Boundedly Rational Dynamic Programming

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August 10, 2015
Preliminary and Incomplete

Abstract

This paper proposes a way to model boundedly rational dynamic programming in a parsimonious and tractable way. The framework is quite general, and has at its core a behavioral version of the Bellman equation, in which the agent uses a simplified model of the world and the consequences of his actions. It is then applied to some of the canonical models in macroeconomics and finance.

In the consumption-savings model, the consumer can pay limited attention to the variables such as the interest rate and his income – this way using a simplified, “sparse” model of the world. Endogenously, the consumer pays little or no attention to the interest rate but pays more attention to his income.

Ricardian equivalence partially fails, because the consumer is only partially attentive to future taxes. The model also yields a behavioral version of the New Keynesian model. In particular, it helps solve the “forward guidance puzzle”, the fact that in that model, shocks to very distant rates have a very powerful impact on today’s consumption and inflation: because the agent is de facto myopic, this effect is muted.

In a Merton-style portfolio choice problem, the agent endogenously pay limited or no attention to the varying equity premium and hedging demand terms. Finally, the paper gives a behavioral version of the canonical neoclassical growth model. Fluctuations are more persistent when agents are more boundedly rational.

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*xgabaix@stern.nyu.edu. I thank David Laibson for a great many enlightening conversations about behavioral economics. For useful comments I also thank Nick Barberis, Robert Barro, John Campbell, Emmanuel Farhi, Harrison Hong, David Laibson, Jennifer La’O, Ali Lazrak, Bentley MacLeod, Ricardo Reis, Michael Rockinger, Thomas Sargent, Alp Simsek and seminar participants at various seminars and conferences. I thank Jerome Williams for very good research assistance. I am grateful to the Dauphine-CAM foundation, the Institute for New Economic Thinking, and the NSF (SES-1325181) for financial support.
1 Introduction

This paper proposes a way to model dynamic programming with boundedly rational agents. It lays out a fairly general procedure to formulate dynamic programming in situations of economic interest. Then, it shows how the framework applies to some of the canonical models in macro-finance and economics more generally: consumption saving problems, the basic New Keynesian model, the baseline neoclassical growth model (the Ramsey-Cass-Koopmans model), general linear-quadratic problems, and investment in risky assets (Merton’s problem). The upshot is that we have a portable, fairly general structure that applies to the some basic machines of macroeconomics, and allows to see where bounded rationality (BR) is important in those situations.

One of the criticisms of traditional economic models is the potential unrealism of the infinitely forward-looking agent who computes the whole equilibrium in her own head. This lack of realism has long been suspected to be the cause of some empirical misfits that we will review below. Behavioral economics aims to provide an alternative. However, the greatest successes of behavioral economics change the agents’ tastes (e.g. prospect theory or hyperbolic discounting) or their beliefs (e.g. overconfidence), but typically keep the assumption of rationality. When tackling the rationality assumption, there is much less agreement and the modeling of bounded rationality is much more piecemeal, different from one paper to the next.

This paper proposes a compromise that keeps much of the generality of the rational approach and injects some of the wisdom of the behavioral approach, mostly inattention and simplification. It does so by proposing a way to insert some bounded rationality into a large class of problems, the “recursive” contexts, i.e. with dynamic programming in some stochastic steady state.

To illustrate these ideas, let us consider a canonical consumption-savings problem. The agent maximizes utility from consumption, subject to a budget constraint, with a stochastic interest rate and stochastic income. In the rational model, the agent solves a complex DP problem with three state variables (wealth, income and the interest rate). This is a complex problem that requires a computer to solve it.

How will a boundedly rational agent behave? I assume that the agent starts with a much simpler model, where the interest rate and income are constant – this is the agent’s “default” model. Only one state variable remains, his wealth. He knows what to do then, but what will he do in a more complex environment, with stochastic interest rate and stochastic income? In the sparse version, he considers parsimonious enrichments to the value function, as in a Taylor expansion. He asks, for each component, whether it will matter enough for his decision. If a given feature (say, the interest rate) is small enough compared to some threshold (taken to be a fraction of standard deviation of consumption), then he drops the feature, or partially attenuates it. The result is a consumption policy that pays partial attention to income, and possibly no attention at all to the interest rate. This does seem realistic.

The result is a sparse version of the traditional permanent-income model. We see that it
is often simpler than the traditional model. Indeed, the agent typically ends up using a rule
which is simpler (e.g., not paying attention to the interest rate).

Let us now turn to macro consequences of this approach. One is a behavioral version
of the traditional New Keynesian model. In that model, “forward guidance” works very
powerful, probably too powerful, emphasized by McKay, Nakamura and Steinsson (2015).
The reason is that the traditional consumer always respects Euler equation, so that a move-
ment of interest far in the future will have a strong impact today. However, in the behavioral
model I put forth, this impact is muted by the agent’s myopia. That makes forward guidance
less powerful. The model, in reduced form, takes the form of a “discounted Euler equation”,
where the agent reacts in a discounted manner to future consumption growth. That appar-
cently small modification, due to bounded rationality, has also powerful implications for the
behavior of an economy at the zero lower bound. Depressions due to the ZLB are moderate,
and bounded, even though they are unboundedly large with rational model (Werning 2012).

I also present a behavioral version of a large class of models, and work out in detail
a BR version of most canonical of them, the neoclassical growth model of Ramsey-Cass-
Koopmans. In this version, agents pay less attention to their own variables, less to
aggregate variables. One upshot is that with BR, macroeconomic fluctuations are larger
and more persistent. I illustrate this proposition, and qualify it, as it appears to hold for
most reasonable values of the parameters, but can be overturned for extreme values. To see
the idea, which is fundamentally quite simple, imagine a economy with only one state
variable, capital. It starts with a steady state amount of capital. Then, there is a positive
shock to the endowment of capital. In a rational economy, agents would consume a certain
fraction of it, say 6%, every period. That will lead the capital stock to revert quickly to its
mean. However, in a an economy with sparse agents, investors will not pay full attention
to the additional capital. They will consume less of it than a rational agent would. Hence,
capital will be depleted more slowly and will mean-revert more slowly. The shock has more
persistent effects.

Given that shocks are more persistent, past shocks accumulate more. Mechanically, this
leads to larger average deviations of capital from its trend. As a consequence, the interest
rate and GDP also have larger, and more persistent, deviations from trend.

The model allows us to express those ideas in simple, quantitative ways. It allows us to
explore them in richer environments, e.g. with shocks to both productivity and the capital
stock.

Another application is a Merton-style dynamic portfolio choice problem, i.e. allocating
one’s wealth between stocks and bonds when the expected returns are stochastic and cor-
related with past returns. This is a notoriously complicated problem for a rational agent.
I study how a sparse agent would handle it. The sparse agent first anchors his action by
imagining he’s facing a simpler problem – a world with a constant equity premium. Then,
he can sparsely enrich his model to take into account the more complex features (the sto-
chasticity of the equity premium, its correlation with past returns, which creates a hedging
demand). Hence the agent will take these complex features into account only partially, or
not at all. This may be a more satisfying description than the hyper-rational model of how
people behave in a complex environment. At the very least, it is important to have a concrete alternative to that hyper-rational model.

Here are some conclusions for individual decision-making in macro:

1. The consumer has a higher marginal propensity to consumer for current and near payment, rather than distant payments.

2. The Euler equation fails, though it holds under the agent’s subjective, sparse model (difficult, though not in principle impossible to access, by the econometrician).

3. Agents create a too small buffer of savings, as they’re (partially) myopic to the risk of income fluctuations.

4. This consumer starts saving “too late” for retirement.

5. When choosing their portfolio, agents are pay less attention to the hedging demand motive.

Here are some conclusions for aggregate macro:

1. Ricardian equivalence partly fails. If the government gives a dollar today, and takes it back later (plus interest), consumption today increases – though it should not react in the simplest rational model.

2. Monetary policy’s impact is changed. In the traditional NK model, if the central bank announces today that it will raise rates for one period in $T$ periods, the impact today increases with $T$ – the more remote the policy action, the greater the impact: that is the “forward guidance puzzle”. This is not the case here.

3. The Zero Lower Bound is much less costly. In the traditional NK model, a very long period at the ZLB has an unboundedly large recession. This impact is moderate, and finite, if agents are myopic enough.

4. Fluctuations are amplified in the most basic Ramsey-Cass-Koopmans model.

5. The agent looks like a hybrid between a neoclassical agent, a New Keynesian agent, and an old Keynesian agent – in the sense that he’s myopic and adopts simple decision rules. However, unlike the truly old Keynesian agents, those decisions rules are not arbitrary — they are microfounded, and reaction to permanent changes in the economy. Unlike the neoclassical and NK agent, this sparse agent is partially myopic, does not react to all things.

**Literature review.** There already are a few partial models of behavioral intertemporal choice. Krusell Smith (1998), Caballero (1995), Campbell-Mankiw (1989), Sims (2003),

One key difference here is that the model is here much more systematic. It explicitly applies, in a unified manner, to a wide class of models. It relies on an earlier paper (Gabaix 2014) that proposes a “sparse max”, a behavioral, less than fully rational and attentive of the traditional max operator. That paper was concerned with static cases, here we explore dynamic ones. Similarly, that paper allows to give a behavioral formulation of some basic chapters of the microeconomic textbooks (consumer theory, equilibrium theory, Arrow-Debreu). The present paper allows to give a behavioral version of some chapters of the macroeconomics textbooks (mostly, consumption-savings problem and the basic neoclassical growth model). Hence, we have a more unified view of bounded rationality in micro and macro, whether those other papers are more piecemeal.

The other approaches have not (yet) yielded a systematic approach of dynamic programming, or of those basic building blocks. One partial exception is Maćkowiak and Wiederholt (forthcoming). They work out a New Keynesian model with an entropy-based penalty for precision à la Sims (2003), and show that it is quantitatively successful. The present paper is more analytical, develops tools that apply to those of other situations. It is also not quantitative, but more systematic in terms of theory.

The rest of the paper is as follows. Section 2 presents the general procedure. Then, we apply to a variety of canonical examples. Section 3 presents basic partial-equilibrium building blocks: the basic consumption-savings problem, including variants such as failure of Ricardian equivalents and the Merton portfolio problem. Section 6 works out the neoclassical growth model, e.g. a general equilibrium situation. Section 7 develops other models: the Merton dynamic consumption-investment model, linear-quadratic models, and the Becker-Murphy rational addiction model. Section 9 concludes. The Appendix and Online Appendix contain further extensions and proofs.

2 General Framework

2.1 The sparse max operator for static problems: quick review

In Gabaix (2014), I defined a sparse max or smax operator, which is a behavioral, partially inattentive version of the max operator. The agent faces a maximization problem which is, in its rational version, \( \max_a u(a, x) \). I state here the sparse max, using slightly different notations.

There is an attention vector \( m \), and an attention-dependent extension of the utility function, \( u(a, x, m) \). For instance, defining the Hadamard (component-wise) product as:

\[
m \odot x := (m_i x_i)_{i=1...n}
\]

then

\[
u(a, x, m) = u(a, m \odot x)
\]

5
is the perceived utility function when the consumer is partially inattentive to \( x_i \). When \( m_i = 1 \), the agent fully perceives dimension \( i \), when \( m_i = 0 \), the agent is fully inattentive to it. There is a default attention vector \( m^d \), taken to be 0 in most applications, and a default action \( a^d := \arg \max_a u (a, x, m^d) \). We call \( a_{m_i} = \frac{\partial u}{\partial m_i} \), evaluated at \( (a, m) = (a^d, m^d) \). Hence, \( a_{m_i} = -u_{aa}^{-1} u_{am_i} \). When (2) holds, \( a_{m_i} = a_{x_i} \).

There is a nonnegative parameter \( \kappa \), which is a taste for sparsity. When \( \kappa = 0 \), the agent is the traditional agent (unless the matrix \( \Lambda \) of Definition 2.1 is singular – in that case, the iterated sparse max below helps). The \( x_i \) are viewed by the agent as being drawn with a standard deviation \( \sigma_i \), and covariance \( \sigma_{ij} \).

**Definition 2.1** (Sparse max operator, Gabaix 2014) The sparse max,

\[
\text{smax} u (a, x, m)
\]

is defined by the following procedure.

**Step 1:** Choose the attention vector \( m^* \):

\[
m^* = \arg \min_{m \in [0,1]^n} \frac{1}{2} \sum_{i,j=1 \ldots n} (1 - m_i) \Lambda_{ij} (1 - m_j) + \kappa \sum_{i=1 \ldots n} g (m_i - m_i^d)
\]

with the cost-of-inattention factors \( \Lambda_{ij} := -E [a_{m_i} u_{aa} a_{m_j}] \).

**Step 2:** Choose the action

\[
a^* = \arg \max_a u (a, x, m^*)
\]

and set the resulting utility to be \( u^* = u (a^*, x) \). In the expressions above, derivatives are evaluated at \( m = m^d \) and \( a^d = \arg \max_a u (a, x, m^d) \).

In other terms, the agent solves for the optimal \( m^* \) that trades off a proxy for the utility losses (the first term in the right-hand side of equation (3)) and a psychological penalty for deviations from a sparse model (the second term on the left-hand side of equation (3)). Then, the agent maximizes over the action \( a \), as if \( m^* \) were the true model.

This leads to define the attention function:

\[
\mathcal{A}_g (v) := \sup \left[ \arg \min_{m \in [0,1]} \frac{1}{2} (m - 1)^2 |v| + g (m) \right].
\]

This represents the optimal attention to a variable with variance \( |v| \) an impact of 1 on the decision, with the cost of thinking \( \kappa \) is 1.

The following Lemma derives the main case.

**Lemma 2.1** (Gabaix 2014) When variables are perceived to be uncorrelated, the \text{smax} operator yields:

\[
a^* = \arg \max_a u (a, m_1^* x_1, \ldots, m_n^* x_n)
\]
with

\[ m_i^* = A_g \left( \sigma_i^2 a_{x_i} u_{aa} a_{x_i} / \kappa \right) \]  \hspace{1cm} (5)

and \( a_{x_i} = \frac{\partial a}{\partial x_i} = -u_{aa}^{-1} \cdot u_{a,x_i} \). In the expressions above, derivatives are evaluated at \( x = 0 \) and \( a^d = \text{arg max}_a u (a, 0) \).

The intuition is that the \( x_i \)'s are truncated. If \( \left| \frac{\partial a}{\partial x_i} \right| \) is small enough, so that \( x_i \) shouldn’t matter much any way, then \( m_i^* = 0 \), and the agent doesn’t pay attention to \( x_i \) (if \( m_i^d = 0 \)).

This leads to the defining the truncation function:

\[ \tau_g (b, k) := b A_g \left( \frac{b^2}{k^2} \right) \]  \hspace{1cm} (6)

It is the coefficient \( b \), times the attention to the coefficient, divided by the scaled cognition cost \( k \).

The following lemma gives a more explicit version of the action.

**Lemma 2.2** If the rational action is:

\[ a^s (x) = a^d + \sum_i b_i x_i + O (\|x\|^2) \]

then the sparse action is

\[ a^s (x) = a^d + \sum_i \tau \left( b_i, \frac{\kappa_a}{\sigma_i} \right) x_i + O (\|x\|^2) \]  \hspace{1cm} (7)

with \( \kappa_a := (\kappa / |u_{aa}|)^{1/2} \).

When attention is chosen after seeing \( x \) ("ex post"), we use the same expressions, with \( \sigma_i := |x_i| \). For instance, the ex-post action becomes:

\[ a^s (x) = a^d + \sum_i \tau (b_i x_i, \kappa_a) + O (\|x\|^2) \]  \hspace{1cm} (8)

We see the contrast. In the first procedure, the slope is chosen before seeing \( x_i \). Hence, the policy is still linear in \( x_i \). In the second policy, the truncation is chosen after seeing the \( x_i \). The policy is now non-linear in \( x_i \). The linearity of policies make the first procedure useful for macro.
Attention and Truncation Functions  Here are some good truncation functions. In Gabaix (2014), I study attention functions $A_\alpha (\sigma^2)$ corresponding to $g(m) = m^\alpha 1_{m > 0}$. For instance, for the values $\alpha = 0, 1, 2$, we have (Gabaix 2014):

$$A_0 (\sigma^2) = 1_{\sigma^2 \geq 2}, \quad A_1 (\sigma^2) = \max \left( 1 - \frac{1}{\sigma^2}, 0 \right), \quad A_2 (\sigma^2) = \frac{\sigma^2}{2 + \sigma^2} \quad (9)$$

Hence the truncation functions $\tau_\alpha (b, k)$:

$$\tau_0 (b, k) = b \cdot 1_{b^2 \geq 2k^2}, \quad \tau_1 (b, k) = b \max \left( 1 - \frac{k^2}{b^2}, 0 \right), \quad \tau_2 (b, k) = \frac{b^2}{b^2 + k^2} \quad (10)$$

Figure 1 plots the attention functions, and Figure 2 the corresponding truncation functions.

