfy the CAPM using X as the pricing portfolio.

### 6 Implementation of the Passive Fund

We have designed a new index, RAMP, such that if a passive fund commits to following it, there exists an equilibrium in which all investors hold exactly one share of the fund. Since RAMP includes positive positions of all traded assets, investors participate in all assets markets via the fund in the equilibrium. However, the fund needs to have full knowledge about the parameter values of the capital market to construct RAMP. Hence, if there is no single individual in the economy who knows all the parameter values, then this knowledge is dispersed. As we have assumed, for each asset, there is a positive measure of investors who know the parameter value for that asset. In this section, we show that the passive fund can still be implemented via a fund of funds.

Suppose that the set of all traded assets can be partitioned into M subsets. In the partition $j$, there are $m_j \geq 1$ assets. We assume that there is a positive measure of investors, who know all parameters about assets in partition $j$ but do not have any private signals about the payoffs of such assets. We call these investors “Group $j$ uninformed
investors.” (In the extreme case, $M = N$, and so, in each partition, there is only one asset. Then, we are in the setting described in Section 2.)

We consider the following equilibrium. Each of the Group $j$ uninformed investors commits to offering a portfolio $Y_j$. Here, $Y_j$ can be seen as a “local” fund, which includes only assets in partition $j$. For each asset $n$ included in asset partition $j$, $Y_j$ includes exactly the same position as in the portfolio $X$.

First, the fund fee will be zero in an equilibrium. Since there are infinitely many funds who are committing to offering $Y_j$, the $j$th local fund industry is perfectly competitive. Hence, the fund fee should be the same as the marginal cost of offering $Y_j$, which is zero.

Second, and more importantly, as required in the existing index fund industry, all local funds are required to disclose their asset holdings at the end of the period. Then, if a local fund of Group $j$ that deviates from $Y_j$, its portfolio holding will differ from other Group $j$ local funds portfolio holdings. Hence, such a deviation is observable ex post and verifiable. Ex post, once a fund’s deviation is detected, We assume that the fund will be heavily punished or incur a large reputation cost. It follows that no local fund is willing to deviate from its commitment to invest in $Y_j$.

Finally, any investor will first buy one share of the Group $j$ local fund, for each $j$. By doing so, any investor will form an asset holding $(Y'_1, Y'_2, \ldots, Y'_M)' = X'$, which is exactly the passive fund specified in equation (6). Then, investors will hold their own information-based portfolios. Obviously, investors are effectively holding one share of the passive fund and their own information-based portfolios, which are their optimal investment strategies in the equilibrium described in Proposition 2.

7 Concluding remarks

A leading explanation for nonparticipation puzzles is investor ambiguity aversion. This literature focuses on direct trading of assets by investors in the face of model uncertainty. We study here whether ambiguity aversion can still solve the puzzle when an appropriately designed passive fund is available run by a manager who observes the model parameters that investors are uncertain about (though the manager does not observe any private information signals about fundamentals). We show that when there is a passive fund that offers the risk-adjusted market portfolio (RAMP), all investors prefer to hold the fund and thus participate in all asset markets, even if they do not know the
passive fund’s composition. This conclusion arises from applying a new portfolio information separation theorem which holds in a setting without model uncertainty and implies that assets’ equilibrium risk premia conditional only on public information satisfy the CAPM, with the passive fund (i.e., \textit{RAMP}) as the pricing portfolio.

Since ambiguity aversion does not, by itself, explain the limited market participation puzzle, what does? Dimmock et al. (2016) find that ambiguity aversion is associated with lower stock market participation. Their tests do not distinguish participation via funds versus direct investment in individual stocks, so their finding does not speak specifically to how ambiguity aversion affects the choice between these alternatives. However, our findings suggest that to understand the Dimmock et al. evidence, it is important to investigate what additional frictions or irrationalities might contribute to nonparticipation and prevent the passive fund solution to ambiguity aversion from working perfectly.

With regard to frictions, there could be heavy trading costs, though as discussed in the introduction, it seems unlikely that this is the full explanation for nonparticipation puzzles.

A second possibility is that existing funds may not offer \textit{RAMP}, so that our solution to nonparticipation is unavailable to investors. This may be because, prior to this paper, it was not understood that \textit{RAMP} solves the problem. (Most existing index funds offer proxies for the value weighted market rather than \textit{RAMP}.) Each active fund provides a different portfolio strategy, so at most only one of these (and probably none) closely matches \textit{RAMP}. This possibility suggests a policy implication of our approach: that fund firms should introduce \textit{RAMP} portfolios as a service to ambiguity averse investors who can benefit from diversification and risk sharing.

A third, and related, possibility derives from agency problems. If investors cannot be sure that a fund manager who claims to hold \textit{RAMP} really does, then our conclusion of full participation does not follow. Whether disclosure policies could address agencies problems is an open question.

Finally, it could be that imperfect rationality explains the puzzle. This possibility has a bearing on the argument that providing investors with better information might encourage market participation. Unfortunately, when there is psychological bias, providing more information could make the problem worse. More information does \textit{not} always debias decision makers, since extraneous information can be distracting or overwhelming. For example, providing extensive information about numerous assets could
make investors feel less competent about evaluating their investments. This could exacerbate ambiguity aversion. Similarly, such information might push investors toward the use of simple judgment heuristics such as narrow framing, which is another leading possible explanation for nonparticipation.

Other forms of irrationality provide a further possible reason for nonparticipation. Our finding that the availability of RAMP induces full participation is based on the premise that ambiguity aversion is their only mistake. However, investors may make psychological errors other than those that come just from ambiguity aversion. If, for example, some investors do not perfectly understand the concept of a market equilibrium, they may regard participation as too risky even if RAMP is available. Furthermore, if it is common knowledge that some investors will fail to hold the fund, the equilibrium will be different. We can no longer conclude that other ambiguity averse investors (even those who do understand the concept of equilibrium) will be willing to participate via a fund.

This suggests a further normative implication, that it is valuable to educate investors more deeply about the concept of market equilibrium. Specifically, as our model makes clear, in equilibrium participation can be much safer than it might otherwise seem. It would be possible to explain to investors at a nontechnical level why holding even assets that one knows little about can sometimes improve reward/risk ratios. In particular, it is intuitive that in equilibrium prices need to be set so that even a very risky asset becomes attractive enough for some investors to want to hold it. Finally, even if there is some aspect of the world that an investor feels she knows almost nothing about, it is intuitive that a trustworthy agent (the caveat ‘trustworthy’ being crucial) could choose on the investor’s behalf the weights on the affected assets that the investor would herself have chosen if she knew what the agent knows.
A  A Formal Model of Investor Knowledge Updates

In this appendix, we introduce a formal model of investor unawareness. Following [Li (2009)] we denote by $q$ the question “For asset $q$, is $\tau_q = 0$?” We further denote by $1_q$ and $0_q$ the affirmative answer and the negative answer to question $q$, respectively. Then, the full state space can be written as

$$B = \prod_{q \in Q} \{1_q, 0_q\} \times \{\Delta\},$$

(19)

where $\Delta$ stands for “cogito ergo sum,” and $\{\Delta\}$ is an investor’s subjective state space at the state $\{1_q, \Delta\}$.

Consider an investor, Lucy, who is subject to model uncertainty. Denote by $Q_L$ the set of assets that Lucy knows. That is, for any $q \in Q_L$, Lucy knows $\tau_q > 0$ and, therefore, her information partition about asset $q$ is $\{(1_q, \Delta), (0_q, \Delta)\}$. Obviously, since $q \in Q_L$, Lucy knows the true state is $(0_q, \Delta)$.

However, for any asset $q \in Q \setminus Q_L$, Lucy does not know $\tau_q$. Therefore, for such an asset, she has the state space $\{\Delta\}$, where the answer to question $q$ is dropped. That is, because Lucy does not realize question $q$, she is unaware of the answer to such a question. Hence, having the subjective state space $\{\Delta\}$, Lucy does not know the possibility of $\tau_q = 0$. And more importantly, she does not know she does not know the possibility of $\tau_q = 0$.

In Section 3.1, we argue that before the news in Television and Newspapers, the investors do not realize question $q$, and therefore they don’t know they don’t know that the random supply of asset $q$ could be extremely volatile. In such a case, we say that the investors are unaware of the possibility that the random supplies of asset $q$ could be extremely volatile.

The discussion about the market volatility in the media will then make Lucy’s knowledge about asset $q$ update from $\{\Delta\}$ to $\{(1_q, \Delta), (0_q, \Delta)\}$. Since Lucy, after realizing the question $q$, cannot exclude the possibility that $\tau_q = 0$, Lucy’s subjective knowledge is $(1_q, \Delta)$ where her belief support about $\tau_q$ becomes $(0, \bar{\tau}_q)$.}

We now provide an example of Lucy’s prior about $\tau_q$, the precision of the random supply of asset $q$ about which she is subject to model uncertainty. This example helps illustrate that despite of the specific distribution, if the lower bound of support of an investor’s prior about $\tau_q$ is 0, the investor is extremely ambiguity averse.
Consider the following sequence of the priors of Lucy:

\[ \tau_p^L \begin{cases} \sim U(0, 1), & \text{with probability } \frac{1}{k}; \\ = 1, & \text{with probability } 1 - \frac{1}{k}. \end{cases} \]

In this example, along the sequence, the lower bound of the support of Lucy’s prior about \( \tau_q \) is 0, while the mean is

\[ \frac{1}{2K} + \left( 1 - \frac{1}{K} \right) = 1 - \frac{1}{K}. \]

Obviously, as \( K \) increases, such a mean is monotonically increasing and converges to 1. That is, when \( K \) becomes larger, Lucy believes that \( \tau_q \) is more and more likely to be close to 1. However, in the sequence, the lower bound of her prior support is always 0. In the paper, we show that to Lucy, \( \tau_q \to 0 \) is the worst-case scenario. Therefore, only the lower bound matters, and the specific distribution of Lucy’s prior does not matter for Lucy’s investment decision.
B Omitted Proofs

Proof of Proposition 1

Because investor \( i \) is uniformed about asset \( n \), by assumption, \( \kappa_i = 0 \). Hence, investor \( i \)'s only information about the distribution of asset \( n \)'s payoff is its price, which may partially aggregate informed investors’ private signals. Suppose the uninformed investors’ aggregate demand for asset \( n \) is \((1 - \lambda_n)D(p_n)\). Since uninformed investors do not observe \( \tau_n \), \( D(p_n) \) is not a function of \( \tau_n \).