Another useful cost function is $g_{L_1} (m) = -\kappa \ln (1 - m)$, which generates $A_{L_1} (\sigma^2) = \max \left( 1 - \frac{1}{\sigma^2}, 0 \right)$, and $\tau_{L_1} (b, \kappa) = sign(b) \max (|b| - |\kappa|, 0)$. The subscript $L_1$ denotes that it often arises when doing an $L_1$ regularization, as in the sparsity literature in statistics (Candès and Tao 2006).

Equipped with this piece of machinery, we turn to dynamic problems.

2.2 A motivating example

Here I state the basic motivating example. The agent has utility $\mathbb{E} \sum_{t=0}^{\infty} \beta^t c_t^{1-\gamma} / (1 - \gamma)$. Wealth $w_t$ evolves as:

$$w_{t+1} = (1 + r + \tilde{\eta}_t) (w_t - c_t) + \gamma + \tilde{y}_t \quad (11)$$
$$\tilde{\eta}_{t+1} = \rho_t \tilde{\eta}_t + \varepsilon_{t+1} \quad (12)$$
$$\tilde{y}_{t+1} = \rho_y \tilde{y}_t + \varepsilon_{t+1} \quad (13)$$
Figure 2: Three truncation functions. Because it gives sparsity and continuity, the $\tau_1$ function is recommended.

That is, wealth at $t+1$ is savings at $t$, $w_t - c_t$, invested at rate $r_t = \tau + \tilde{r}_t$, plus current income, $y_t = \tilde{y} + \tilde{y}_t$. Here $\tilde{r}_t$ and $\tilde{y}_t$ are deviations of the interest rate and income from their means, respectively, and follow AR(1) processes, where $\varepsilon_{t+1}^r$ and $\varepsilon_{t+1}^y$ are disturbances with mean zero and no correlation across periods. For simplicity, assume here that $\overline{R} = 1 + \tau = \beta^{-1}$.

This is a complex problem, with 3 state variables $(w_t, \tilde{r}_t, \tilde{y}_t)$. What will the agent do? In the default model, the agent assumes that $m_r^d = m_y^d = 0$; this is, he assumes that future interest rate and income will be constant. Then, the optimal consumption is

$$c^d (w_t) = \frac{\tau w_t + \overline{y}}{\overline{R}}$$

and the value function is

$$V^d (w_t) = A \left( \frac{\tau w_t + \overline{y}}{\overline{R}} \right)^{1-\gamma}$$

with $A = \frac{1}{\overline{R}^{1-\gamma}}$.

I wish to capture two notions. First “the agent may not want think about the interest rate”. Second “the agent may wish to imagine simplified dynamics for the process”, e.g. supposing that there is no noise, or he may replaced the dynamics of the interest rate by another, simpler or more familiar, process.

To capture those ideas, I imagine that the agent contemplates the following model:

$$w_{t+1} = F^w (c_t, z_t, m) = (1 + \tilde{r} + m_r \tilde{r}_t) (w_t - c_t) + \tilde{y} + m_y \tilde{y}_t$$

$$\tilde{r}_{t+1} = F^r (c_t, z_t, m) = m_r \left( \rho_r \tilde{r}_t + \varepsilon_{t+1}^r \right) + (1 - m_r) \rho_r \varepsilon_{t+1}^r$$

$$\tilde{y}_{t+1} = F^y (c_t, z_t, m) = m_y \left( \rho_y \tilde{y}_t + \varepsilon_{t+1}^y \right) + (1 - m_y) \rho_y \varepsilon_{t+1}^y$$

For instance, if $m_r = 0$ the agent doesn’t pay attention to the interest rate. If $m_y = 0$, the agent replaces interest process by another one, with no noise, and a possibly different speed
of mean-reversion. In the notation \( m_{r'} \), \( r' \) means “value of \( r \) next period” and \( m_{r'} \) refers to
the attention to the transition function of \( r, F^r \).

We call \( m = (m_r, m_{r'}, m_y, m_y') \) and \( \mu = (1, 1, 1, 1) \) parametrizes the objective model.

The next section defines the sparse max. In particular, what does it mean to “have the following model” in mind, in particular, what’s the agent’s anticipation of his future actions, so we have a well-defined notion of dynamic programming? Next, how will attention be allocated? Then, we will study consequences of this behavior for classic macro questions.

2.3 Sparse Dynamic Programming: Definition

We express the notions when there is a finite horizon \( T \). The infinite-horizon case is similar with \( T = \infty \), but as it is more technical it is delegated to Appendix 10.1. The agent’s rational problem is:

\[
\max \left( \alpha_t \right) \sum_{t=0}^{T-1} \beta^t u (a_t, Z_t) \quad \text{s.t.} \quad Z_{t+1} = F^z (a_t, Z_t, \varepsilon_{t+1})
\]

and a terminal condition \( Z_T \in \mathcal{F}^T \) for a given set \( \mathcal{F}^T \). Here state variable \( Z_t \) and action \( a_t \) are vectors, while \( \varepsilon_{t+1} \) is a mean-zero innovation. The state vector include the calendar date \( t \) if functions are time-dependent.

The rational version of the dynamic programming (DP) problem is a series of value functions \( V^{r,t} \) satisfying the Bellman equation:

\[
V^t (Z) = \max_a u (a, Z) + \beta \mathbb{E} V^{r,t+1} (F^z (a, Z, \varepsilon_{t+1}))
\]

for \( t = 0, \ldots, T - 1 \), and with \( V^{r,T} (z) = 0 \). A policy is then a function \( a(Z) \). Actually, one can drop the index \( t \), as it is redundant: e.g., \( V(Z) = V(z, t) = V^t (z) \).

This way, the traditional Bellman equation can be simply written:

\[
V^r (Z) = \max_a u (a, Z) + \beta \mathbb{E} V^r (F^Z (a, Z, \varepsilon_{t+1}))
\]

In the smax version, we are given attention-dependent functions \( u(a, Z, m), F^Z (a, Z, \varepsilon_{t+1}, m) \) – in a way that will be detailed later.

We can formulate the BR Bellman equation.

**Definition 2.2** (Sparse dynamic programming, finite horizon) The action of the behavioral agent is given by:

\[
a(Z) = \arg \max_{a,m,m'^d} [u(a, Z, m) + \beta \mathbb{E} V^p (F^z (a, Z, \varepsilon_{t+1}))]
\]

\footnote{To fix ideas, we could take \( u(a, Z; m) = u(a, m \odot Z, t) \), and for the \( k \)-th component of vector \( F^z \)

\[
F^{zk} (a, Z, \varepsilon, m) = F^{zk} (a, m^k \odot z, \varepsilon)
\]

where \( m^k \in \mathbb{R}^{\text{dim } z} \) denotes the attention to factors; generally \( (m^k)^d = 1 \): when predicting the future values of variables \( z_k \), full attention is paid to its initial value.
where $V^p$ is a proxy given value function. In the most basic sparse max, $V^p = V^r$. In the $q$–iterated sparse max, $V^p = V^s$, the objective value function.

The objective value function $V^s$ satisfies:

$$V^s(Z) = u(a(Z), Z, m) + \beta E V^s(F^a(a(Z), Z, \varepsilon_{t+1}))$$

(21)

with the terminal condition $z_T \in \mathcal{F}_T$.

[XX explain the $q$–iteration] This is the same formulation as in the rational version, but with a smax rather than a max operator. The definition gives a construction of the value function by backward induction: starting from $V^T = 0$, we successively calculate $V^{T-1}, ..., V^0$.

In an infinite-horizon problem, the Bellman equation is much the same, but it is a more involved to guaranty existence. This is discussed in Section 10.1.

The problem may look a bit complicated, but in many cases it is actually simple. Before proceeding to the examples, we present some results that help calculate the smax solution. The reader is encouraged to first skip to the applications starting in Section 3.

### 2.4 Some Tools for Sparse Dynamic Programming

This subsection present tools to compute BR dynamic programming. The appendix’s section 10 contains more tools. The reader is invite to skim it, read the main examples shown later, and then come back to it with those examples in mind.

#### 2.4.1 Taylor expansion of policy and value functions

We decompose the vector of state variables into: $z = (w, x)$ where $w$ is a vector of variables that are fully taken into account ($m^d_i = 1$), in the default mode, while $x$ is a vector of variables not taken into account ($m^d_i = 0$).

**Proposition 2.1** For small $x$, we have:

$$V(w, x) = V^r(w, x) + x' \phi(w, x) x$$

where matrix $\phi(w, x)$ is continuous in $(w, x)$ and twice differentiable at $x = 0$, with $\phi(w, 0)$ negative semi-definite. In other words, the sparse value function and the rational value functions differ only by second order terms in $x$.

This basically generalizes the envelope theorem. It implies that, at $x = 0$:

$$V_w = V^r_w, \quad V_{ww} = V^r_{ww}, \quad V_x = V^r_x, \quad V_{wx} = V^r_{wx}$$

(22)

However, in most situations we have $V_{xx} \neq V^r_{xx}$.

This leads to a simple proposition to calculate the value function.
Proposition 2.2 (Calculation of the optimal sparse policy). Consider the first order expansion of the optimal policy for small \( x \),

\[
a^\ast (w, x) = a^d (w) + \sum_i b_i (w) x_i + O \left( x^2 \right)
\]

Then, the sparse policy is, with ex-ante attention allocation:

\[
a^a (w, x) = a^d (w) + \sum_i \tau \left( b_i (w), \frac{\kappa_i}{\sigma_i} \right) x_i + O \left( x^2 \right) \tag{23}
\]

and with ex-post attention allocation:

\[
a^e (w, x) = a^d (w) + \sum_i \tau \left( b_i (w) x_i, \kappa_a \right) x_i + O \left( x^2 \right) \tag{24}
\]

This proposition will be quite useful. To derive policies, first we can simply do a Taylor expansion of the rational policy around the default model, and then truncate term by term.

3 Intertemporal Consumption: Behavioral Version

I now work out a few explicit examples to develop flesh out the above procedure.

3.1 A 3 period model

I start with simple 3-period model. There is no discounting \( (R = \beta = 1) \), and utility is

\[
\sum_{t=0}^{2} u (c_t).
\]

Calling \( w_t \) the wealth at the beginning of period \( t \), the budget constraints at times \( t = 0, 1, 2 \) are:

\[
w_1 = w_0 - c_0, \quad w_2 = w_1 - c_1, \quad 0 = w_2 + x - c_2.
\]

The agent starts with an endowment \( w_0 \), and receives \( x \) at time 2. For instance, \( x \) could represent a negative income shock, such a tax to pay, or a decrease in income as retirement.

How much attention will the agent pay attention to time-2 payment \( x \)?

The rational solution is to smooth consumption: total resources are \( w_0 + x \) (initial wealth \( w_0 \) and time-2 payment \( x \)), and they should be consumed equally in all periods:

\[
c_t = \frac{w_0 + x}{3} \quad \text{for } t = 0, 1, 2
\]

The corresponding dynamic policy is:

\[
c_0 = \frac{w_0 + x}{3}, \quad c_1 = \frac{w_1 + x}{2}, \quad c_2 = w_2 + x.
\]
as we will verify soon.

The rest of this subsection derive the BR solution. We first state the result, then derive it.

**Proposition 3.1** Take the 3-period life-cycle problem. The BR policy is
\[
c_0 = \frac{w_0 + m_0 x}{3}, \quad c_1 = \frac{w_1 + m_1 x}{2}, \quad c_2 = w_2 + x.
\]
where \( m_t \) are the attention values given in (27) and (32). If \( |x| \) is not too large, they satisfy \( m_0 \leq m_1 \). In particular, this implies
\[
\frac{\partial c_0(w_0, x)}{\partial x} \leq \frac{\partial c_1(w_0, x)}{\partial x} \leq \frac{\partial c_2(w_0, x)}{\partial x}
\]
with at least one strict inequality if \( \kappa \) large enough. If the agent was rational, we would have:
\[
\frac{\partial c_0(w_0, x)}{\partial x} \leq \frac{\partial c_1(w_0, x)}{\partial x} \leq \frac{\partial c_2(w_0, x)}{\partial x}.
\]

**Derivation of Proposition 3.1** We apply the smax procedure of Definition 2.2, using backward induction.

At time 2, the agent consumes all his disposable wealth:
\[ V^2(w_2, x) = u(w_2 + x) \]

At time 1, the agent’s problem is:
\[ \text{smax} v^2(c_1, x, m_1) \text{ with } v^2(c_1, x, m_1) := u(c_1) + V^2(w_1 - c_1, m_1 x). \]

The first order condition \( v^2_{c_1} = 0 \) reads:
\[ u'(c_1) = V^2_w(w_1 - c_1, m_1 x) = u'(w_1 - c_1 + m_1 x) \]
so \( c_1 = w_1 - c_1 + m_1 x \), and
\[ c_1 = \frac{w_1 + m_1 x}{2} \quad (26) \]

Hence, the agent pays partial attention \( m_1 \) to the time-2 income \( x \).

To calculate attention \( m_1 \), we apply (5). Noting that \( v^2_{xx}(c, x, m_1)|_{m_1=0} = 2u''(c_1') \), with \( c^d = \frac{w_2}{2} \) is the optimal consumption with \( m_1 = 0 \), we have \( m_1 = A \left( \frac{1}{2\kappa} u''(c_1') \text{ var} \left( \frac{x}{2} \right) \right) \), so
\[ m_1 = A \left( \frac{1}{2\kappa} u'' \left( \frac{w_1}{2} \right) \sigma_x^2 \right). \quad (27) \]

At time 0. Let us first take the Definition 2.2, with \( V^p = V^r \). Then, the value function (assuming future rationality) is
\[ V^{1,p}(w, x) = 2u \left( \frac{w + x}{2} \right). \quad (28) \]
Under the procedure with \( q = 0 \) iteration, this is the value function. If we used the
procedure with \( q = 1 \) iteration, the agent uses the objective value function, which is:

\[
V^1(w_1, x) = u \left( \frac{w_1 + m_1 x}{2} \right) + u \left( \frac{w_1 + (2 - m_1) x}{2} \right)
\]

As per the envelope’s theory, the two value function differ only by second order terms:

\[
V^1(w_1, x) = V^{1,p}(w, x) + O(x^2).
\]

Next, we use the simplest procedure, with \( V^{1,p} \) as in (28).

At time 0, the agent does \( \max_{c_0, m} v^0(c_0, x, m_0^x) \), with

\[
v^0(c_0, w_0, x, m_0) := u(c_0) + V^{1,p}(w_0 - c_0, m_0^x) \tag{30}
\]

The FOC is \( v^0_{c_0} = 0 \) with, i.e.

\[
u'(c_0) = V^{1,p}_{w}(w_0 - c_0, m_0^x) = u' \left( \frac{w_0 - c_0 + m_0^x}{2} \right)
\]

i.e. \( c_0 = \frac{w_0 - m_0^x}{2} \) and

\[
c_0 = \frac{w_0 + m_0^x}{3} \tag{31}
\]

To determine attention \( m_0 = m_0^x \), we again use (5); we calculate:

\[
v^0_{cc} = u''(c^d) + V^1_{w,w|m=0} = u''(c^d) + \frac{1}{2} u''(c^d) = \frac{3}{2} u''(c^d)
\]

so that

\[
m_0 = A \left( \frac{1}{\kappa} u^0_{cc, var}(c_0) \right) = A \left( \frac{13}{6} u''(c^d) \frac{\sigma_x^2}{x} \right)
\]

i.e.

\[
m_0 = A \left( \frac{1}{6\kappa} u'' \left( \frac{w_0}{3} \right) \frac{\sigma_x^2}{x} \right). \tag{32}
\]

The consumptions are:

\[
c_0(w_0, x) = \frac{w_0}{3} + \frac{m_0}{3} x, \quad c_1(w_0, x) = \frac{w_0}{3} + \left( \frac{m_1}{2} - \frac{m_0}{6} \right) x, \quad c_2(w_0, x) = \frac{w_0}{3} + \left( 1 - \frac{m_0 + m_1}{2} \right) x. \tag{33}
\]

Comparing (27) and (32), we see that \( m_0 \leq m_1 \) iff \( \frac{1}{6} \left| u'' \left( \frac{w_0}{3} \right) \right| \leq \frac{1}{2} u'' \left( \frac{w_1}{2} \right) \). When \( x = 0 \), this is automatically verified, as \( \frac{w_0}{3} = \frac{w_1}{2} \). Hence, we have \( m_0 \leq m_1 \) iff \( x \) is not too large.\(^2\)

Given \( 0 \leq m_0 \leq m_1 \leq 1 \), equation (33) implies (25). When \( \kappa \) is very large, then \( m_0 = m_1 \to 0 \), so that \( \frac{\partial c_0(w_0, x)}{\partial x} \to 0 \) and \( \frac{\partial c_2(w_0, x)}{\partial x} \to 1 \), so that one inequality in (25) is strict.

\(^2\)If \( x \) was very large and positive, we could have the following effect: the agent realizes at time 1 that he’s actually quite wealthy, so pays less attention. This effect needs a very large \( x \), so is not operative in most situations.
This example (Proposition 3.1) illustrates a few general features that are specific to a dynamic setting.

Sparse agents are locally myopic like in hyperbolic agents, but globally patient like rational agents. As a result, they differ from both the rational and hyperbolic agents. Indeed, Agents here invest their wealth \( w \) very patiently here, exactly like rational agents. At the same time, they tend to be myopic about the future small shocks (the time-2 shock \( x \)), as in models of hyperbolic discounting (Laibson 1997, O’Donoghue and Rabin 1999). In other terms, in the present model, agents are only partially myopic (e.g. don’t react to a schedule increase in taxes).

Agents react more to “near” shocks than to “distant” shocks: That’s equation (25). The main reason is that, normatively, the shock \( x \) should impact \( c_0 \) as \( \frac{x}{2} \left( \frac{\partial c_1(w,x)}{\partial x} = \frac{1}{3} \right) \), while it should impact \( c_1 \) as \( \frac{x}{2} \left( \frac{\partial c_1(w,x)}{\partial x} = \frac{1}{2} \right) \). Hence, attention is greater to the last period shock \( x \) is lower at earlier dates (\( t = 0 \)) than at late dates (\( t = 1 \)).

The Euler equation fails. The Euler equation holds under the BR-perceived consumption, but not under the actual consumption. For instance, at time 1, if \( m_1 = 0 \), then the agent expects to consume \( (c_1, c_2) = \left( \frac{w_1}{2}, \frac{w_2}{2} \right) \), but actually consumes \( (c_1, c_2) = \left( \frac{w_1}{2}, \frac{w_2}{2} + x \right) \). The traditional Euler equation will hold only if \( m_1 = 1 \).

The sparse agent exhibits partial sophistication in this understanding of its future actions. The rational value function (29) endows the agent with a perfectly sophisticated understanding of his future actions (in particular, the agent understand that he will not fully optimize at period 1. However, the simplified value function \( V^1 \) in (100) with \( m^V = 0 \) gives him a rougher understanding of his future actions: the agent is more like a naive agent. Hence, the agent is sophisticated with \( m^V = 1 \) and more naive when \( m^V = 0 \). The agent optimizes on the degree of sophistication.

### 3.2 Life-cycle problem with infinite horizon

Here I propose a version of the canonical model. We have \( x_t \) the state vector of disturbances, which follows a process:

\[
x_t = F^x(x_{t-1}, \varepsilon_t^x)
\]

The interest rate and income are a linear function of that state vector:

\[
\hat{r}_t = k^r \cdot x_t, \quad \hat{y}_t = k^y \cdot x_t
\]

for some vectors \( k^r, k^y \).

In the AR(1) specification,

\[
x_t = F^x(x_{t-1}, \varepsilon_t^x) = Ax_{t-1} + \varepsilon_t^x
\]

while in the model subjectively perceived by the agent,

\[
x_t = A(m) x_{t-1} + \varepsilon_t^x
\]

\(3\) This is related to the “perceived law of motion” of the literature on learning. But the concept of a mental representation also holds in static contexts, and the model is not about learning.
where the components of transition matrix $A(m)$ are for instance parametrized as:

$$(A(m))_{ij} = m_i^d A_{ij} + (1 - m_i^d) A_{ij}$$

with $A_{ij}$ being a default transition function. Hence, we have

$$F^w(Z, m) = (\tilde{R} + m_r \tilde{c}_t) (w_t - c_t) + \bar{y} + m_y \tilde{y}_t$$

$$F^x(Z, m) = A(m) x + \varepsilon^x$$

One useful particular case is that of diagonal AR(1) processes (i.e., the $A(m)$ matrix is diagonal). This is, then, $x_t = (\tilde{r}_t, \tilde{y}_t)$, so $k^r = (1, 0)$, $k^y = (0, 1)$, and

$$F^w(Z, m) = (\tilde{R} + m_r \tilde{r}_t) (w_t - c_t) + \bar{y} + m_y \tilde{y}_t$$

$$F^y(Z, m) = (m_y \tilde{y}_t + (1 - m_y) \tilde{y}_t) \tilde{y}_t + \varepsilon^y_{t+1}$$

$$F^r(Z, m) = (m_r \tilde{r}_t + (1 - m_r) \tilde{r}_t) \tilde{r}_t + \varepsilon^r_{t+1}$$

The value of $F^y$ reflects that the agent may not well perceived the speed of mean-reversion of $y$.

Another particular case is that of announced shocks. Call $f_t^y$ the vector of future flows, whose $s$-th component will arrive in $s$ periods. This is, $E_t [y_{t+s}] = (f_t^y)_s = k_s^y \cdot f_t$, with $k_s^y = (0, ..., 0, 1, 0, 0, ...)$ the vector selecting the $s$-th component. In the rational model,

$$f_{t+1}^y = L f_t + \varepsilon_{t+1}^y$$

where $L$ is the left-shift operator $L (f_1, f_2, f_3, ...) = (f_2, f_3, ...)$. An innovation $\varepsilon_{t+1}^y$ codifies announcement. For instance, if it’s announced at $t$ that a lump-sum of $\mathcal{S} 7$ arrives in 3 periods (i.e., at $t + 3$), then $\varepsilon_t^y = (0, 0, 7, 0, 0, ...)$. In terms of (36), this means that $A_{ij}^y = 1_{i=j+1}$, $m_i^{A} = m_{fz}$.

Then, with $Z = (w, r_t, f_t^y, f_t^r)$, the life-cycle problem is, under the subjective model:

$$F^w(Z, m) = (\tilde{R} + m_r \tilde{r}_t) (w_t - c_t) + \bar{y} + m_y \tilde{y}_t$$

$$F^x_{t+1} = m_{fz} L f_t^x + \varepsilon_{t+1}^x$$

$$\hat{x}_t = k^x_1 \cdot f_t^x$$

for $x = \tilde{r}, \tilde{y}$

Here, the value of $m_{fz}$ “dampens” the appreciation of future movements in variable $x$.

Intuitively, because the future is harder to predict, its simulations are dampened.

We start with a simple lemma. We use the timing convention for $r_t$ that $w_t = (1 + r_t) (w_{t-1} - c_{t-1})$.

**Lemma 3.1** (Traditional consumption function) **In the rational policy, the optimal consumption is:** $c_t = c_t^d + \hat{c}_t$, with $c_t^d = \frac{r w_t + \bar{y}}{R}$ and

$$\hat{c}_t = \mathbb{E}_t \left[ \sum_{\tau > t} \frac{1}{R^{\tau - t}} \left( b_r (w_t) \tilde{r}_t + b_y \tilde{y}_t \right) \right]$$

$$b_r (w_t) := \frac{\bar{R} (w_t - \bar{y}) - \psi c^d}{R}, \quad b_y := \frac{r}{R}$$

(37)  

(38)
up to second order terms. In matrix form, with $c^c(w_t) := b_r(w_t) k^r + b_y k^y$, and up to second order terms:

$$
\hat{c}_t = c^c \cdot A (R - A)^{-1} x_t = c^c(w_t) \cdot \mathbb{E}_t \sum_{\tau > t} \frac{A^{\tau-t}}{R^{\tau-t}} x_t
$$

We record a consequence.

**Lemma 3.2** In the AR(1) model (12)-(13), the rational policy is:

$$
c^r(w_t, \hat{y}_t, \hat{r}_t) = c^d(w_t) + B_r(w_t) \hat{r}_{t+1} + B_y \hat{y}_t + O \left( \|x\|^2 \right)
$$

where

$$
B_r(w_t) = \frac{b_r(w_t)}{R - \rho_r}, \quad B_y = \frac{b_y}{R - \rho_y}.
$$

We can now derive the behavioral policy.

**Proposition 3.2** (Sparse consumption function) In the behavioral model,

$$
\hat{c}_t^\sigma = \mathbb{E}_t^{\sigma} \left[ \sum_{\tau > t} \frac{1}{R^{\tau-t}} (b_y m_y \hat{y}_t + b_r(w_t) m_r \hat{r}_t) \right]
$$

where $\mathbb{E}_t^{\sigma}$ is the transition function under the subjective model. In matrix form, with $k^{c-s}(w_t) := m_y b_y k^y + m_r b_r(w_t) k^r$, we have $\hat{c}_t = k^{c-s} \cdot A(m) (R - A(m))^{-1} x_t$.

We specialize this to the AR(1) model:

**Proposition 3.3** Under the AR(1) model, the behavioral policy is have:

$$
c^s(w_t, \hat{y}_t, \hat{r}_t) = c^d(w_t) + B_r^s(w_t) \hat{r}_{t+1} + B_y^s \hat{y}_t + O \left( \|x\|^2 \right)
$$

$$
B_r^s(w_t) = \frac{b_r(w_t) m_r}{R - \rho_r(m)}, \quad B_y^s = \frac{b_y m_y}{R - \rho_y(m)}.
$$

**Proposition 3.4** In the lag-in-announcement specification, the behavioral policy is:

$$
\hat{c}_t^s = \mathbb{E}_t \sum_{\tau > t} \left[ m_r b_r(w_t) \frac{m_r^{\tau-t}}{R^{\tau-t}} \hat{r}_\tau + m_y b_y \frac{m_y^{\tau-t}}{R^{\tau-t}} \hat{y}_\tau \right],
$$

All those expressions hold up to second order terms.

Formulation (45) encapsulates two difference forms of inattention. First, the agent may not think about income at all if $m_r = 0$. Second, he may discount future news, if $m_r < 1$. Indeed, he discounts future news arriving in $T$ periods by a factor $m_r^T$. In addition, this discounting is source-specific: if news about future interest rates are less important than news about future income (something we will compute soon), they are (cognitively) discounted more.
Endogenizing attention to the interest rate and income To calculate the BR policy, we first use Proposition 10.2. This gives the following Lemma, shown in the appendix. The agent does a sparse truncation of (40), as in Proposition 2.2. Hence, we obtain the following.

**Proposition 3.5** A sparse agent has the following consumption policy, up to second order terms:

\[ c^s_t = c^d(w_t) + B^\tau \hat{r}_t + B^\gamma \hat{y}_t \]

where (for \( X = \hat{r}, \hat{y} \)) \( B^X := \tau \left( B_X, \frac{\kappa_x}{\sigma_x} \right) \) and \( B_X \) are in (41).

Equation (46) shows a “feature-by-feature” truncation. It is useful because it embodies in a compact way the policy of a sparse agent in quite a complicated world. Note that the agent can solve this problem without solving the 3-dimensional (and potentially 21-dimensional, say, if there are 20 state variables besides wealth) problem. Only local expansions and truncations are necessary.

In this manner, we arrive at a quite simple way to do sparse dynamic programming. There is just one continuously-tunable parameter, \( \kappa \). When \( \kappa = 0 \), the agent is (to the leading order) the traditional rational agent. When \( \kappa \) is large enough, the agent is fully sparse, and does not react to any variable. Hence, we have a simple, smooth way to parametrize the agent, from very sparse to fully rational.

**Numerical illustration** To get a feel for the effects, consider a calibration with (using annual units): \( \gamma = 1, r = 5, \varpi = 2\bar{\sigma}, \bar{\sigma} = 1, \sigma_r = 0.8\%, \sigma_y = 0.2\bar{\sigma}, \rho_y = 0.95, \rho_\tau = 5\% \), and \( \rho_r = 0.7 \): as income shocks are persistent, they are important to the consumer’s welfare. We use the \( \tau_1 \) truncation function.

Then, Figure 3 shows the impact of a change in the interest rate and income on consumption. Consider the left panel, \( B^\tau \). If the cost of rationality is \( \kappa = 0 \), then the agent reacts like the rational agent: if interest rates go up by 1%, then consumption falls by 2.8% (the agent saves more). However, for a sparsity parameter \( \kappa \approx 0.5 \), the agent essentially does not respond to interest rates. Psychologically, he thinks “the interest rate is too unimportant, so let me not take it into account.” Hence, the agent does not react much to the interest rate, but will react more to a change in income (right panel of Figure 3), which is more important: the sensitivity of consumption to income remains high even for a high cognitive friction \( \kappa \). Note that this “feature-by-feature” selective attention could not be rationalized by just a fixed cost to consumption, which is not feature-dependent.

The same reasoning holds in every period. The above describes a practical way to do sparse dynamic programming. In some cases, this is simpler than the rational way (as the agent does not need to solve for the equilibrium), and it may also be more sensible.

**Active decision: Consumption or Savings?** Here we assume that the active decision was one of consumption. One could imagine that it would be in savings. Does this matter?
First, for many variables, it does not matter: the impact of interest rates, future taxes, future income shocks etc. are the same whether a sparse agent uses the consumption frame or saving frame. However, the frame does matter for one variable: current income. Indeed, take the permanent-income setup.4

Which frame does the agent use? One might posit that the agent takes the frame that yields the higher expected utility. To analyze this, we note the following result.

**Proposition 3.6** (Welfare under the consumption vs savings frame) The consumption frame yields greater utility than the savings frame if and only if \( \phi_y > r \), i.e. if income shocks mean-revert faster than the interest rate.

When \( \phi_y > r \) (which is probably the relevant case), the “consumption” frame is indeed better for the agent. The reason is that consumption should be smooth, while savings could be bumpy as they absorb transitory income shocks. When the agent chooses consumption in an inattentive manner, it makes consumption automatically rather smooth. However, if the agent chooses savings inattentively, he makes savings smooth, but consumption needs to absorb the shocks, hence is quite volatile. Hence, generally, to keep consumption smooth,

---

4 Recall that \( \tilde{c}_t = \frac{r}{r + \phi} \hat{y}_t \), so

\[
\tilde{c}_t = \frac{r}{r + \phi} m \hat{y}_t \quad \text{under the consumption frame}
\]

However, if the consumer choose savings, \( S_t \), and then consumes \( c_t = w y_t - S_t \), the rational amount is \( \tilde{S}_t = \hat{y}_t - \tilde{c}_t \), i.e. \( \tilde{S}_t = \frac{\phi}{r + \phi} \hat{y}_t \). Hence, the savings of a sparse agent is \( \tilde{S}_t = \frac{\phi}{r + \phi} m \hat{y}_t \), and the deviation of consumption is: \( \tilde{c}_t = \hat{y}_t - \tilde{S}_t \), i.e.

\[
\tilde{c}_t = \left( 1 - \frac{m \phi}{r + \phi} \right) \hat{y}_t \quad \text{under the savings frame}
\]

which is generally not the same as \( \tilde{c}_t \) under the consumption frame.
choosing consumption inattentively is better than choosing savings inattentively. However, when income shocks are a random walk ($\phi_y = 0$), the savings frame is better. An inattentive agent will keep a constant savings, and let consumption react one for one to income shock, which is the normatively correct behavior when income shocks are completely persistent.

4 Partial Failure of Ricardian Equivalence

General idea Intuitively, a sparse agent will violate Ricardian equivalence (Barro (1974)). I study the magnitude and dynamics of that violation. Let me specialize (45):

$$\hat{c}_t = \frac{r}{R}(w_t + \hat{y}_t + \hat{g}_t) + \sum_{\tau>t} \left( m_y b_y \frac{m_y^{\tau-t}}{R^{\tau-t}} \hat{y}_\tau + m_y b_y \frac{m_y^{\tau-t}}{R^{\tau-t}} \hat{g}_\tau \right)$$

(47)

I keep interest rates constant, and call $\hat{g}_t$ the transfers from the government to the agent.

Suppose that $m_y = 1$ but $m_g < 1$. Suppose that the government gives $\hat{g}$ at $t$, and $-\hat{g} R^T$ in $T$ periods. A rational consumer would not change consumption, as the present value is unchanged. However, a BR consumer increases consumption by $\frac{r}{R} (1 - m_g) \hat{g}$: the positive shock increases it by $\frac{r}{R} \hat{g}$, and the negative shock decreases it by $\frac{r}{R} m_g \hat{g}$.

Proposition 4.1 (Failure of Ricardian equivalence) A behavioral consumer increases consumption at $t$ by $\frac{r}{R} (1 - m_g) \hat{g}$. This way, Ricardian equivalence does not hold, unless attention is full. The further away the increase in taxes (keeping their present value constant), the lower the reaction.

A worked out example For simplicity, I use continuous time. The interest rate is $r = -\ln \beta$. The government needs to collect a present value of $G/r$. This could be done by taxing the population (of size normalized to 1) by $H = Ge^{rT}$, starting at a period $T$. Hence, the path of taxes is: $0$ for $t < T$, and $H$ for $t \geq T$.

What is a consumer’s response at time $t < T$? If the consumer is perfectly attentive, then he should start saving at time 0. However, a sparse agent might not pay attention to those future taxes increases, and start cutting on consumption only later, or indeed perhaps just when the tax cuts are enacted.

Let us analyze this more in detail. At $T$, the tax $H$ is enacted, so that for $t \geq T$, the agent is aware of it. This yields: $\hat{c}_t = r \hat{w}_t - H$.