Given any \( P \) and any \( \tau_n \in (0, \bar{\tau}_i) \), we derive investor \( i \)'s expected utility conditional on \( P \) as follows. Suppose asset \( n \)'s pricing function in a linear equilibrium is

\[
f_n = a + bp_n + cz_n,
\]

where \( a \), \( b \), and \( c \) are undetermined parameters. Since informed investors know \( \tau_n \), they can extract information from the price without any ambiguity. Therefore, any informed investor \( j \)'s demand is

\[
D_j = \rho \left[ \kappa_n s_j + \frac{\tau_n}{c^2} a + \frac{\tau_n}{c^2} (b - r)p_n - r\kappa_n p_n \right].
\]

Then, the informed investors’ aggregate demand will be

\[
\lambda_n \rho \left[ \kappa_n f_n + \frac{\tau_n}{c^2} a + \frac{\tau_n}{c^2} (b - r)p_n - r\kappa_n p_n \right].
\]

Then, the market clearing condition implies that

\[
\lambda_n \rho \left[ \kappa_n f_n + \frac{\tau_n}{c^2} a + \frac{\tau_n}{c^2} (b - r)p_n - r\kappa_n p_n \right] + (1 - \lambda_n)D(p_n) = w_n + z_n.
\]

Matching the coefficient of the market clearing condition and the pricing function, we have

\[
a = \frac{w_n}{\lambda_n \kappa_n \rho} - \frac{\tau_n}{c^2 \kappa_n} a \\
bp_n = -\frac{(1 - \lambda_n)D(p_n)}{\lambda_n \kappa_n \rho} - \frac{\tau_n}{c^2 \kappa_n} (b - r)p_n + rp_n \\
c = \frac{1}{\lambda_n \kappa_n \rho}
\]

Therefore, for any given \( \tau_n \in (0, \bar{\tau}_i) \), conditional on the price \( P_n \), \( |\mathbb{E}(f_n - rp_n|p_n)| < +\infty \). On the other hand, the variance of asset \( n \)'s payoff conditional on \( p_n \) is

\[
\mathbb{V}(f_n|p_n) = c^2 \tau_n^{-1},
\]

30
which diverges to $+\infty$ as $\tau_n$ goes to 0. Hence, any non-zero position $D_i$ of asset $n$ brings investor $i$ a utility
\[
- \exp \left( -\frac{1}{\rho} w_i r p_n \right) \exp \left[ -\frac{1}{\rho} D_i \mathbb{E} (f_n - r p_n | p_n) + \frac{D_i^2}{2\rho^2} \mathbb{V} (f_n | p_n) \right],
\]
which goes to $-\infty$ as $\tau_n$ goes to 0. Therefore, if investor $i$ is uninformed about asset $n$, and $\tau^i_n = 0$, investor $i$ refrains from participating in the market of asset $n$.

**Q.E.D.**

**Proof of Proposition 2**

We first verify that the market clearing condition holds. Each investor $i$'s effective risky assets holding is
\[
d_i^* X + D_i^* = \left[ I + \frac{1}{\rho^2} (\Sigma U)^{-1} \right]^{-1} W + \rho \Omega_i (S_i - rP).
\]
Then, using the pricing function (equation (7)), the aggregate demand can be calculated as
\[
\int_0^1 (d_i^* X + D_i^*) \, di = \left[ I + \frac{1}{\rho^2} (\Sigma U)^{-1} \right]^{-1} W + \rho \Sigma (F - rP)
\]
\[
= \left[ I + \frac{1}{\rho^2} (\Sigma U)^{-1} \right]^{-1} W + \rho \Sigma \left( \frac{1}{\rho} (\Sigma + \rho^2 \Sigma U \Sigma)^{-1} W + \frac{1}{\rho} \Sigma^{-1} Z \right)
\]
\[
= \left[ I + \frac{1}{\rho^2} (\Sigma U)^{-1} \right]^{-1} W + \left[ I + \rho^2 \Sigma \right]^{-1} W + Z
\]
\[
= \rho^2 \Sigma U \left[ I + \rho^2 \Sigma \right]^{-1} W + \left[ I + \rho^2 \Sigma \right]^{-1} W + Z
\]
\[
= W + Z.
\]
Therefore, the market clears.

Now, for any investor $i$, we consider a general investment strategy $d_i X + D_i$. Denote by $D_{in}$ investor $i$'s direct holding of asset $n$. Suppose that investor $i$ is informed about asset $n$. Then, the pricing function (7) implies that investor $i$'s optimal holding of asset $n$ is
\[
\left[ 1 + \frac{1}{\rho^2} (\lambda_n \kappa_n \tau_n)^{-1} \right]^{-1} w_n + \rho \kappa_n (s_{in} - r p_n) = x_n + \rho \kappa_n (s_{in} - r p_n).
\]
Therefore, any combination of $d_i$ and $D_{in}$ such that
\[ d_i x_n + D_{in} = x_n + \rho \kappa_n (s_{in} - r p_n) \]
can lead to the optimal holding of asset $n$ for investor $i$.

Now, consider an asset $n$ that investor $i$ is uninformed about. For any given $d_i$ and $D_{in}$, investor $i$ is effectively holding a position $d_i x_n + D_{in}$ of asset $n$. Then, for any given $\tau_n$ such a holding will bring investor $i$ a utility
\[ -\exp \left( -\frac{1}{\rho} w_{in} r p_n \right) \exp \left[ -\frac{1}{\rho} (d_i x_n + D_{in}) E (f_n - r p_n | p_n) + \frac{(d_i x_n + D_{in})^2}{2 \rho^2} \mathbb{V} (f_n | p_n) \right]. \] (21)

There are two cases. In the first case where $\tau_n^i = 0$, similarly to Proposition 1 if $D_{in} \neq 0$, the infimum of such a utility is $-\infty$, since $\mathbb{V} (f_n | P_n) \to +\infty$ as $\tau_n \to 0$. Therefore, $D_{in}^* = 0$. Next, substituting $X_n$ into equation (21), the investor’s utility given $\tau$ is
\[ -\exp \left( -\frac{1}{\rho} w_{in} r p_n \right) \exp \left[ - \left( d_i - \frac{1}{2} d_i^2 \right) \frac{\rho \tau_n \lambda_n^2 \kappa_n^2 w_n^2}{\lambda_n \kappa_n + \rho^2 \tau_n \lambda_n^2 \kappa_n^2} \right]. \] (22)

It follows from equation (22) that for any $d_i$, the infimum of the investor’s utility is at most $-\exp \left( -\frac{1}{\rho} w_{in} r P_n \right)$. Since the investor can get the utility at least $-\exp \left( -\frac{1}{\rho} w_{in} r P_n \right)$ by employing the investment strategy $d_i^* = 1$, there is no profitable deviation.