Before the enactment of taxes ($t < T$), will the consumer think of the tax $H$? That tax lowers the present value of his income by $He^{-r(T-t)}$, so the consumer’s response is:

$$\hat{c}_t = r \hat{w}_t - \tau \left( He^{-r(T-t)} \right)$$

If taxes are collected later, then to guarantee the same present value, they need to be larger by a factor $e^{rT}$. 20
Figure 4: Reaction of consumption and wealth to an increase of future taxes, for different level of $\kappa$. Notes. At time 0, it is announced that taxes will be paid start at time $T = 10$. This Figure plots the change in consumption and wealth. The solid line is the prediction of the rational model (i.e. $\kappa = 0$), the other lines the reaction for different value of $\kappa$ ($\kappa = 0.01$ (blue, dotted), $\kappa = 0.025$ (red, dashed-dotted), $\kappa = 0.1$ (green, dashed)). The very BR agents does not react at first, but starts reacting when he is closer to $T$. He reacts even more when taxes are in effect. As he delayed his savings, he needs to cut more on consumption when taxes start. Units are percentage points of previous steady state consumption. The amount is $G = 2\%$ of permanent income.

Hence, the consumer will not think about the tax increase $H$ when $He^{-r(T-t)} \leq \kappa$. Call $s \in [0, T)$ the first moment when he thinks about them (if it exists, i.e. if $H > \kappa$), otherwise we set $s = T$.

The next Proposition details the dynamics.

**Proposition 4.2** (Myopic behavior and failure of Ricardian equivalence) *Suppose that taxes will go up at time $T$. While a rational agent would cut consumption at time 0, a sparse agent cuts consumption later, at a time $s = \max \left( 0, \min \left( T, \frac{1}{r} \ln \frac{\kappa}{He^{-r}} \right) \right)$. His consumption path is:

$$
\hat{c}_t = \begin{cases} 
0 & \text{for } t < s \\
-He^{-r(T-t)} + \kappa \left( 1 - r(t-s) \right) & \text{for } s \leq t < T \\
r\tilde{w}_T - H & \text{for } t \geq T 
\end{cases}
$$

with $\tilde{w}_T = \frac{H}{r} \left( 1 - e^{-r(T-s)} \right) - \kappa(T-s)$.

Let us take an example illustrated in Figure 4, with $r = 5\%$, $G = 2\%$, $T = 10$ years. This Figure plots the change in consumption and wealth for the rational actor $\kappa = 0$ (black, solid), and progressively less rational agents: $\kappa = 0.01$ (blue, dotted), $\kappa = 0.025$ (red, dashed-dotted), $\kappa = 0.1$ (green, dashed). The traditional Ricardian consumer ($\kappa = 0$) immediately decreases his consumption by $2\%$, which leads to wealth accumulation at until time $T$. In contrast the very BR consumer ($\kappa = 0.1$) doesn’t react at all until $T = 10$ (hence he doesn’t
accumulated any wealth), and then cuts a lot on consumption. The value $\kappa = 0.01$ and $\kappa = 0.025$ display an intermediary behavior. For $\kappa = 0.025$, the consumer initially doesn’t pay attention to the future tax. However, at a time $s = 4.5$ years, (i.e., when there are 3.6 years remaining until the taxes are effective), he starts paying attention, and starts savings for the future taxes. As the tax looms larger, the agent saves more. As the agent delayed his savings, he ends up cuttings down on consumption more drastically when taxes are in effect.

Smaller taxes generate a more delayed reaction. Controlling for the PV of taxes, consumers are better off with early rather than delayed taxes (as this allows them to smooth more).

## 5 A Boundedly Rational New Keynesian model

### 5.1 A Discounted Euler Equation

Here I explore the simplest specification of BR dynamic programming. I find that it leads to a natural modification of the New Keynesian framework, with important consequences for “forward guidance”.

The agent’s dynamic programming problem is as in Section 3.2, but I assume that all $m$’s are equal to some $\bar{m}$

$$m_r = m_y = m_{r'} = m_{y'} = \bar{m} \quad (48)$$

For instance, suppose that if the true model can be linearized as:

$$w_{t+1} = (R + \hat{\rho}_t) (w_t + y - c_t) + \hat{y}_{t+1}$$
$$z_{t+1} = A z_t + \varepsilon_{t+1}$$

where $z_t$ is a state vector, and for some vectors $k^y, k^r$, the agent’s subjective model of the world is:

$$w_{t+1} = (R + \bar{m} \hat{r}_t) (w_t + y - c_t) + \bar{m} \hat{y}_{t+1}$$
$$z_{t+1} = \bar{m} A z_t + \varepsilon_{t+1}$$
$$\hat{r}_t = k^r z_t, \quad \hat{y}_{t+1} = k^y z_{t+1}$$

i.e. future events have an extra discounting term in their transition function. For now, imagine $\hat{r}_t = \hat{r}_t$, so that the interest rate is perceived correctly.

In that case, we then have the following Euler equation.

**Proposition 5.1** (Discounted Euler equation) *If the consumer follows the BR dynamic programming approach, then we have the (linearized) discounted Euler equation:*

$$ME_t [\hat{c}_{t+1}] - \hat{c}_t = \bar{\sigma} \hat{r}^s_t + [(M - 1) + M (\hat{r}_t - \hat{r}_t)] \frac{R}{\hat{r}_t} w_t \quad (49)$$
wealth has changed by so his consumption is dissaved, so spoore ratim1). In addition, he now perceives a non income term, innovation in wealth, This gives the term in the Euler equation.6 The there are more terms. One, \((M - 1) \hat{r}_t w_t\) is a mechanical compensation term: recall that the consumer consumes a fraction \(m\) of it, so feels richer by the present value \(\frac{m^T}{R} \hat{y}_T\). Given the MPC is \(\frac{r}{R}\), he changes consumption by \(\hat{c}_0 = \frac{r}{R} \hat{m} \hat{y}_T\). At time 1, the consumer’s financial wealth has changed by \(\hat{w}_1 = -(R + \hat{r}_0) \hat{c}_0 \approx -R \hat{c}_0\) to the leading order (the consumer had dissaved, so is poorer at time 1). In addition, he now perceives an income \(\frac{m^T}{R} \hat{y}_T\) in \(T - 1\) periods, whose present value is \(\frac{m^T}{R} \hat{y}_T\). So he consumes a fraction \(\frac{r}{R}\) of this perceived wealth:

\[
\hat{c}_1 = \frac{r}{R} \left( \hat{w}_1 + \frac{m^{T-1}}{R^{T-1}} \hat{y}_T \right) = \frac{r}{R} \left( -R \hat{c}_0 + \frac{m^{T-1} R^{T+1}}{m R^T} \hat{c}_0 \right) = \left( -r + \frac{R}{m} \right) \hat{c}_0 = -\frac{Rm + R}{m} \hat{c}_0 = \frac{1}{M} \hat{c}_0
\]

This gives \(M \hat{c}_1 = \hat{c}_0\), a discounted Euler equation.

On the right-hand side of (49), consider the first term, \(\hat{\sigma} \hat{r}_t^*\). It is the perceived interest rate, not the actual one that matters for the consumption decision, hence for the term \(\hat{\sigma} \hat{r}_t^*\) in the Euler equation.6 There are two more terms. One, \((M - 1) \hat{r}_t^* w_t\) is a mechanical compensation term: recall that the consumer consumes a fraction \(\frac{r}{R}\) of wealth; if there’s an innovation in wealth, \(c_t = \frac{r}{R} w_t = c_{t+1}\), so that \(ME_t [\hat{c}_{t+1}] - \hat{c}_t = (M - 1) \frac{r}{R} w_t\). The last term, \(M (\hat{r}_t - \hat{r}_t^*) \frac{r}{R} w_t\), reflects “surprise income”: the interest rate income at time \(t + 1\) is higher by \((r_t - r_t^*) w_t\) at time \(t + 1\), and a fraction \(\frac{r}{R}\) of it is consumed at time \(t + 1\). That gives the extra adjustment term.

The distinction between \(m\) and \(M\) is not very important: they are same up to second order terms. In the limit of small time intervals, write \(m = 1 - \xi\) for a small \(\xi\). Then, \(M = \frac{1 - \eta}{1 + r - r(1 - \xi)} \approx 1 - \xi + O(\xi r)\).

---

6To understand that expression, suppose that there is a disturbance \(\hat{r}_0\), lasting for 1 period, and take the simplest case with no initial wealth. In the rational model, we should have \(\hat{c}_0 = -\frac{\psi e^d}{R} \hat{r}_0\) (To see this, observe that \(\hat{c}_0\) creates \(\hat{w}_1 = -R \hat{c}_0\), so \(\hat{c}_1 = \frac{r}{R} \hat{c}_0 = -r \hat{c}_0\); the Euler equation imposes \(\psi e^d \frac{r}{R} = \hat{c}_1 - \hat{c}_0 = -R \hat{c}_0\), hence \(\hat{c}_0 = -\frac{\psi e^d}{R} \hat{r}_0\)). The BR agent instead uses a perceive interest rate \(\hat{r}_0^*\), and discounts by \(m\) it’s perceived effects, so his consumption is \(\hat{c}_0 = -\frac{m \psi e^d \hat{r}_0^*}{R}\). As \(\hat{c}_1 = -r \hat{c}_0\),

\[
M \hat{c}_1 - \hat{c}_0 = -(Mr + 1) \hat{c}_0 = -\frac{RM}{m} \hat{c}_0 = -M \frac{m \psi e^d}{R} \hat{r}_0^* = \hat{\sigma} \hat{r}_0^*.
\]

This gives the term \(\hat{\sigma} \hat{r}_0^*\) on the RHS of (49).
Model with discounted Euler equation for firms This subsection can be skipped at the first reading. Here I explore what happens if firms also don’t fully process the future. Suppose indeed the same setup for firms that for consumers, but now with a coefficient $M^f$ for firms. That is, given their state variable $z_t$, the firm’s dynamic programming problem is:

$$V(z_t) = \max_{p^*_t, N_t, m} v(N_t, z_t) + \beta \mathbb{E} [V(F^z(m \odot z_t))]$$

(53)

where $v(N_t, z_t)$ is the profit function, $N_t$ is the quantity of labor to employ, and with $m = (m_w, m_x)$ with $m_w = 1$ and $m_x = M^f(1, \ldots, 1)$, i.e. $m_x$ uniformly depends all quantities (this can be relaxed later, and gives similar results, though in a more complex way).

We apply this to the NK model, as exposited in Gali (2015, Chapter 3). Then, calculations show that the firm’s optimal price is:

$$p^*_t = (1 - \beta \theta) \sum_{k=0}^{\infty} (\beta \theta)^k E_t^{BR}[\hat{p}_{t+k} - \Theta \hat{\pi}_{t+k}]$$

(54)

where $E_t^{BR}$ is the expectation with the beliefs induced by the partially inattentive model: $z_{t+1} = F^z(m \odot z_{t+1})$. This gives:

$$p^*_t = (1 - \beta \theta) \sum_{k=0}^{\infty} (\beta \theta M^f)^k [\hat{p}_{t+k} - \Theta \hat{\pi}_{t+k}]$$

By following the same steps as Gali, we have (with $\kappa = (1 - \theta) (1 - \beta \theta M^f) / \Theta$)

$$\pi_t = \beta M^f E_t[\pi_{t+1}] + \kappa x_t$$

(55)

This is the NK Phillips curve, with partially attentive firms. The case $M^f = 1$, fully attention, is the traditional curve.

I should emphasize that those firms can be fully attentive to all idiosyncratic terms (something that will be easy to include in a future version of the paper), e.g. idiosyncratic part of productivity of demand. They simple have pay attention $M^f$ to macro outcomes. If we include idiosyncratic terms, and firms are fully attentive to them, the aggregate NK curve doesn’t change.

5.2 Implication of the discounted Euler equation for the Behavioral NK model

In the traditional NK model, there is no capital or government spending, so GDP is consumption. Calling $x_t = (c_t - c^*_t) / c^d$ the output gap, the traditional NK model is:

$$x_t = E_t[x_{t+1}] - \sigma (i_t - E_t \pi_{t+1} - r_t^n)$$

(55)

$$\pi_t = \beta E_t[\pi_{t+1}] + \kappa x_t$$

(56)
where $r^n_t$ is the real risk-free rate in the frictionless economy. I now state its inattentive version.

**Proposition 5.2** This behavioral version of the New Keynesian model yields:

\[ x_t = ME_t[x_{t+1}] - \sigma (i_t - E_t \pi_{t+1} - r^n_t) \quad (IS \ curve) \tag{58} \]

\[ \pi_t = \beta M^I E_t[\pi_{t+1}] + \kappa x_t \quad (Phillips \ curve) \tag{59} \]

where $M, M^I \in [0, 1]$ are the attention of consumers and firms, respectively, to macroeconomic outcomes. In the traditional model, $M = M^I = 1$. In addition, $\sigma = M \psi \frac{1}{\bar{r}}$.

**Proof.** The most challenging part is (58). To derive it, we just apply Proposition 5.1, with $w_t = 0$: there is no wealth in the NK model. □

The behavioral “New Keynesian IS curve”, (58). It is the more drastic modification of the framework. It implies, if $\hat{r}_t^s = \hat{r}_t$,

\[ \hat{c}_t = \hat{\sigma} \sum_{s \geq t} M^{s-t} E_t[i_t - E_t \pi_{t+1} - r^u_t] \]

i.e. it’s the discounted value of future interest rates that matters, rather than the undiscounted sum. This will be important soon when we study forward guidance.

In particular, in the NK model with the corrective tax, there’s no wealth, so that we plainly have

\[ ME_t[\hat{c}_{t+1}] - \hat{c}_t = \hat{\sigma} \hat{r}_t^s \]

where $\hat{r}_t^s$ is the interest rate perceived at $t$. McKay, Nakamura and Steinnsson (MNS, 2015) do argue that this equation fit better. They provide a microfoundation based on heterogeneous rational agents with limited risk sharing. In their model, wealthy, unconstrained agents with no unemployment risk would still satisfy the usual Euler equation. Piergallini (2006), Nistico (2012), Del Negro, Giannoni, Patterson (2015), other micro-foundations with heterogeneous mortality shocks, as in a perpetual-youth models (this severely limit how myopic agents can be, given that life expectancies are quite high). Here I keep the representative agent, but make him more boundedly rational; wealth, behavioral agent would still satisfy this discounted Euler equation.

\[ 0 = -\gamma (Ec_{t+1} - c_t) + r_t - \rho_t, \]

we set $\sigma = \psi$, and $r^u_t = \rho_t$ and $r_t = i_t - E_t \pi_{t+1}$.

---

\[ ^7 \text{See e.g. Gali (2015) for a textbook exposition. In the traditional proof, output is consumption (there’s no investment), and the Euler equation holds: } 1 = E \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \beta_t R_t, \text{ which leads to (with } \beta_t = e^{-\rho_t}) \]

\[ 0 = -\gamma (Ec_{t+1} - c_t) + r_t - \rho_t, \quad \text{and} \]

\[ c_t = E_tC_{t+1} - \psi (r_t - \rho_t) \tag{57} \]

we set $\sigma = \psi$, and $r^u_t = \rho_t$ and $r_t = i_t - E_t \pi_{t+1}$.
Figure 5: This Figure shows the response of current inflation to forward guidance about interest rate in $T$ periods, compared to an immediate rate change of the same magnitude. Units are yearly. The left panel is the traditional New Keynesian model, the right panel the behavioral model. Parameters are the same in both models, except that (annualized) attention is $M = e^{-\xi} = 0.7$ in the behavioral model, and $M = 1$ in the traditional model.

The modified Philipps curve (59) Its change is less drastic, as it simply changes $\beta$ into $\beta M^T$. Let me reiterate that firms are still forward looking (with discount parameter $\beta$) in the deterministic steady state. It’s only there sensitivity to deviations around the deterministic steady state that they’re partially myopic.

We now study two impact of this modifications, to the forward guidance puzzle.

5.3 Application to the Forward Guidance Puzzle

5.3.1 An announcement of a rate cut at time $T$

Here I follow McKay, Nakamura and Steinsson (2015). Suppose that the central bank announces a time 0 that it will cut the rate at time $T$, following a policy $\delta_t = 0$ for $t \neq T$, $\delta_T < 0$. What is the impact?

We have $x_t = M x_{t+1} - \sigma \delta_t$, so $x_t = -\sigma M^{T-t} \delta_T$ for $t \leq T$ and $x_t = 0$ for $t > T$. Next, inflation is

$$\pi_0 = \kappa \sum_{t \geq 0} \beta^t x_t = -\kappa \sigma \sum_{t=0}^{T} \beta^t M^{T-t} \delta_T = -\kappa \sigma \frac{M^{T+1} - \beta^{T+1}}{M - \beta} \delta_T$$

In the traditional case, $M = 1$, so that a rate cut in a very long future has a big impact on inflation, $\pi_0 = -\frac{\kappa \sigma}{1-\beta}$. In contrast, when $M < 1$, the effect is just 0.