In the second case, $\tau_n^i > 0$. We first assume that any investor $i$ has min-max utility, and then finally show that her max-min utility is the same as her min-max utility, which implies a strong min-max property. Then, investor $i$’s optimal investment strategy with a max-min utility is the same as the optimal investment strategy with a min-max utility. Since investor $i$ does not know $\tau_n$, $d_i$ and $D_{in}$ are not functions of $\tau_n$. For any given $\tau_n$, we can solve $d_i^*$ and $D_{in}^*$ by the first order condition of the following maximization problem:
\[ \max_{d_i, D_{in}} \left( d_i x_n + D_{in} \right) \frac{w_n}{\rho \left( \lambda_n \kappa_n + \rho^2 \tau_n \lambda_n^2 \kappa_n^2 \right)} - \frac{(d_i x_n + D_{in})^2}{2 \rho} \frac{1}{\rho^2 \lambda_n^2 \kappa_n^2 \tau_n}. \] (23)

The second order condition of such a maximization problem holds, because the utility function in equation (23) is strictly concave.

Differentiating the utility function in equation (23) with respect to $d_i$, we get one of the first-order conditions:
\[ x_n \frac{w_n}{\rho \left( \lambda_n \kappa_n + \rho^2 \tau_n \lambda_n^2 \kappa_n^2 \right)} - \frac{(d_i x_n + D_{in}) x_n}{\rho} \frac{1}{\rho^2 \lambda_n^2 \kappa_n^2 \tau_n} = 0. \]
So,
\[ d_{i}x_{n} + D_{in} = d_{i} \frac{\rho^{2} \tau_{n} \lambda_{n}^{2} \kappa_{n}^{2}}{\lambda_{n} \kappa_{n} + \rho^{2} \tau_{n} \lambda_{n}^{2} \kappa_{n}^{2}} w_{n} + D_{in} = \frac{\rho^{2} \tau_{n} \lambda_{n}^{2} \kappa_{n}^{2}}{\lambda_{n} \kappa_{n} + \rho^{2} \tau_{n} \lambda_{n}^{2} \kappa_{n}^{2}} w_{n} + D_{in}. \]

Then, \( d_{i}^{*} = 1 \) and \( D_{in}^{*} = 0 \), because they are not functions of \( \tau_{n} \). Therefore, with a min-max utility, if an investor \( i \) is uninformed about asset \( n \), she will hold exactly one share of the passive fund and a zero position of asset \( n \).

Because investor \( i \)'s optimal investment strategy \((d_{i}^{*}, D_{in}^{*}) = (1, 0)\) is constant across all possible \( \tau_{n} \), we have
\[
\min_{\tau_{n}} \max_{d_{i}, D_{in}} u((d_{i}, D_{in}), \tau_{n}) = \min_{\tau_{n}} u((1, 0), \tau_{n}) \leq \max_{d_{i}, D_{in}} \min_{\tau_{n}} u((d_{i}, D_{in}), \tau_{n}).
\]

Generally, by the min-max utility, we have
\[
\min_{\tau_{n}} \max_{d_{i}, D_{in}} u((d_{i}, D_{in}), \tau_{n}) \geq \max_{d_{i}, D_{in}} \min_{\tau_{n}} u((d_{i}, D_{in}), \tau_{n}).
\]

Then, we have
\[
\min_{\tau_{n}} \max_{d_{i}, D_{in}} u((d_{i}, D_{in}), \tau_{n}) = \max_{d_{i}, D_{in}} \min_{\tau_{n}} u((d_{i}, D_{in}), \tau_{n}).
\]

This implies a strong min-max property, and hence, \( (d_{i}^{*}, D_{in}^{*}) = (1, 0) \) is also the optimal investment strategy of investor \( i \), when she has a max-min utility.

In sum, given the pricing function specified in equation (7), it is optimal for any investor \( i \) to choose the investment strategy \( d_{i}^{*} = 1 \) and
\[
D_{in}^{*} = \begin{cases} 
0, & \text{if she is uninformed about asset } n; \\
\rho \kappa_{n} (S_{in} - rP_{n}), & \text{if she is informed about asset } n.
\end{cases}
\]

Q.E.D.

Proof of Theorem 3

Let’s first prove a more general version of Proposition 3 when investors hold a common prior belief about \( F \), \( F \sim \mathcal{N}(\overline{F}, V) \). As is standard in the literature of rational expectations equilibrium, we consider the linear pricing function
\[
F = A + BP + CZ, \quad \text{with } C \text{ nonsingular.} \tag{24}
\]
If and only if $B$ is nonsingular, equation (24) can be rearranged to

$$P = -B^{-1}A + B^{-1}F - B^{-1}CZ,$$  \hfill (25)

which solves for prices. Recall that $S_i = F + \epsilon_i$, so conditional on $F$, $P$ and $S_i$ are independent. Therefore, we can write down assets’ payoffs’ posterior means and posterior variances conditional on all information that are available to investor $i$ as follows.

First consider investor $i$’s belief about $F$ conditional on $P$. Conditional on $P$, $F$ is normally distributed with mean $A + BP$ and precision $[CU^{-1}C']^{-1}$. On the other hand, conditional on $S_i$, investor $i$’s belief about $F$ is also normally distributed, with mean $S_i$ and precision $\Omega_i$. Therefore, investor $i$’s belief about $F$ conditional on what the investor observes, $P$ and $S_i$, is also normally distributed. The mean of the conditional distribution of $F$ is the weighted average of the expectation conditional on the price $P$, the expectation conditional on investor $i$’s private signal $S_i$, and the prior mean $\bar{F}$. Therefore, the conditional mean of $F$ is

$$\left[(CU^{-1}C')^{-1} + \Omega_i + V^{-1}\right]^{-1}\left[(CU^{-1}C')^{-1}(A + BP) + \Omega_i S_i + V^{-1}\bar{F}\right].$$  \hfill (26)

The precision of the conditional distribution of $F$ is

$$(CU^{-1}C')^{-1} + \Omega_i + V^{-1}.$$  \hfill (27)

Then, from any investor $i$’s first order condition, investor $i$’s demand is

$$D_i = \rho \left[(CU^{-1}C')^{-1} + \Omega_i + V^{-1}\right]$$

$$\left\{\left[(CU^{-1}C')^{-1} + \Omega_i + V^{-1}\right]^{-1}\left[(CU^{-1}C')^{-1}(A + BP) + \Omega_i S_i + V^{-1}\bar{F}\right] - rP\right\}$$

$$= \rho \left\{\left[(CU^{-1}C')^{-1}(A + BP) + \Omega_i S_i + V^{-1}\bar{F}\right] - \left[(CU^{-1}C')^{-1} + \Omega_i + V^{-1}\right] rP\right\}$$

$$= \rho \left\{(CU^{-1}C')^{-1}(B - rI) - r\Omega_i - rV^{-1}\right\} P$$

$$+ \rho \Omega_i S_i + \rho [(CU^{-1}C')^{-1} A + V^{-1}\bar{F}].$$  \hfill (28)

Integrating across all investors’ demands gives the aggregated demand as

$$\int_0^1 D_i di = \rho \left\{(CU^{-1}C')^{-1}(B - rI) - r \left(\int_0^1 \Omega_i di\right) - rV^{-1}\right\} P$$

$$+ \rho \left(\int_0^1 \Omega_i S_i di\right) + \rho [(CU^{-1}C')^{-1} A + V^{-1}\bar{F}].$$  \hfill (29)
By equation (11), we have \( \int_0^1 \Omega_i \, di = \Sigma \). Also, note that
\[
\int_0^1 \Omega_i S_i \, di = \Sigma F.
\]
Therefore, from the market clearing condition, we have
\[
\int_0^1 D_i \, di = Z + W. ~ (30)
\]
In an equilibrium, both equation (24) and equation (30) hold simultaneously for any realized \( F \) and \( Z \), therefore, by matching coefficients in these two equations, we have
\[
\rho \left[ (CU^{-1}C')^{-1} A + V^{-1} \bar{F} \right] - W = -C^{-1} A ~ (31)
\]
\[
\rho \left[ (CU^{-1}C')^{-1} (B - rI) - r \Sigma - rV^{-1} \right] = -C^{-1} B ~ (32)
\]
\[
\rho \Sigma = C^{-1} ~ (33)
\]
Therefore, from equation (33), we have
\[
C = \frac{1}{\rho} \Sigma^{-1}
\]
Obviously, \( C \) is positive definite and symmetric. Then from equation (31), we have
\[
[\rho^2 (\Sigma U \Sigma) + \Sigma] A = \frac{1}{\rho} W - V^{-1} \bar{F}.
\]
Because both \( (\Sigma U \Sigma) \) and \( \Sigma \) are both positive definite, we have
\[
A = [\rho^2 (\Sigma U \Sigma) + \Sigma]^{-1} \left( \frac{1}{\rho} W - V^{-1} \bar{F} \right).
\]
From equation (32), we have
\[
[\rho^2 (\Sigma U \Sigma) + \Sigma] (B - rI) = rV^{-1}.
\]
Again, because \( [\rho^2 (\Sigma U \Sigma) + \Sigma] \) is positive definite, we have
\[
B = rI + r[\rho^2 (\Sigma U \Sigma) + \Sigma]^{-1} V^{-1}.
\]
Obviously, \( B \) is invertible. By substituting \( A, B, \) and \( C \) into equation (25), we solve the equilibrium pricing function.
Now, let’s look at any investor $i$’s holding. Substituting the coefficients into investor $i$’s holding function \((28)\), we have

$$D_i = \left( I + \frac{1}{\rho^2} (\Sigma U)^{-1} \right)^{-1} W + \rho \left[ I + \rho^2 \Sigma U \right]^{-1} V^{-1} (F - rP) + \rho \Omega_i (S_i - rP).$$

Finally, because the pricing function $P$ and any investor $i$’s demand function $D_i$ are continuous in $V^{-1}$, we can substitute $V^{-1} = 0$ to get Proposition 3.

Q.E.D.