Figure 5 illustrate the effect.
5.3.2 A persistently binding ZLB

In the behavioral NK model, write $M = e^{-\xi \Delta t}$, where $\xi$ is cognitive discounting parameter due to myopia. The model (58) becomes:

$$\dot{x}_t = \xi x_t + \sigma (i_t - r_t - \pi_t)$$  \hspace{1cm} (60)

$$\dot{\pi}_t = \rho \pi_t - \kappa x_t$$  \hspace{1cm} (61)

When $\xi = 0$, we recover Werning (2012)’s formulation, in which agents are all rational. Also, $\rho = -\ln (\beta M^T)$ is the discount rate inclusive of firm’s myopia.

A ZLB scenario. I follow a thought experiment in Werning (2012), but with behavioral agents. I take $r_t = \underline{r}$ for $t \leq T$, and $r_t = \bar{r}$ for $t > T$, with $\underline{r} < 0 < \bar{r}$. I assume that for $t > T$, the central bank implements, $x_t = \pi_t = 0$, by setting $i_t = \bar{r}$. At time $t < T$, I suppose that the CB is at the ZLB, so that $i_t = 0$.

**Proposition 5.3** In the traditional case ($\xi = 0$), we obtain an unboundedly intense recession as the length of the ZLB is longer:

$$\lim_{t \to -\infty} x_t = -\infty.$$  

This also holds when myopia is mild, $\xi \in [0, \frac{\sigma \rho}{\kappa}]$.

However when cognitive myopia is strong enough ($\xi > \frac{\sigma \rho}{\kappa}$), we obtain a boundedly intense recession

$$\inf_{t \in (-\infty, T]} x_t = \lim_{t \to -\infty} x_t = \frac{\rho \sigma \underline{r}}{\rho \xi - \sigma \kappa} < 0.$$  

We see how impactful myopia can be. We see that myopia has to be stronger when the agent are highly sensitive to the interest rate ($\sigma$ high) and price flexibility is high (high $\kappa$). High price flexibility makes the system very reactive, and a high myopia is useful to counterbalance that.

The dynamics are illustrated in Figure 6. The left panel shows the traditional model, the right one the behavioral model. The parameters are the same in both models, except that attention is lower (set an annualized rate of $M = e^{-\xi} = 0.7$) in the behavioral model (against its value $M = 1$ in the traditional model).\footnote{Note that the way to read (60) is as a forward equation. Calling $\delta_t := i_t - r_t - \pi_t$ the excess interest rate. We take into account the fact that the model is stationary ($\lim_{t \to -\infty} x_t = 0$). We have:

$$\dot{x}_t = -\sigma \int_t^\infty e^{-\xi(s-t)} \delta_s ds$$  \hspace{1cm} (62)

Hence, the effect of future rate changes is dampened by myopia.}

\footnote{The other parameters are: $\rho = 3\%$, $\kappa = 0.1$, $\sigma = 0.2$, $\underline{r} = -5\%$.}
Figure 6: This Figure shows the output gap $x_t$. The economy is at the Zero Lower Bound during times $0$ to $T = 15$ years. The left panel is the traditional New Keynesian model, the right panel the behavioral model. Parameters are the same in both models, except that (annualized) attention is $M = e^{-\xi} = 0.7$ in the behavioral model, and $M = 1$ in the traditional model.

6 Neoclassical Growth Model: A Boundedly Rational Version

We can now study a BR version of the baseline neoclassical growth model, the Ramsey-Cass-Koopmans model.

6.1 Setup

The utility function is still $E \sum_{t} \beta^{t} C_{t}^{1-\gamma} / (1 - \gamma)$, and we again call $\psi = \frac{1}{\gamma}$ and $\beta = e^{-\rho}$. In the aggregate, the capital stock follows:

$$K_{t+1} = f (K_{t}, L) + (1 - \delta) K_{t} - C_{t} + \epsilon_{t+1} \tag{63}$$

where $\epsilon_{t+1}$ are mean-zero shocks, whose distribution we’ll specify later. This way, there is just one state variable in the economy, the capital stock. In the most basic neoclassical model, $\epsilon_{t+1}$ is always 0, and $L$ is fixed.

This is a textbook example, which can be found e.g. in Acemoglu (2009, Chapter 8), Blanchard-Fischer (1989, Chapter 2), Romer (2012, Chapter 2); it introduces generations of students to macroeconomics. However, it looks somewhat odd (in my opinion), with these infinitely-rational forward looking agents that calculate the whole macroeconomic equilibrium in their heads. I present here an alternative to that model.

**Let us first review some mechanics of convergence.** If there were no shocks, the economy would be at the steady state, with capital stock $K^*$. I use the hat notation for deviations from the mean, e.g. $\hat{K}_t = K_t - K^*$. The law of motion for capital (63) is, in
Figure 7: This Figure shows the traditional approach to the neoclassical growth model. Arguably, this is psychologically quite absurd. The present paper proposes a more behavioral approach.

**Linearized Form:**

\[
\hat{K}_{t+1} = (1 + r) \hat{K}_t - \hat{C}_t + \varepsilon_{t+1}
\]

where \( r \) is the steady state interest rate, \( r = \beta^{-1} - 1 \).

As there is one state variable, the linear policy function of the agent (rational or not) is:

\[
\hat{C}_t = b\hat{K}_t
\]

for some constant \( b \) to be determined.

Plugging this into (64) we obtain:

\[
\hat{K}_{t+1} = (1 + r - b) \hat{K}_t + \varepsilon_{t+1}, \text{ i.e.}
\]

\[
\hat{K}_{t+1} = (1 - \phi) \hat{K}_t + \varepsilon_{t+1}
\]

where \( \phi \) is the speed of mean-reversion:

\[
\phi = b - r.
\]

When agents are more reactive to shocks (when \( b \) is higher), the economy mean-reverts faster to the steady state (\( \phi \) is higher).

Finally, squaring equation (66), we obtain: \( \text{var}\hat{K}_{t+1} = (1 - \phi)^2 \text{var}\hat{K}_t + \sigma^2 \). As in the steady state, \( \text{var}\hat{K}_{t+1} = \text{var}\hat{K}_t, \text{var}\hat{K}_t = \frac{\sigma^2}{1 - (1 - \phi)^2} \), i.e., in the limit of small time intervals:

\[
\sigma_K = \frac{\sigma_\varepsilon}{\sqrt{2\phi}}
\]

When shocks mean-revert more slowly (lower \( \phi \)), the average deviation of the stock price from trends is higher (shocks “pile up” more).
The rational agent has a value function $V(K_t)$, which satisfies:

$$V(K) = \max_c u(c) + \beta \mathbb{E}[V(K')]$$

with $K' = F(K, L) - c + \varepsilon_t$

where $F(K) := f(K, L) - \delta K$ is output net of depreciation.

The steady state is at $K = K^*, C_t = C^*$ with:

$$F'(K^*) = \rho$$

which determines $K^*$ and consumption is determined by:

$$C^* = F(K^*)$$

We define:

$$\xi = \frac{u'(C^*)}{u''(C^*)} F''(K^*) = -\psi C^* F''(K^*) \geq 0$$

which plays an important role later.

### 6.2 Boundedly Rational Version

The agent has wealth $k_t$, and we normalize the population to be 1, so that in equilibrium will be equal to $k_t = K_t$. The agent’s wealth evolves as:

$$k_{t+1} = (1 + r_t) (k_t + y_t - c_t)$$

where $y_t = F(K_t) - K_t F'(K_t)$ is labor income, and $r_t = F'(K_t)$ is the interest rate. Taylor expansion yields, to the leading order:

$$\hat{r}_t = F''(K^*) \hat{K}_t, \quad \hat{y}_t = -K^* F''(K^*) \hat{K}_t$$

(70)

In the agent’s model, income and interest rate evolve as:

$$\hat{r}_{t+1} = (1 - \phi^s) \hat{r}_t, \quad \hat{y}_{t+1} = (1 - \phi^y) \hat{y}_t$$

where $\phi^s$ is the perceived speed of mean-reversion. We again parametrize it as: $\phi^s = (1 - m_\phi) \phi^d + m_\phi \phi^e$, where $\phi$ is the equilibrium speed of mean-reversion, and $\phi^d$ is a default value – perhaps coming from some empirical experience, saying that “business cycles” have a half-life of a few years.

**Rational agent**  Lemma 3.2 reads here:

$$\hat{c}_t = r \hat{k}_t + \frac{r}{r + \phi} \hat{y}_t + \frac{r k^* - c^* \psi}{r + \phi} \hat{r}_t$$

and using (70) and (69) we obtain:

$$\hat{c}_t = r \hat{k}_t + \frac{-r K^* F''(K^*) + (r k^* - c^* \psi) F''(K^*)}{r + \phi} \hat{K}_t$$

30
i.e., as \( k^* = K^* \),

\[
\widehat{c}_t = r\widehat{k}_t + \frac{\xi}{r + \phi}\widehat{K}_t
\]  

(71)

Hence, the aggregate consumption follows \( \widehat{C}_t = b\widehat{K}_t \) with:

\[
b = r + \frac{\xi}{r + \phi}
\]

and using (67), the speed of mean reversion is:

\[
\phi = \frac{\xi}{r + \phi}
\]

Solving for the fixed point in \( \phi \):

\[
\phi = \frac{-r + \sqrt{r^2 + 4\xi}}{2}.
\]  

(72)

**Behavioral agent**  The behavioral agent partially does not think about those aggregate shocks \( \widehat{K}_t \). Hence, instead of (71), the BR agent pays an attention \( m_K \) to the capital stock, and perceives that it mean-reverts at a speed

\[
\phi^s = (1 - m_\phi) \phi^d + m_\phi \phi.
\]

where \( \phi^d \) is a default speed of mean-reversion (which in practice could be the empirical speed of mean-reversion of business cycles). This gives:

\[
\widehat{c}_t = r\widehat{k}_t + \frac{\xi}{r + \phi^s} m_K \widehat{K}_t
\]  

(73)

Endogenizing \( m_K \), we obtain:

\[
\widehat{c}_t = r\widehat{k}_t + \frac{\xi}{r + \phi^s} m_K \widehat{K}_t
\]

Hence, \( b = r + \tau \left( \frac{\xi}{r + \phi^s} \frac{\kappa_c}{\sigma_K} \right) \widehat{K}_t \), and using (67) we obtain:

\[
\phi = \tau \left( \frac{\xi}{r + \phi^s} \frac{\kappa_c}{\sigma_K} \right).
\]

Now, what is \( m_\phi \), the attention to the accurate speed of mean-reversion? Given (73), \( \frac{\partial c}{\partial \phi} |_{m_K = 0} = 0 \). This means that using the sparse max, \( m_\phi = 0 \). (With the iterated sparse max, we can have \( m_\phi > 0 \) if \( \pi \) is small enough). Hence, we have

\[
\phi = \tau \left( \phi_0, \frac{\kappa_c}{\sigma_K} \right), \quad \phi_0 := \frac{\xi}{r + \phi^d}
\]  

(74)
When $\kappa$ is exogenous, then recalling that $\kappa_c = \sqrt{\frac{u_{cc}}{u_{cc}}}$ and (68), we have:

$$\phi = \tau \left( \phi_0, \frac{\kappa_c}{\sigma_K} \right) = \tau \left( \phi_0, \frac{\sqrt{\kappa_{|u_{cc}|}}}{\sigma \sqrt{2\phi}} \right) = \tau \left( \phi_0, \sqrt{B \phi} \right), \quad B := \frac{2\kappa}{|u_{cc}| \sigma^2}$$

For simplicity, we use the $\tau_1$ truncation function here (see equation 10): $\tau (b, k) = b \max \left( 1 - \frac{k^2}{b^2}, 0 \right)$, so $\phi = \phi_0 - \frac{B \phi}{\phi_0}$, and

$$\phi = \frac{\phi_0}{1 + \frac{B \phi}{\phi_0}}.$$  \hfill (75)

Another solution is to use the “scale-free” version of $\kappa$, equation (95). This gives $\frac{\kappa_c}{\sigma_K} = \frac{\pi_{cc}}{\kappa_{|u_{cc}|}} = \pi (r + \phi_0)$, and

$$\phi = \tau \left( \phi_0, \pi (r + \phi_0) \right)$$  \hfill (76)

Note that $\phi$ has the same comparative statics as the rational case: $\frac{\partial \phi (\xi, r)}{\partial k} \geq 0$, $\frac{\partial \phi (\xi, r)}{\partial r} \leq 0$.

There is a new comparative statics: $\frac{\partial \phi}{\partial \kappa} \leq 0$.

**Proposition 6.1** The speed of mean-reversion of the economy is:

$$\phi = \frac{\xi}{r + \phi^d - \pi r} +$$

If the default perception $\phi^d$ is the actual $\phi$ (as in rational expectations), $\phi^d = \phi$, so that: $\phi = \frac{\xi}{r + \phi - \pi r}$, whose solution is

$$\phi = \frac{1}{2} \left( -r (1 + \pi) + \sqrt{r^2 (1 - \kappa)^2 + 4 \xi} \right) +$$

In particular, $\phi$ is decreasing in $\pi$, $\phi (\pi = 0) = \phi^r$. We have $\phi > 0$ iff $\pi \leq \pi^* := (1 + r^2 / \xi)^{-2}$.

**The impact of fluctuations** Hence, (provided $\phi^d \geq \phi$) the variance of shocks will be larger in the sparse economy than in the rational economy.

**Proposition 6.2** Suppose that $\phi^d \geq \phi$. Then, $\phi \leq \phi^r$, so that in the sparse economy, the speed of mean-reversion is slower, and the variance of shocks is bigger, than in the rational economy.

## 7 Behavioral Version of a Few Other Models

To probe the validity of the framework, we study here a few other models.
7.1 The life-cycle model

By life-cycle model, I mean a model that features a finite life (unlike the previous infinite-horizon model), and emphasizes the need to save for retirement. I develop the BR version here.

The agent lives for $T$ periods, receives income $y_t = y$ for times $t \in [0, L]$, (where $L$ is the time where labor income shocks) then $y_t = y + \hat{y}$ (with $\hat{y} < 0$) for $t \in [L, T]$. We call $B = T - L$ the length of retirement. Utility is: $\sum_{t=0}^{T-1} u(c_t)$. The interest rate and the discount rate are both 0.

In the rational model, the optimal policy is to smooth consumption: consume

$$c_t = \frac{w_0 + L y + B (y + \hat{y})}{T} = \frac{w_0}{T} + y + \frac{B}{T} \hat{y}$$

Proposition 7.1 In the BR life-cycle model, the optimal consumption policy is, before retirement ($t < L$)

$$c_t = \frac{w_t}{T - t} + \tau \left( \frac{B \hat{y}}{T - t}, \kappa \right) + y.$$

and after retirement $c_t = \frac{w}{T - t} + y + \hat{y}$ for $t \geq L$. Hence, when $\kappa > 0$ and $\hat{y} < 0$, consumption weakly falls over time, and discretely falls at retirement. After retirement, consumption is is constant.

7.2 Dynamic Portfolio Choice

I now study a Merton problem with dynamic portfolio choice. The agent’s utility is: $E \left[ \frac{1}{1 - \gamma} \int_0^\infty e^{-\gamma s} c_s^{1-\gamma} ds \right]$, and his wealth $w_t$ evolves according to:

$$dw_t = (-c_t + rw_t) \, dt + w_t \theta_t (\pi_t \, dt + \sigma dZ_t)$$

where $\theta_t$ is the allocation to equities.

I start by describing the rational problem, then the behavioral solution. I call $\psi = \frac{1}{\gamma}$ the IES. Though for simplicity I use a CRRA utility function, I try to write the expressions in a way that involves both $\gamma$ and $\psi$, in a way that would generalize correctly to Epstein-Zin utility, where the two notions are disentangled.

7.2.1 Taylor expansions of the value function: rational case

We examine the problem in the rational case first, with a reminder of notions of portfolio choice. In a deterministic context with interest rate $r_t$, the SDF is simply $M_t = e^{-\int_0^t r_t \, ds}$. Next, suppose that there is a stochastic opportunity set: A set of assets with risk premium $\pi_t$, and covariance matrix $\Sigma_t$. In a static maximization, the optimal portfolio the certainty equivalent is a return: $R_t(\theta_t) = r_t + \theta_t \pi_t - \frac{1}{2} \theta_t \Sigma_t \theta_t$, so that the (static) optimal portfolio
choice is $\theta_t = \arg\max_\theta R_t(\theta)$, i.e. $\theta_t = \frac{1}{\gamma} \Sigma_t^{-1} \pi_t$, and the certainty equivalent is finally:

$$R_t = r_t + \frac{1}{2\gamma} \Lambda_t$$  \hspace{1cm} (77)

where

$$\Lambda_t = \pi_t \Sigma_t^{-1} \pi_t$$  \hspace{1cm} (78)

the “squared Sharpe ratio” of the investment opportunity set. Suppose that the process is driven by a Brownian motion $B_t$ (which may be multidimensional). If the price of risk is $\lambda_t$ (so that $\Lambda_t = \|\lambda_t\|^2$), the stochastic discount factor can be represented as:

$$M_t = \exp \left[ - \int_0^t \left( r_s + \frac{\Lambda_t}{2} \right) ds - \lambda_s dB_s \right]$$  \hspace{1cm} (79)

The value function is as follows.