Proof of Proposition 4

By equations \((16)\), \((17)\), and \((18)\), we have

$$\frac{1}{P X} \text{diag}(P)^{-1} C U^{-1} C X \ X' A = \left( \frac{1}{P X} \right)^2 X' C U^{-1} C X \frac{P' X}{P X} = \frac{\text{diag}(P)^{-1} C U^{-1} C X}{X' C U^{-1} C X} X' A.$$

This is the RHS of the Security Market Line relation. We want to show that this equals the difference between the risky assets’ rates of return and the riskfree asset’s rate of return, which is shown to be $\text{diag}(P)^{-1} A$ from equation \((15)\).

Then, we have

$$\frac{\text{diag}(P)^{-1} C U^{-1} C X}{X' C U^{-1} C X} X' A = \text{diag}(P)^{-1} A \Leftrightarrow \text{diag}(P)^{-1} C U^{-1} C X X' A = \text{diag}(P)^{-1} A X' C U^{-1} C X \Leftrightarrow C U^{-1} C X X' A = A X' C U^{-1} C X.$$

The last equation holds because $X = \rho (C U^{-1} C)^{-1} A$ and $(C U^{-1} C)^{-1}$ is a symmetric matrix.

Q.E.D.
References


C Online Appendix

To evaluate the robustness of our conclusions about investor participation, we now consider two possible model generalizations. First, investors may be unaware of certain traded assets, making it unattractive or infeasible for them to hold such assets. Previous literature considers this another important possible reason for limited participation. Second, investors may have heterogeneous risk tolerances. These cases also suggest further empirical implications.

C.1 Uncertainty about Other Parameters and Unawareness

Section 2 assumed that investors were uncertain about the precisions of assets’ supply shocks, and maximized their CARA utilities based upon worst-case scenarios. In addition, we assumed that the number of the risky assets is common knowledge. Hence, all investors know the existence of all assets, and can observe their prices.

Investors unawareness is an important alternative possible explanation for nonparticipation. Specifically, investors may not know certain traded risky assets, and so they do not observe such assets’ prices. It is infeasible for investors to directly hold assets they are unaware of (Merton 1987; Easley and O’Hara 2004). For example, if an investor has never heard of FLIR Systems (an S&P 500 firm), it seems natural for the investor not to participate in this market.

Such a definition of investor unawareness is rather restricted. We relax the definition of investor unawareness to allow for extreme ignorance about some of the asset’s characteristics, even when the investor can observe its price. This makes it physically possible (though not necessarily attractive) for an investor to hold an asset the investor is unaware of.

Formally, we say that investor $i$ is unaware of asset $n$, if she holds a diffuse uniform prior about the precision of asset $n$’s supply shock\footnote{If we assume that the prior about $\tau_n$ is $\tau_n \sim U(0, \bar{\tau}_n)$ for some $\bar{\tau}_n \in \mathbb{R}^{++}$, Proposition 5 below still holds.} that is, $\tau_n \sim U(0, +\infty)$. Here, we allow investor $i$ to know all characteristics of asset $n$ (other than the precision of the supply shock) and observe asset $n$’s price.

Since our focus is now on unawareness, not ambiguity aversion, instead of maximizing her CARA utility in the worst case scenario, investor $i$ maximizes her ‘average’ CARA utility over the set of all possible precisions of asset $n$’s supply shock. Proposition 5 below shows that in this setting, the investor will not hold asset $n$ directly.

**Proposition 5** When there is no passive fund, investors will not participate in the markets of assets they are unaware of.

**Proof of Proposition 5**
Because there is no passive fund, \( d_i = 0 \). Consider \( D_i \neq 0 \) (any non-zero direct holding of asset \( n \)). For any given \( \tau_n \in (0, +\infty) \), conditional on asset \( n \)'s price \( P_n \), investor \( i \)'s utility is

\[
- \exp \left( -\frac{1}{\rho} w_i r P_n \right) \exp \left[ -\frac{1}{\rho} D_i \mathbb{E} \left( f_n - r P_n \middle| P_n \right) + \frac{D_i^2}{2\rho^2} \mathbb{V} \left( f_n \middle| P_n \right) \right].
\]

Similarly to the proof of Proposition 1, \( |\mathbb{E} \left( f_n - r P_n \middle| P_n \right)| \) is bounded, and the variance of asset \( n \)'s payoff conditional on \( P_n \) is

\[
\mathbb{V} \left( f_n \middle| P_n \right) = c^2 \tau_n^{-1},
\]

where \( c = 1 / (\lambda_n \kappa_n \rho) \) is independent of \( \tau_n \). Then, by Jensen’s inequality, we have

\[
\lim_{h \to +\infty} \int_0^h -\frac{1}{h} \exp \left( -\frac{1}{\rho} w_i r P_n \right) \exp \left[ -\frac{1}{\rho} D_i \mathbb{E} \left( f_n - r P_n \middle| P_n \right) + \frac{D_i^2}{2\rho^2} \mathbb{V} \left( f_n \middle| P_n \right) \right] \, d\tau_n
\leq - \exp \left( -\frac{1}{\rho} w_i r P_n \right) \exp \left[ \lim_{h \to +\infty} \int_0^h -\frac{1}{h} \left( -\frac{1}{\rho} D_i \mathbb{E} \left( f_n - r P_n \middle| P_n \right) + \frac{D_i^2}{2\rho^2} \mathbb{V} \left( f_n \middle| P_n \right) \right) \, d\tau_n \right].
\]

The right-hand side of this inequality diverges to \( -\infty \), implying that the left-hand side, which is the investor’s average CARA utility, is also \( -\infty \). Therefore, \( D_i \neq 0 \) is dominated by \( D_i = 0 \); hence, investors will not directly hold the assets they are unaware of.

Q.E.D.