**Lemma 7.1 (Value function, traditional case)** Suppose that the interest rate $r_t$ and the price of risk $\lambda_t$ are deterministic, and that the agent is the traditional rational agent. The value function is

$$V_w(w_t, x_t) = (\mu_t w_t)^{-\gamma}$$

and the optimal policy is to consume $c_t = \mu_t w_t$, where:

$$\mu_t^{-1} = E_t \left[ \int_0^\infty e^{-\psi \rho s} \left( \frac{M_t + \psi}{M_t} \right)^{1-\psi} ds \right] = E_t \left[ \int_0^\infty e^{-\mu_s - (1-\psi) R_s} du ds \right]$$  \hspace{1cm} (80)

where

$$R_t = r_t + \frac{1}{2\gamma} \Lambda_t.$$

is the certainty equivalent of expected portfolio returns (comprising stocks and bonds), with $\Lambda_t = \|\lambda_t\|^2$ is the square Sharpe ratio of the investment opportunity set.

When the opportunity set is constant, we have $R_t = R_*$ and $\mu_t = \mu_*$ with

$$\mu_* = \psi \rho + (1 - \psi) R_*.$$  \hspace{1cm} (81)

When it is not constant, we have, up to second order terms:

$$\mu_t = \psi \rho + (1 - \psi) \bar{R}_t$$  \hspace{1cm} (82)

where $\bar{R}_t = \mu_*, V_t^R$ is the average future portfolio returns, and $V_t^R$ is the present value of future portfolio returns.

$$V_t^R := E_t \left[ \int_t^\infty e^{-\mu_s (s-t)} R_s ds \right]$$  \hspace{1cm} (83)
Here \( \bar{R}_t \) is the future average return of the portfolio (including stocks and bonds). Hence, the marginal propensity to consume is a weighted average (with weights \( \psi \) and \( 1 - \psi \)) of the pure rate of time preference \( \rho \), and the average future return of the portfolio.

Lemma 7.1 summarizes and somewhat generalizes well-known notions, particularly from the work of Campbell and Viceira (2002). It indicates that what matters is the risk-adjusted rate of return of the portfolio, \( R_t \): it is the safe short-term rate \( r_t \), plus the square Sharpe ratio \( \Lambda_t \), divided by two times risk aversion. The future average return \( \bar{R}_t \) is key to capture the (leading order of) the value function.

To structure the problem, suppose that the vector of asset returns \( d\tilde{r}_i \) (where \( d\tilde{r}_i \) is the return of asset \( i \)) follows:

\[
d\tilde{r} = (r + \pi_s + \hat{\pi}_t) dt + \sigma dZ_t
\]

\[
\hat{\pi}_t = f'X_t
\]

where \( X_t \) is a vector of factors, following an AR(1):\(^{10}\)

\[
dX_t = -\Phi X_t dt + \sigma X dZ_t
\]

and \( f \) is a matrix of weights. We call

\[
\Sigma^{r,X} = \text{cov}(d\tilde{r}, dX_t')/dt = \sigma \sigma^{X'}.
\]

the matrix of covariance, i.e. \( \Sigma^{r,X}_{ij} = \text{cov}(d\tilde{r}_i, dX_{jt})/dt \). We define \( \theta_* = \frac{1}{\gamma} \Sigma^{-1}_s \pi_s \) the portfolio choice in the model with constant variance and expected returns.

Then, the portfolio return is

\[
R_t = \frac{1}{2\gamma} (\pi_s + \hat{\pi}_t)' \Sigma^{-1}_t (\pi_s + \hat{\pi}_t) = \frac{1}{2\gamma} \pi_s \Sigma^{-1}_s \pi_s + \theta'_s \hat{\pi}_t + O(\|X_t\|^2)
\]

\[
= R_s + \theta'_s \hat{\pi}_t
\]

\[
= R_s + \theta'_sf'X_t = R_s + b'X_t
\]

i.e. the return is augmented by \( \theta'_s \hat{\pi}_t \), with

\[
b := f\theta_*.\]

Then, the present value of returns (83) is:

\[
V^R_t = \frac{R_t}{\mu_*} + b' (\mu_s I + \Phi)^{-1} X_t
\]  

(84)

where \( I \) is the identity matrix of the \( X \)'s dimension.

For instance, if \( X_t \) is a one-dimensional, so that \( bX_t = \hat{R}_t := R_t - R_s \), and \( \bar{R}_t := R_s + \frac{\mu_s}{\mu_* + \Phi R} \hat{R}_t \).

\[
\mu_t = \mu_* + (1 - \psi) \frac{\mu_*}{\mu_* + \Phi} \hat{R}_t
\]

(85)

Hence, we obtain a tractable representation of the value function, to the leading order.

\(^{10}\)Or \( X_t \) could be a linearity-generating twisted-AR(1), so that the derivations below can be exact (Gabaix 2009).
7.2.2 The hedging demand

We can calculate the hedging demand.

**Lemma 7.2** (Hedging demand, rational) The stock demand is

\[ \theta_t = \frac{1}{\gamma} \Sigma_t^{-1} (\pi_t + H_t) \]  

(86)

where \( H_t \) is the hedging demand premium, equal to (up to second order terms):

\[ H_{it} = (1 - \gamma) \text{cov} (d\tilde{r}_i, dV_t^R) \]  

(87)

i.e. \( H_{it} \) is \((1 - \gamma)\) times the covariance between asset \( i \)'s return \((d\tilde{r}_i)\) and the present value of future returns \( V_t^R \) (equation 83).

In the AR(1) framework above,

\[ H_t = (1 - \gamma) \Sigma r'X (\mu_s I + \Phi')^{-1} b. \]  

(88)

Suppose that returns mean-revert, i.e. \( \text{cov} (d\tilde{r}_{it}, d\tilde{R}_t) < 0 \). So, if \( \psi < 1 \), then investors load more on stocks because of the hedging demand.

We next state the modification of the value function.

**Lemma 7.3** (Value function with hedging demand, rational) In the hedging demand context, we have:

\[ \mu_t = \psi \rho + (1 - \psi) (\tilde{R}_t + \theta' H_t) \]  

(89)

where \( \tilde{R}_t = \mu_s V_t^R \) is the expected present value of returns, and \( H_t \) is the hedging demand term; they are explicit in (84) and (88).

The intuition for (86) is that \( H_t \) is a risk-adjusted risk premium of asset \( i \). This intuition carries over to (89). Compared to (82), the expression for \( \mu (X_t) \) offers one more term, the term \((1 - \psi) \theta' H_t\).

**A tractable case** The equity premium \( \pi_t = \pi + \tilde{\pi}_t \) has a variable part \( \tilde{\pi}_t \), which follows

\[ d\tilde{\pi}_t = -\phi_R \tilde{\pi}_t dt - \chi_t \sigma_d Z_t^1 + \sigma_{\tilde{\pi}} dZ_t^2 \]

where the return is \( d\tilde{r}_t = (r_t + \pi_t) dt + \sigma dZ_t^1 \). The parameter \( \chi_t \geq 0 \) indicates that equity returns mean-revert: good returns today lead to lower returns tomorrow. That will create a hedging demand term.

We call \( \theta_s := \frac{\pi}{\pi \sigma^2} \) the standard, myopic demand for stocks.
7.2.3 The sparse agent’s investment and consumption

We can calculate the sparse agent’s demand. Recall that $\psi = 1/\gamma$ is the IES.

**Proposition 7.2** (Behavioral dynamic portfolio choice) The fraction of wealth allocated to equities is, with $\theta_* := \frac{\psi}{\xi}$

$$\theta^*_t = \theta_* + \tau \left( \frac{\widehat{\pi}_t}{\gamma \sigma^2}, \kappa \right) + \tau \left( \frac{H_t}{\gamma \sigma^2}, \kappa_{\theta} \right)$$

while consumption is: $c^*_t = \mu^*_t w_t$ with

$$\mu^*_t = \mu_* + \tau \left( 1 - \psi \right) \frac{\mu_*}{\mu_* + \Phi} \theta_* \widehat{\pi}_t, \kappa_{c/w} + \tau \left( 1 - \psi \right) \theta_* H_t, \kappa_{c/w}$$

where $H_t$ is the hedging demand term (90)

$$H_t = (1 - \gamma) \text{cov} \left( dr_t, dV^R_t \right) = -(1 - \gamma) \theta_* \frac{1}{\mu_* + \Phi} \sigma^2 \chi_t$$

**Proof** We first calculate the rational values. In that case

$$\widehat{R}_t = r_* + \frac{\Lambda_*}{2\gamma} + \theta_* \frac{\mu_*}{\mu_* + \Phi} \widehat{\pi}_t$$

$$H_t = (1 - \gamma) \text{cov} \left( dr_t, d \left( \frac{\widehat{R}_t}{\mu_*} \right) \right) = -(1 - \gamma) \theta_* \frac{1}{\mu_* + \Phi} \sigma^2 \chi_t \quad (90)$$

so that

$$\theta_t = \frac{\pi_* + \widehat{\pi}_t + H_t}{\gamma \sigma^2}$$

In addition

$$\mu_t = \psi \rho + (1 - \psi) \left( \widehat{R}_t + \theta'_* H_t \right) = \mu_* + (1 - \psi) \left( \theta_* \frac{\mu_*}{\mu_* + \Phi} \widehat{\pi}_t + \theta'_* H_t \right)$$

As in Proposition 2.2, with ex-post attention, the BR agent just truncates those terms.

Proposition 7.2 predicts the choice of a sparse agent. When $\kappa = 0$, it is the policy of a fully rational agent, e.g. as in Campbell and Viceira (2002). When $\kappa > 0$, it is the policy of a sparse agent. When $\kappa$ is larger, portfolio choice becomes insensitive to the change in the equity premium, $\widehat{\pi}_t$, and the agent thinks less about the mean-reversion of asset, the $B\chi$ terms.

In addition, the agents’ consumption function pays little attention to the mean-reversion of assets. [Next iteration should have a calibration.]
7.3 Linear-Quadratic models

A lot of economic problems can be conveniently expressed as linear-quadratic (LQ) models (Ljungqvist and Sargent 2012). We show here how to systematically derive a BR version of those models.

We again write \( \zeta = (\omega, \xi) \), where \( \omega \) is the set of variables known under the default model, and \( \xi \) is the set of variables that are not-considered in the default model. Utility is:

\[
u(\zeta, \alpha) := \frac{1}{2} \begin{pmatrix} \omega' \\ a \end{pmatrix}' \begin{pmatrix} U_{zz} & U_{za} \\ U_{az} & U_{aa} \end{pmatrix} \begin{pmatrix} \zeta \\ a \end{pmatrix}
\]

and the law of motion is:

\[
z' = F^z(z, a) := \Gamma^z_z z + \Gamma^z_\alpha a
\]

where \( U \) and \( \Gamma \) are constant matrices. The rational value function is also LQ

\[
V(z) = -\frac{1}{2} z'V_{zz} z = -\frac{1}{2} (w'V_{ww} w + 2w'V_{wx} x + x'V_{xx} x)
\]

Under the default model, \( V_{ww} \) is known, and

\[
a^d(w) = A_w w
\]

for \( A_w \) a constant. Our goal is to find \( V_{wx} \), which affects the value function. To do so, we apply from (287).

**Lemma 7.4** In the linear-quadratic problem, the cross-partial of the value function is

\[
V_{wx} = V_{wx} = \left[ 1 - \beta (D_w w') \cdot \Gamma^x_z \right]^{-1} [U_{xx} + U_{xa} A_w + \beta \Gamma^w_x V_{ww} (D_w w')].
\]

where \( D_w w' = \Gamma^w_w + \Gamma^w_a A_w \). The impact on the action is \( a = A_w w + A_x x \), where \( A_w \) is the default value, and

\[
A_x = -\Psi^{-1}_a \Psi_x,
\]

where

\[
\Psi_a = U_{aa} + \beta \Gamma^w_a V_{ww} \Gamma^w_a
\]

\[
\Psi_x = U_{xa} + \beta V_{wx} \Gamma^w_a
\]

This illustrates that the value function can be written:

\[
V(z) = -\frac{1}{2} z'V_{zz} z = -\frac{1}{2} w'V_{ww} w + w'V_{wx} x + O (\|x\|^2)
\]

with matrix \( V_{wx} \) is expressed in closed form above.

Hence, the BR value function is simply:

\[
V^*(z, m) = -\frac{1}{2} z'V_{zz} z = -\frac{1}{2} w'V_{ww} w + w'V_{wx} M(m) x + O (\|x\|^2)
\]

for the diagonal attention matrix \( M(m) = diag(m_x) \).
Proposition 7.3 (Behavioral version of linear-quadratic problems) In a linear-quadratic problem, the optimal attention is

\[ m_{x_i} = A \left( A_{x_i} \Psi_a A_{x_i} \sigma_i^2 / \kappa \right) \]  \hspace{1cm} (92)

and the optimal sparse action is

\[ a = A_w w + A_x M x \]

where \( M = \text{diag}(m_{x_i}) \). Here we use the notations of Lemma 7.4.

8 \ Discussion

8.1 Other Ways to Model “Myopia”

Here I list some alternative models. To examine them, I use the following criteria.

1. Do the models apply also to bounded rationality in static contexts? If the goal (which is mine, at least) is to have a quite unified way of talking about bounded rationality in both static (e.g., doing static consumer theory with \( n \) goods) and dynamic contexts, this is a desirable property.

2. Can the model keep a representative agent?

3. Do we keep deterministic actions, e.g. for a given problem, do have a definite, or a distribution of random actions?

4. Do we have source-dependent inattention?

5. Do we keep the higher MPC for current income rather than future income (keeping the present values constant)

<table>
<thead>
<tr>
<th>Model</th>
<th>BR in static contexts</th>
<th>Rep. agent allowed</th>
<th>Determin. actions</th>
<th>Source depdt attention</th>
<th>Bayesian</th>
<th>Textbook micro?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rule-of-thumbs</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Hyperbolics</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Credit constraints</td>
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<td>No</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Sticky info.</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Noisy info</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Entropy penalty</td>
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<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>BGS salience</td>
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<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
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<tr>
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<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>

39
Rule-of-thumb consumers. Campbell and Mankiw (1989) assume that a fraction $\lambda$ are “rule-of-thumb” consumers who just consume their current income, $c_t = \hat{y}_t$. Hence, aggregate consumption is

$$\hat{c}_t = \lambda \hat{y}_t + (1 - \lambda) \frac{r}{R} \sum_{r \geq t+1} \frac{1}{R^{r-t}} \hat{y}_r$$

So we do obtain the higher MPC for current income than for discounted future income.

Hyperbolic consumers. Laibson (1997) models hyperbolic consumers (fully rational or fully naive), O’Donoghue and Rabin (1999) partially naive consumers. There are some key differences. One is that a hyperbolic consumer is still a wealth-maximizer, so his consumption function depends only on the present value of future income, without a higher marginal propensity to consume out of current rather than future income (Barro 1999) – unless they are credit constrained. Another is that sparse agents have source-dependent myopia, unlike (the most common) hyperbolic consumers.

Credit constrained consumers. Those consumers are rational, but credit constrained now or in the near future. As a result, their MPC out of current income is higher than out of future income. A big difference with sparse agent is that again sparse agents have source-dependent myopia. Hsieh (2003) provide evidence for source-dependent myopia: his consumers consume “regularly arriving” government transfers (from the Alaskan oil fund) rationally, but react in myopic way (high MPC) to other transitory income shocks.

Sticky information. In the sticky information model, agents update their information set every now, e.g. every $D$ periods or in a Poisson way (Gabaix and Laibson 2001, Mankiw and Reis 2002).

Sims’ entropy proposal. In Sims’ approach to inattention, the penalty is an entropy penalty, which results in parsimonious (but very complex to solve) models, which however does not allow for source-dependent inattention. Other papers (Maćkowiak and Wiederholt 2009, forth.) deviate from this doctrine by accepting source-dependent inattention. Even though there are many differences of details, there is a common spirit. The noisy-signal theories actions are stochastic: the same problems lead to stochastic actions by agents. As a result we lose the representative agents in the noisy signals models. It can be recovered by taking linear approximations. In contrast, in the sparsity theory actions are deterministic, which is one factor is making this model easier to analyze. In addition, and relatedly, the sparsity model applies well to basic micro (Gabaix 2014), which is so far too difficult to noisy signal models.

The Hansen-Sargent (2007) robustness framework generates a sophisticated form of risk aversion and pessimism, but not myopia per se.