We now analyze whether a passive fund that offers RAMP can lead to full participation in the setting with investors’ unawareness. This is not a trivial question, since investors still need to assess the expected return and the risk of holding the passive fund, when they allocate their initial wealth among the passive fund, the risky assets they are aware of, and the riskfree asset.

We assume that any investor \( i \) believes that the number of all traded assets is equally likely to be any integer that is greater than or equal to the number of assets she is aware of. For an asset investor \( i \) is unaware of, investor \( i \) holds diffuse uniform priors about all its parameters she does not know\(^{15} \). These priors consist of a uniform uninformative prior about its endowment over the support \((0, +\infty)\), a uniform uninformative prior about the precision of the average private information about its payoff over the support \((0, +\infty)\), and a uniform uninformative prior about the precision of the supply shock in its market over the support \((0, +\infty)\). Investors know that all random variables about assets’ characteristics are independent.

We further assume that all investors are aware of the riskfree asset, and that there is a fund that invests in the risky assets. Investors know the existence and name of the

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\(^{15}\)The assumption of diffuse uniform priors is not necessary for Proposition 6 below. Indeed, from its proof, we can see that Proposition 6 holds for any subjective priors investors may have.
fund and are aware of its price, but are unaware of (have diffuse priors about) its return characteristics. So an investor who is unaware of some assets can have extremely poor information about the distribution of returns on this fund.

The fund manager observes the characteristics and prices of all assets and therefore is able to construct and offer to investors the portfolio \( X \) as specified in equation (6). It is common knowledge that this is the portfolio offered by the fund.

In such a setting, an investor \( i \)'s investment strategy is a function mapping from her information set to positions in the passive fund and the other assets. As we argue above, if investor \( i \) is unaware of asset \( n \), and decides not to hold the passive fund, then investor \( i \) will have a zero holding of asset \( n \), because either it is infeasible for investor \( i \) to hold asset \( n \), or holding asset \( n \) is infinitely risky to the investor.

We define an \textit{admissible world} of an investor as the union of the set of assets she is informed about and a possible set of assets she is uninformed of and hypothesized possible characteristics for these assets. Specifically, consider any investor \( i \). We divide all traded assets into two groups, \( \Gamma_{i1} \) and \( \Gamma_{i2} \). Suppose that investor \( i \) is informed about \( \Gamma_{i1} \) assets only, and so she knows all characteristics of \( \Gamma_{i1} \) assets. However, she is uninformed about or unaware of \( \Gamma_{i2} \) assets. In particular, she knows the existence of \( \Gamma_{i2}' \) assets and she can contemplate a possible set of assets \( \Gamma_{i2}'' \). Denote by \( \tilde{\Gamma}_{i2} \) the union of \( \Gamma_{i2}' \) and \( \Gamma_{i2}'' \). The combined asset set \( \Gamma_{i1} \cup \tilde{\Gamma}_{i2} \), together with an hypothesized vector of characteristics for each asset in \( \tilde{\Gamma}_{i2} \), constitutes an admissible world. The set \( \tilde{\Gamma}_{i2} \) is associated with a number \( \tilde{N} \geq \#(\Gamma_{i2}') \) of assets in \( \tilde{\Gamma}_{i2} \). For each asset \( n \in \tilde{\Gamma}_{i2} \), the possible world specifies the specific parameters values characterizing asset \( n \): the endowment \( \tilde{W}_n > 0 \), the average precision of private information \( \tilde{\lambda}_n \tilde{\kappa}_n \), and the precision of the supply shock \( \tilde{\tau}_n \).

We assume that conditional upon an admissible world, the investor has the CARA utility function. However, investors maximize their average CARA utilities over all admissible worlds, when making investment decisions.

Proposition 6 below shows that in such a general case, investors will hold exactly one share of the passive fund, and thus participate in all assets’ markets.

\textbf{Proposition 6} In the general model where investors are uncertain about several characteristics of the traded assets, including the number of assets:

1. There exists an equilibrium in which all investors hold one share of the passive fund and their own information-based portfolios.

2. Asset prices and investors’ effective risky assets holdings are identical to those in the model without any model uncertainty.

3. Generically all investors take non-zero positions in all traded assets.

\textit{Proof of Proposition 6}
Consider any investor $i$, who is aware of assets in $\Gamma_{i1}$ but is uninformed about or unaware of assets in $\Gamma_{i2}$. Then, any of investor $i$’s admissible world $\tilde{\Gamma}$ consists of $\Gamma_{i1}$ assets and possible $\tilde{\Gamma}_{i2}$ assets; that is, $\tilde{\Gamma} = \Gamma_{i1} \cup \tilde{\Gamma}_{i2}$.

The strategy profile under consideration prescribes that all investors buy one share of the fund and hold their own information-based portfolios. Hence, in $\tilde{\Gamma}$, by The Information Separation Theorem, all other investors’ portfolio choices are effectively the same as in equation (12), because the fund offers the risk-adjusted market portfolio in $\tilde{\Gamma}$. Therefore, in $\tilde{\Gamma}$, the pricing function will be the same as in equation (8). Then, for any given price vector, investor $i$’s optimal portfolio choice will be the same as in equation (12) too. Such a portfolio choice can be implemented by holding the passive fund and investor $i$’s own information-based portfolio based only on her knowledge about $\Gamma_{i1}$ assets. Therefore, in $\tilde{\Gamma}$, it is optimal for investor $i$ to hold the fund and her information-based portfolio, when all other investors do the same thing.