9 Conclusion

I presented a practical way to do boundedly rational dynamic programming. It is portable and tractable, and sheds light on a number of puzzles reviewed in the introduction.
Given that it seems easy to use and sensible, we can hope that this model may be useful for other extent issues in macroeconomics and finance.
10 Appendix: Some more advanced notions and tools

10.1 Infinite horizon

Call $Z = (w, x, \theta)$, $z = (w, x)$ are variables and $\theta$ is a parametrization of the world. We are given $F(Z, m)$ and $u(Z, m)$. We have $F^\theta(Z, a, m) = \theta(m)$ as perceived $\theta$. For instance, if $\theta = \phi_x$, then $\theta(m) = \phi^d + m (\phi^c - \phi^d)$; and $F^x = (1 - \theta(m)) x$, for the transition function. The value $m = \mu$ indicates the true model. Typically we normalize $\mu = (1, ..., 1)$. In a model time-dependance (e.g., a finite-horizon model), then calendar time $t$ is just one of the components of $Z$, say $Z^t$, and the transition from time $t$ to time $t + 1$ is simply coded by $F^{Z^t}(Z, m) = Z^{t+1}$.

We have $V^r(Z)$ the traditional value function assuming full rationality, under $m = \mu$. It satisfies the usual Bellman equation, $V^r(Z) = \max_a u(a, Z, \mu) + \beta V^r(F(a, Z, \mu))$.

We call $V^p$ and $V^o$ the (pre-simplification) perceived and objective value functions.

**Definition 10.1** (Sparse action and value function) The sparse policy $a(Z, V^p)$ is defined by:

$$a(Z, V^p) = \arg \max_{a, m | m^d} u(a, z, m) + \beta \mathbb{E}[V^p(F(a, z, m))]$$ (93)

This defines the transition operator $T(V, V^p)$ by:

$$T(V, V^p)(Z) := u(a(Z, V^p), Z, \mu) + \beta \mathbb{E}[V(F(a(Z, V^p), \mu))]$$

If there are domain conditions for $a(Z, V^p)$, we just add them to the definition of $a(Z, V^p)$, as in the static smax with budget constraints (Gabaix 2014).

**Lemma 10.1** (Monotone contraction) The operator $T(V, V^p)$ is a monotone $\beta$-contraction as a function of $V$. More explicitly, for any functions $V, \tilde{V}$ (i) $\| T(V, V^p) - T(\tilde{V}, V^p) \|_\infty \leq \beta \| V - \tilde{V} \|_\infty$ (ii) if $V(Z) \leq \tilde{V}(Z)$ for all $Z$, then $T(V, V^p) \leq T(\tilde{V}, V^p)$. As a result, there is a unique fixed point $V^o$ to the equation $V^o = T(V^o, V^p)$.

Given this Lemma 10.1, the objective value function satisfies the fixed point relation:

$$V^o = T(V^o, V^p).$$

**Definition 10.2** The basic dynamic sparse max action is $a(Z, V^r)$ defined above. The $q$-iterated smax action is $a(Z, V^{(q)})$, where $V^{(0)} = V^r$, and for $q > 1$, $V^{(q)}$ is characterized by $V^{(q)} = T(V^{(q)}, V^{(q-1)})$.

In practice, we will just take $q = 0$ in most problems of interest: this captures the essence of the economics, which keeping the model quite easy to use. Other examples will suggest that the higher-iterations are quite demanding in rationality.
When we have $T$ periods, we can just use i.e. the level $q = 0$. $a (Z)$—assuming rational behavioral later. Indeed, that’s quite simpler. In the “terminal $x$” problem, we just have $V_t^1 (w, x) = u' \left( \frac{w+x}{2} \right)$, quite simply. Whereas, it’s more complex at the level 1. To get the full rational expectation $a_t (Z)$, we just iterate $q = T - 1$ times.

Note that at all iterations, $V^p (Z) = V^r (Z) + O \left( \|x\|^2 \right)$ and $V^o (Z) = V^r (Z) + O \left( \|x\|^2 \right)$. In practice, for infinite horizon problems, there’s just 1 iteration. $a = \arg \max u + \beta V^p$. Lots of “simplified” iterations, e.g. $S_{\text{Taylor},1} V (w, x) =$ same up to second order terms; or using the certainty equivalent.

We also have an “admissible perceived value function”, which is a function that satisfies $V^p (Z, \mu) = V^r (Z, \mu) + O \left( \|x\|^2 \right)$.

In all cases, we have: If $V^p$ induces $a (z, \mu) = a^d (z, \mu) + O \left( \|x\| \right)$, then

$$T \left( V^0, V^p \right) = T \left( V^0, V^d \right) + O \left( \|x\|^2 \right).$$

That implies [xx give conditions]:

$$F^o \left( V^p; F^o \right) = F^o \left( V^d; F^o \right) + O \left( \|x\|^2 \right)$$

Also, $a \left( F^p, V^p \right)$ is 1st order dependent on $F^p$, but 2nd order dependent on $V^p$.

### 10.2 Some ancillary results

Call $G (Z, m) = Z (m)$ a transformation function for $Z$. E.g. in the basic lifecycle example, $G (w, \hat{y}, \hat{r}, m) = (w, m_y \hat{y}, m_r \hat{r})$. [Note: below, the notation bar isn’t ideal, as bar refers to means; perhaps tilde would be better] When can we express the perceived model as a rational model, with different utility and transition functions? The following Lemma gives the answer.

**Lemma 10.2** *Let $G (Z, m)$ be a function and define $\bar{Z}_t = G (Z_t, m)$. Suppose that we can write

$$u \left( a, Z, m \right) = \bar{u} \left( z, G (Z, m) \right)$$

$$G \left( F \left( a, Z, m \right), m \right) = \bar{F} \left( a, G (Z, m) \right)$$

for two functions $\bar{u}, \bar{F}$. Then the model evaluated at $m$ is the same as a rational model with state variables $\bar{Z}_t$, utility $\bar{u}$, transition function $\bar{F}$. We also have $V (Z) = \bar{V} \left( G (Z, m) \right)$*  

**Proof** We have

$$\bar{Z}_{t+1} = G \left( Z_{t+1}, m \right) = G \left( F \left( a, Z_t, m \right), m \right) = \bar{F} \left( a, G \left( Z_t, m \right) \right) = \bar{F} \left( a, \bar{Z}_t \right)$$

The value function $\bar{V}$ satisfies, with $Z$ s.t. $\bar{Z} = G (Z, m)$

$$\bar{V} \left( Z \right) = \max_a \bar{u} \left( a, Z \right) + \beta \bar{V} \left( F \left( a, Z \right) \right)$$

$$= \max_a u \left( a, Z, m \right) + \beta \bar{V} \left( G \left( F \left( a, Z, m \right), m \right) \right)$$

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Define $V(Z) := \bar{V}(G(Z, m))$. Then,

$$V(Z) = \bar{V}(Z) = \max_a u(a, Z, m) + \beta V(F(a, Z, m), m)$$

So indeed $V$ satisfies the Bellman equation. □

Also, we have $F^w(Z, m) = ((R + \bar{r})(w - \bar{y}), \rho)$

gives

$$\hat{y}_{t+1} = m_F \rho_y \hat{y}_t + \varepsilon_{t+1}^{y}$$

i.e.

$$F^{\bar{y}}(a, Z) = m_y \rho_y (\bar{y}_t + m_y \varepsilon_{t+1}^{y})$$

There is just a $O \left( ||x||^2 \right)$ loss: Towards a the proof  
Suppose that $|a(Z) - a^r(Z)| \leq H \|X\|$, and defining

$$v(a, Z) := u(a, Z, \mu) + \beta V^r(F(a, Z, \mu))$$

Given that $v_0(a^r(Z), Z) = 0$, and $v(a, Z)$ is $C^2$, we have the following Lipschitz condition: there is an $L > 0$ such that for all $a'$,

$$|v(a^r(Z), Z) - v(a', Z)| \leq L |a^r(Z) - a'|^2$$

Suppose that $0 \leq V^r(Z) - V(Z) \leq K$ for some $K$. Then,

$$T(V, V^p)(Z) = u(a, Z, \mu) + \beta V(F(a, Z, \mu))$$

$$\geq u(a, Z, \mu) + \beta V^r(F(a, Z, \mu)) - \beta K \text{ by Lemma 10.1}$$

$$\geq u(a^r, Z, \mu) + \beta V^r(F(a^r, Z, \mu)) - L |a(Z) - a^r(Z)|^2 - \beta K$$

i.e.

$$L H^2 \|X\|^2 + \beta K \geq V^r(Z) - V^o(Z)$$

Hence, taking $K = \sup_Z V^r(Z) - V^o(Z)$, we have $L H^2 \|X\|^2 + \beta K \geq K$, i.e.

$$K \leq K_* := \frac{L H^2 \|X\|^2}{1 - \beta}.$$  

Hence, we have:

$$\|V^r(Z) - V^o(Z)\|_\infty \leq \frac{L H^2 \|X\|^2}{1 - \beta}.$$  

I now note a simple corollary.

**Proposition 10.1** Suppose that $V^p(w, x) = V^r(w, x) + O \left( ||x||^2 \right)$. Then, given any level of iteration $q$, calling $V = V^{w,q}$,

$$V(Z) = \arg \max_{a,m,m^q} u(a, Z, \mu) + \beta \mathbb{E} \left[ V(F(z, a, \mu)) \right] + O \left( ||x||^2 \right)$$

(94)
Proof Given that \( V^{p,0}(w, x) = V^r(w, x) + O(\|x\|^2) \), for every \( q \), \( V^{p,q}(w, x) = V^{p,q-1}(w, x) + O(\|x\|^2) \). Hence, we also have the announced Proposition. □

It would be interesting to know if we can actually prove (under some conditions) the existence of a function \( \bar{\psi} \) such that (94), without the \( O(\|x\|^2) \). This seems doable, but delicate. Even though the operator \( V \mapsto T(V, V^\nu) \) is a monotone contraction by Lemma 10.1, it is not generally the case that the operator \( V \mapsto T(V, V) \) itself is a monotone contraction. However, that may hold in some local sense, i.e. for small \( \|x\| \). I differ this mathematically interesting issue to future research.

How to prove that \( \bar{\psi} \) is \( \mathcal{C}^k \)? Note that here the policy is not optimal, which makes the problem different from those in the tradition of Benveniste and Scheinkman (1979). Under policy \( a(Z) \), we have \( Z_{t+1} = G(Z_t) \) with \( G(Z) = F(a(Z), Z) \) (removing noise here). Call \( \rho(H) \) the spectral norm of a matrix \( H \)– the largest absolute value (or modulus in case of a complex value) of its eigenvalues. Then, we call

\[
v(Z) := u(a(Z), Z) \\
\rho_G := \sup_Z \rho(G'(Z)) \\
V^{[n]} := \sum_{t=0}^n \beta^t v(Z_t)
\]

As \( \left\| \frac{dZ_t}{dZ_0} \right\|_\infty \leq \rho_G \), the chain rule implies:

\[
\left\| \frac{dZ_t}{dZ_0} \right\|_\infty \leq \rho_G
\]

\[
V^{[n]}_Z = \sum_{t=0}^n v'(Z_t) \beta^t \frac{dZ_t}{dZ_0}
\]

\[
\left\| V^{[n]}_Z \right\|_\infty \leq \sum_{t=0}^n \left\| v'(Z_t) \right\|_\infty \beta^t \rho_G
\]

We suppose that \( \beta \rho_G < 1 \), which means that the dynamic system \( (Z_t) \) doesn’t diverge locally too fast. We have the convergence of \( V^{[n]}_Z \) to \( V^{(\infty)}_Z \), and \( V \) is \( \mathcal{C}^1 \).

The same argument shows that if \( \beta \sup_Z \rho(G^{(k)}(Z)) < 1 \), (with \( G^{(k)} \) the \( k \)-th derivative), then \( V \) is \( \mathcal{C}^k \).

I conclude with a remark which will be useful later, drawing again on Gabaix (2014). As \( \kappa \) has the units of utils. One can make it more endogenous with the primitive, unitless parameter \( \bar{\kappa} \), by setting:

\[
\kappa = \bar{\kappa}^2 \var(u(a^d(x), x, m^d))^{1/2}
\]

(95)
10.3 Tools to Expand a Simple Model Into a More Complex one

Here I develop the method to derive the Taylor expansion of a richer model, when starting from a simpler one. Here the methods are entirely paper and pencil. They draw from the techniques surveyed by Judd (1998, Chapter 14), who has a more computer-based perspective.

The state variables evolve according to:

\[ w' = F^w (w, x, a), \quad x' = F^x (x) \]

where variable \( x \) (which again is a vector) is like a macro disturbance, such as the deviation of the interest from trend, which evolves independently of the actions and state variable of the agent \((w, a)\).

Consider the fully rational model:

\[ V^r (w, x) = \max_a u (w, x, a) + \beta \mathbb{E} V^r (F^w (w, x, a), F^x (x)) \]

We start with a simpler model, where \( x \) is always 0: \( x \equiv 0 \), \( F^{x'} = 0 \), i.e.

\[ V^d (w) = \max_a u (w, 0, a) + \beta \mathbb{E} V^d (F^{w'} (w, 0, a)) \]

We use the notation

\[ D_w f = \partial_w f + (\partial_a f) \frac{da}{dw} \quad (96) \]

which is the total derivative with respect to \( w \) (e.g. the full impact of a change in \( w \), including the impact it has on a change in the action \( a \)).

**Proposition 10.2** The impact of a change \( x \) on the value function is:

\[ V_{w,x} (w, 0) = \frac{D_w u_x + \beta D_w \left[ F^{w'} (w, 0, a) V_{w'} (w') \right]}{1 - \beta F^{x'} D_w w'} \quad (97) \]

The impact of a change \( x \) on the optimal action is:

\[ da = -\Psi^{-1}_a \Psi_x dx \]

\[ \Psi (a, x) = u_a (w, a) + \alpha V^a w' \]
\[ \Psi_a = u_{aa} + \beta F^{w'} F^{w'} F_{aa} + \beta V^a w' F^{w'} \]
\[ \Psi_x = u_{ax} + \beta V^a w' F^{w'} + \beta V^a w' F^{w'} \]

They depend only on the transition functions and the derivatives of the simpler baseline value function \( V_{w'} (w') \).
Proof Differentiating the Bellman equation (first with respect to the new variable \( x \), then with respect to the default variable \( w \)), we obtain:

\[
V_w (w, x) = u_x + \beta V'_w F'_w (w, x, a) + \beta V'_w F'_x
\]

\[
V_{w,x} (w, x) = D_w u_x + \beta D_w \left[ V'_w F'_w (w, x, a) \right] + \beta F'_x V'_{w,x} D_w w'
\]

so

\[
V_{w,x} (w, 0) = \frac{D_w u_x + \beta D_w \left[ F'_w (w, 0, a) V'_w (w', 0) \right]}{1 - \beta F'_x D_w w'}
\]

\(\square\)

The same procedure can be followed when \( x' = F'x' (w, x, a) \), with more complex algebra.

We next show a useful consequence.

### 10.4 Simplification of functions

We develop here a bit of simple machinery to reflect how the agent can “simplify” a function (in practice a value function), by forcing them to have a given functional form.

**A motivating example.** Suppose that the agent consumes

\[
\begin{align*}
\psi_1 &= \frac{w}{2} + y_1 \\
\psi_2 &= \frac{w}{2} + y_2,
\end{align*}
\]

where \( y = (y_1, y_2) \) can be viewed as small. His value function is:

\[
v(y) = u \left( \frac{w}{2} + y_1 \right) + u \left( \frac{w}{2} + y_2 \right)
\]

The agent may wish to use a simplified representation of this function. We observe that

\[
v(y) = v^S(y) + O(\|y\|^2)
\]

with

\[
v^S(y) := 2u \left( \frac{w + y_1 + y_2}{2} \right)
\]

We shall take this function \( V^S \) as a “simplified” representation of \( v \). We can then form a more general function:

\[
v(y, m^V) := (1 - m^V) v^S(y) + m^V v(y).
\]

If \( m^V = 1 \), the agent uses the rational value function. If \( m^V = 0 \), the agent uses the proxy value function \( v^S \), which is in some sense simpler.

The following Definition generalizes that thought and codifies the creation of a “simplified” value function.

**Definition 10.3** (Simplifying function) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a function such that \( f_{x_i} (x)|_{x=0} \neq 0 \) for all \( i \), and \( \phi : \{1, \ldots, p\} \to \{1, \ldots, n\} \). Call \( \mathcal{E}^f := \{ v \in C^1 (\mathbb{R}^p, \mathbb{R}) \text{ such that } v(0) = f(0) \} \).

We define the simplification function \( S_{f,\phi} : \mathcal{E}^f \to \mathcal{E}^f \) by:

\[
(S_{f,\phi} (v))(y) := f (b \cdot y)
\]

where \( b \) is the uniquely determined matrix \( b \in \mathbb{R}^{n \times p} \) such \( b_{ij} = 0 \) unless \( i = \phi (j) \) and

\[
v(y) = f(b \cdot y) + o(\|y\|)
\]
Furthermore, \( b_{ij} = \frac{v_{ij}(y)}{f_{ij}(x)} \) if \( i = \phi(j), b_{ij}(x) = 0 \) otherwise.