Since the admissible world $\tilde{\Gamma}$ is constructed arbitrarily, the arguments above show that it is optimal for investor $i$ to hold one share of the fund and her own information-based portfolio, when all other investors hold the fund and their own information-based portfolio. By similar arguments, when all other investors hold the fund and their own information-based portfolios, any investor will optimally hold the fund and her own information-based portfolio. Therefore, the strategy under consideration is an equilibrium.

Then, for any realized world, since all investors’ effective holdings are exactly same as in equation (8), the market clearing condition implies that the equilibrium price function is same as in the case where all parameters are common knowledge. In addition, since all investors hold the fund who offers the risk-adjusted market portfolio, all investors will have strictly positive positions of all assets.

\[ Q.E.D. \]

From information separation, the portfolio constructed by the fund described in Proposition 6 is implementable using only public information. So if a passive fund wants to provide investors with RAMP, it does not need to know the private signal of any investor. For investors, buying a fund share is the same as holding RAMP—the first component described by the information separation theorem. Therefore, intuitively, all investors are satisfied to buy fund shares, despite their extreme ignorance about the return distribution of the fund and its assets.

Since we have assumed a uniform uninformative prior for an investor on the number of assets of which he is not aware, one might suspect that this would interfere severely with the investor’s attempt to speculate even on the assets the individual is aware of. However, investors do not need to know the number of assets traded in the market when forming their information-based portfolios. Consider for example any investor $i$. Denote by $N_i$ the number of assets that she is informed about. For any given $\hat{N} \geq N_i$, except the $N_i \times N_i$ block $\Omega_{ii}$, all other blocks in the $\hat{N} \times \hat{N}$ matrix $\Omega_i$ are 0. So lack of knowledge about the number $N$ does not affect investors’ information-based trading.
C.2 Heterogeneous Risk Tolerances

In the model described in Section 2, investors share a same risk aversion coefficient \( \rho \). Such an assumption leads to investors’ homogeneous holdings of the passive fund. Indeed, in the equilibrium characterized in Proposition 2 all investors hold one share of the passive fund. However, it is conceivably that differences in risk tolerances, and investor unawareness of other investors’ risk tolerances, could resurrect investors’ heterogeneous holdings of the passive fund. We extend the model in Section 2 by assuming that any investor \( i (i \in [0, 1]) \) has the risk aversion coefficient \( \rho_i \). Here, \( \rho_i \) is a continuous function of \( i \). Let

\[
\bar{\rho} = \int_0^1 \rho_i di \quad \text{and} \quad \Sigma = \int_0^1 \rho_i \Omega_i di.
\]

Here, \( \bar{\rho} \) is the average risk tolerance, and \( \Sigma \) is the average precision of investors’ private information that is weighted by their risk tolerances. We assume that any investor \( i \) knows \( \rho_i \), but she does not know the distribution of \( \rho_i \) and thus the average risk tolerance \( \bar{\rho} \). The passive fund cannot evaluate each individual investor’s risk tolerance, but it has accurate information about the distribution of investors’ risk tolerances; hence, it knows \( \bar{\rho} \) and \( \Sigma \). Then, the passive fund offers the portfolio

\[
X = \left[ \bar{\rho} + (\Sigma U)^{-1} \right]^{-1} W. \tag{34}
\]

to all investors. Proposition 7 shows that investors with different risk tolerances hold different numbers of shares of the passive fund.

**Proposition 7** In the model with investors’ heterogeneous risk tolerances, there exists an equilibrium in which any investor \( i \) with the risk tolerance \( \rho_i \) holds \( \rho_i \) shares of the passive fund and her own information-based portfolio \( \rho_i \Omega_i (S_i - rP) \).

**Proof of Proposition 7**

We first analyze the model in which investors have heterogeneous risk tolerances and all parameters are common knowledge. We again consider the linear pricing function as in equation (24),

\[
F = A + BP + CZ, \quad \text{with } C \text{ nonsingular}.
\]

Therefore, conditional on the price, assets’ payoffs have the conditional distribution is

\[
F|P \sim \mathcal{N} \left( A + BP, CU^{-1}C' \right).
\]

An investor \( i \) gleans such information from the price. Therefore, an investor \( i \)’s demand is

\[
D_i = \rho_i \left[ (CU^{-1}C')^{-1} (B - rI) - r\Omega \right] P + \rho_i \Omega_i S_i + \rho_i (CU^{-1}C') A. \tag{35}
\]
Then, by integrating all investors’ demands and equalizing the aggregate demand and the total supply (the aggregate endowments and the supply shocks), we can derive the pricing function

\[
P = B^{-1} [F - A - CZ],
\]

(36)

where

\[
A = \left[ \Sigma + \bar{\rho} (\Sigma U \Sigma)^{-1} \right]^{-1} W
\]

(37)
\[
B = r I
\]

(38)
\[
C = \Sigma^{-1}.
\]

(39)

Any investor \(i\)’s risky asset holding is

\[
D_i = \rho_i \left[ \bar{\rho} + (\Sigma U)^{-1} \right]^{-1} W + \rho_i \Omega_i (S_i - rP).
\]

(40)

Because the passive fund provides the portfolio \(\overline{X}\) specified in equation (34), Equation (40) can be rewritten as

\[
D_i = \rho_i \overline{X} + \rho_i \Omega_i (S_i - rP).
\]

(41)

Then, when investors are uncertain about some parameters and thus are subject to ambiguity aversions, they still want to hold the passive fund. In particular, investor \(i\) first buys \(\rho_i\) shares of a passive fund and then use her own private information to form the information-based portfolio \(\rho_i \Omega_i^{-1} (S_i - rP)\). Finally, investor \(i\) invests the rest of her endowments in the riskfree asset.

\emph{Q.E.D.}