This also defines an attention-augmented function
\[
v(y, m^V) := (1 - m^V) v^S(y) + m^V v(y)
\]
where parameter \( m^V \) captures the attention to the true value \( v \).

**Proof** We prove the \( b \) is indeed unique. We want: \( v(y) = f(b \cdot y) + o(\|y\|) \). This is equivalent to:
\[
v_{ij}(y)|_{y=0} = \sum_i f_i b_{ij} = f_{\phi(j)} b_{\phi(i)j}.
\]
Inspecting the Taylor expansions gives the result. \( \square \)

Basically \( f(b \cdot y) \) is like a non-linear Taylor expansion of \( v(y) \). For instance, in our introductory example, \( f(x) = 2u(\frac{a+x}{2}), y = (y_1, y_2), n = 1, p = 2, \phi(j) = 1, \) and \( b = (1, 1) \).

Here are two other variants of the same idea. Suppose that we have a stochastic variable, and a variant of the Black-Scholes model, with say stochastic volatility. Then, we may approximate the value function in by tweaking the implied volatility: \( V(x_t, S, K, r, t) = V^{BlackScholes}(\sigma(x_t) + o(x_t), S, K, r, t) \), where \( V^{BlackScholes} \) is the regular Black-Scholes formula, and \( \sigma(x_t) \) could be affine.

Suppose that the agent estimates a distribution, \( h(y) \), where \( y \) are parameters of the distribution. The agent may wish to replace this distribution by a distribution with a simpler functional form, say a Gaussian: then \( f \) is a Gaussian distribution approximating the distribution \( h \), perhaps by matching \( h \)'s mean and variance.

### 11 Appendix: Extensions

#### 11.1 Model variants

**11.1.1 Iterated Static Sparse Max**

In some cases, it is useful to have a generalization of the basic sparse max.

**Definition 11.1** (Iterated static sparse max) The \( K \)-times iterated sparse max, \( \text{smax}^{K}_{a,m,m^d} u(a, x) \), is defined by the following procedure. Define \( m^d(k)_{k=1} \) to be the initial default attention, \( m^d \).

Start at round \( k = 1 \). At each round \( k \leq K \), apply the regular \( \text{smax} \), using the default \( m^d(k) : \text{smax}_{a,m,m^d(k)} u(a, x, m) \), and call \( m^*(k) \) and \( a^*(k) \) the resulting attention. Define then \( m^d(k + 1) = m^*(k) \).

Stop at the end of round \( k = K \), and return \( m^*(K) \) and \( a^*(K) \), the optimal attention and action at the last iteration.

**Illustration.** Suppose that
\[
u(a, x) = -\frac{1}{2}(a - x_1(1 + x_2))^2
\]
so that the rational policy is \( a^r(x_1, x_2) = x_1(1 + x_2) \). If the agent doesn’t think of \( x_1 \) (so, replaces it by \( x_1 = 0 \)), he should not think about \( x_2 \).

We next apply the iterated \( \text{smax} \) outlined in Definition 11.1, iterating twice \((K = 2)\). Initial default attention is \( m^d(1) = (0, 0) \). We start step \( k = 1 \). We observe that so \( a^r_{x_1} = 1 + x_2, a^r_{x_2} = x_1 \), which gives:

\[
m^*_1(1) = A\left(\frac{\sigma^2_1}{\kappa}\right), \quad m^*_2(1) = 0
\]

So, at the beginning of the second step, the default is \( m^d(2) = m^*(1) \). Applying again the plain \( \text{smax} \) but with that default \( m^d(2) \), we have:

\[
m^*_1(2) = A\left(\frac{\sigma^2_1}{\kappa}\right), \quad m^*_2(2) = A\left(\frac{m^*_1(1)^2 \sigma^2_2}{\kappa}\right)
\]

Hence, the action is \( a = a^r(m^*(2) \odot x) = m^*_1(2)x_1(1 + m^*_2(2)x_2) \). We also see that as \( \kappa \rightarrow 0 \), the action converges to the rational action.\(^{11}\)

We have developed tools to define and compute BR dynamic programming.

### 11.1.2 Finite-difference in the sensitivity to \( m \)

When we calculate \( \frac{\partial a}{\partial m_i} \) in Definition 2.1, the following variant \( \overline{a}_{m_i} a \) is sometimes useful. We need to define the finite-difference operator:

\[
\Delta_{m_i} g(m) := g(m)_{m_i=1, m_{-i}=0} - g(0) = g(0, \ldots, 0, 1, 0, \ldots, 0) - g(0)
\]

where the 1 is at the \( i \)-th coordinate of \( m \). This is the “finite difference” analogue to \( \frac{\partial a}{\partial m_i} g(m) = \frac{\partial g}{\partial m_i | m=0} \). Next, we define:

\[
\overline{a}_{m_i}(m, x) := \Delta_{m_i}(a_x \cdot x)
\]

\[
= \Delta_{m_i}\left( \sum_k \left( \frac{\partial}{\partial x_k} a(m, x)_{|x=0} \right) x_k \right) = (\Delta_{m_i} \partial_x)(a) \cdot x
\]

Note that if \( a(x, m) = \sum b_im_i x_i \), then \( \overline{a}_{m_i}(m, x) = b_i m_i = \partial_{m_i} a(m, x) \). However, the definition using \( \overline{a}_{m_i} \) generalizes better. For instance, if \( m \) is one-dimensional \((m = m_1)\) and

\[
a(m_1, x) = \sum_{i=1}^3 m_i b_i x_i
\]

then \( \overline{a}_{m_1}(m, x) = \sum_{i=1}^3 b_i x_i \), whereas \( \partial_{m_1} a(m, x) = \frac{\partial a(x)}{\partial m_1 | m=0} = b_1 x_1 \). The higher-power terms \( m^2, m^3 \) are “invisible” when using \( \partial_{m_1} \), but “visible” when using \( \overline{a}_{m_1} \).

\(^{11}\) This iterated \( \text{smax} \) suffices for the problems considered in this paper. For other purposes, one could imagine a variant where the default is at say \( m^d = (\varepsilon, \ldots, \varepsilon) \), for some \( \varepsilon > 0 \), so as to better “probe” the importance of all variables.
11.2 Extension of the basic 3-period example

Using the simplification function  The value (29) is a bit complicated. This is where the simplification operator $S$ (defined in Definition 10.3) intervenes. Applying it (with the same notations as in the motivating example before and after Definition 10.3), we obtain $V^{1,S} := S(V^1)$, i.e.

$$V^{1,S}(w, x) = \frac{w_1 + x}{2}$$

The value is the same as $V^1$, up to $O(x^2)$ terms: $V^1(w_1, x) = V^{1,S}(w, x) + O(x^2)$. The attention-augmented value function at time 1 is:

$$V^1(w, x, m^V) = m^V V^1(w, x) + (1 - m^V) V^{1,S}(w, x)$$

At time 0, the agent does $\text{smax}_{c_0, m} v^0(c_0, x, m_0)$, with $m_0 = (m_0^x, m^V)$ and

$$v^0(c_0, w_0, x, m_0) := u(c_0) + V^1(w_0 - c_0, m_0^x x, m^V)$$

The FOC is $v^0_{c_0} = 0$ with

$$v^0_{c_0} = u'(c_0) - V^1_w (w_0 - c_0, m_0^x x, m^V).$$

We have $V^{1,w, m^V} = 0$ at the default $m_0^d = (0, 0)$, so $\frac{\partial v^0}{\partial m^V | m^d = 0} = 0$ and the optimal attention is $m^V = 0$: the agent uses the proxy value function, not the exactly rational one (we will see soon that attention $m^V$ can be non-zero using the 2-step smax, but it is still likely to be 0 if $\kappa$ is not too small).

We note that if $m^V > 0$, the FOC is more complex. The FOC is:

$$u'(c_0) = (1 - m^V) u'\left(\frac{w_1 + m_0 x}{2}\right) + m^V \frac{1}{2} \left[ u'\left(\frac{w_1 + m_1 x}{2}\right) + u'\left(\frac{w_1 + (2 - m_1) x}{2}\right) \right]$$

Still, to the first order, the decision is the same (as per Proposition 2.2). Making the problem simpler at every period, via the $m^V = 0$ device, makes the problem more tractable for both the agent and the economist examining him. We next study this, using the 2-step sparse max.

Using the 2-step sparse max  We have so far used the plain sparse max. This led to $m^V = 0$, the exclusive reliance on the simplified value function. We now calculate what happens when using the twice-iterated sparse max of Definition 11.1.

To endogenize $m^V$, we use the twice-iterated smax: $\text{smax}_{c_0, w}^2 v^0(c_0, w_0, x, m)$ with $m = (m_0^x, m^V)$. At the first round, $v^0_{c_0, m^V} = 0$, so $m^V_0(1) = 0$, and as before $m^V_0(1) = A\left(\frac{1}{6\kappa} u'' \left(\frac{w_0}{3}\right) \sigma^2 \right)$. At the second round, now $m^V = (m^V_0(1), 0)$. The easy part is the attention to $x$, which is slightly different than at step 1:

$$m^x_0(1) = A\left(\frac{1}{6\kappa} u'' \left(\frac{w_0 + m^x_0(0) x}{3}\right) \sigma^2 \right)$$
The more novel part is to calculate $m_V$. We have, with $w_1 = w_0 - c_0$, and calling $x^s := m_0^\tau x$

$$
\nu_{c,mv}^0 (c_0, w_0, x, m_0^\tau, m_V) = \partial_c \left[V^1 (w_0 - c_0, x^s) - V^1 (w_0 - c_0, x^s, m_V = 0)\right] \\
= -\frac{1}{2} u' \left(\frac{w_1 + m_1 x^s}{2}\right) - \frac{1}{2} u' \left(\frac{w_1 + (2 - m_1) x^s}{2}\right) + u' \left(\frac{w_1 + x^s}{2}\right)
$$

Doing a Taylor expansion of the consumptions $\frac{w_1 + m_1 x^s}{2}$ and $\frac{w_1 + (2 - m_1) x^s}{2}$ around their mean $e^d = \frac{w_1 + x^s}{2} = \frac{w_1 + m_0^\tau x}{2}$, we obtain:

$$
\nu_{c,mv}^0 = -\frac{1}{2} u' \left(e^d + (m_1 - 1) \frac{x^s}{2}\right) - \frac{1}{2} u' \left(e^d - (m_1 - 1) \frac{x^s}{2}\right) + u' \left(e^d\right) \\
= -\frac{1}{2} u'' \left(e^d\right) (m_1 - 1)^2 \left(\frac{x^s}{2}\right)^2 \times 2 + O(\delta^2)
$$

Likewise, $\nu_{cc|m=m_0^\tau(1)}^0 = \frac{3}{2} u'' \left(e^d\right)$. So, the impact of $m_V$ is

$$
\frac{\partial c_0}{\partial m_V} = -\frac{\nu_{c,mv}^0}{\nu_{cc}^0} = -\frac{1}{2} \frac{u'' \left(e^d\right)}{6 \ u'' \left(e^d\right)} (m_1 - 1)^2 \left(m_0^\tau x\right)^2 + O(\delta^2)
$$

Hence, for a small $x$, the attention $m_V$ to the difference between the difference between the true and proxy value functions (i.e., $V^1 (w_1, x, m_V)$ for $m^V = 1$ vs $m^V = 0$) is:

$$
m_V^0 = A \left(\frac{1}{\kappa} E \left[\left(\frac{\partial c_0}{\partial m_V}\right)^2 \nu_{cc}^0\right]\right) \\
= A \left(\frac{1}{\kappa} E \left[\frac{1}{6} \frac{u'' \left(e^d\right)}{u'' \left(e^d\right)} (m_1 - 1)^2 \left(m_0^\tau x\right)^2 \right]^2 \frac{3}{2} u'' \left(e^d\right)\right) \\
= A \left(\frac{1}{24 \kappa} \left(\frac{u'' \left(e^d\right)}{u'' \left(e^d\right)}\right)^2 (m_1 - 1)^4 \left(m_0^\tau\right)^4 E \left[x^4\right] u'' \left(e^d\right)\right)
$$

(102)

It is instructive to take the limit of small $\kappa$, using a sparsity-inducing cost function $(g'(0) > 0)$. To have $m_0^\tau > 0$, we need $\frac{\sigma_x^2}{\kappa}$ large enough, so $\sigma_x \geq \kappa^{1/2}$. To have $m_V^0 > 0$, we need $\frac{\sigma_x^2}{\kappa}$ large enough, i.e. $\sigma_x \geq \kappa^{1/4}$, which is a much higher hurdle ($\frac{\kappa^{1/4}}{\kappa^{1/2}} \to \infty$) for small $\kappa$. We formalizing this.
Proposition 11.1 (Attention to a variable, vs attention to the fine properties value function depending on that variable) Suppose a succession of problems (indexed by $\kappa$ going to 0) such that there are positive constants $B, B', \varepsilon$ such that for $\kappa$ small enough: $B\kappa^{1/2-\varepsilon} \leq \sigma_x (\kappa) \leq B'\kappa^{1/4+\varepsilon}$. Then, the agent will have $m_0^0 > 0$ and $m_0^V = 0$ when $\kappa$ is small enough. This is, the agent pays attention to the disturbance $\vartheta$, but not to the subtle difference between the true and proxy value functions (i.e., $V^1(w_1, x, m^V)$ for $m^V = 1$ vs $m^V = 0$).

In plain terms: because thinking about the nuances $m^V$ in $V(x, m^V)$, one needs to think about $x$ at all. Hence, in many situations, we have $m^V = 0$ and $m^x > 0$. Indeed, we cannot have (with just one state variable $m^x = 0$ and $m^V > 0$).

In particular, for our 3-period problem for $\kappa$ small enough but not too small, $m^V = 0$ and $m_0^0 > 0$: the agent uses the simplified value function, as still pays attention to $x$, like in the basic smax case. This is one reason it is useful to use the basic smax: it gets to the essence of the more complex patterns that can later be refined using the iterated smax.

11.3 Continuous time

Calculations are typically cleaner in continuous time, so we develop the continuous-time version of the machinery. We take for now problems without stochastic terms (those should be added later).

The laws of motion are:

$$\begin{align*}
\dot{w}_t &= F^w(w, x, a) \\
\dot{x}_t &= F^x(w, x)
\end{align*}$$

and the Bellman equation is:

$$\rho V(w, x) = u(w, x, a) + V_w(w, x) F^w(w, x, a) + V_x(w, x) F^x(w, x, a)$$

In the more complex case $\dot{x}_t = F^w(w, x, a)$, we need to solve for a matrix Ricatti equation -- but not here.

Call $D_w = \partial_w + a_w \partial_a$ the “total impact” of a change in $w$. Then:

$$\rho V_x = u_x + V_w F^w_x + V_x F^x_x + V_{xx} F^x_x$$  \hspace{1cm} (103)$$

Now, we differentiate and evaluated at $x = 0$:

$$\rho V_{wx} = D_w (u_x + V_w F^w_x) + V_{wx} F^x_x + V_x F^x_{wx}$$

so

$$V_x = (\rho - F^x_x)^{-1} [u_x + V_w F^w_x]$$  \hspace{1cm} (104)$$

$$V_{wx} = (\rho - F^x_x)^{-1} [D_w (u_x + V_w F^w_x) + V_x F^x_{wx}]$$  \hspace{1cm} (105)$$

As $a$ satisfies $\Psi = 0$ with

$$\Psi (a, w, x) = u_a + V_a F^w_a$$
Hence, the impact of $\alpha$ on the optimal action is

$$a_x = -\Psi_a^{-1}\Psi_x$$

$$\Psi_a = u_{aa} + V_w F_{aw}$$
$$\Psi_x = u_{ax} + V_w F_{aw} + V_w F_{ax}$$

Calculation of $\mathcal{K}_{xx}$. We now turn to the more difficult case of $\mathcal{K}_{xx}$. Using $\Delta = \alpha + \alpha a$ the “total impact” of a change in $\alpha$, we have:

$$\rho V_x = D_x u + V_w D_x F_w + V_x F_{xx} + V_{xx} F_{xx}$$
$$= a_x (u_a + V_w F_{aw}) + u_x + V_w F_{ax} + V_x F_{xx}$$

Next, differentiating at $x = 0$,

$$\rho V_{xx} = a_x D_x (u_a + V_w F_{aw}) + D_x [u_x + V_w F_{ax} + V_x F_{xx}] + V_{xx} F_{xx}$$
$$= a_x [u_{ax} + u_{aa} a_x + V_{wx} F_{aw} + V_w F_{ax} + V_w F_{aw} a_x]$$
$$+ u_{xx} + u_{xa} a_x + V_{xw} F_{aw} + V_u D_x F_{ax} + 2 V_{xx} F_{xx} + V_x F_{xx}$$

hence

$$(\rho - 2 F_{xx}) V_{xx} = a_x [u_{ax} + u_{aa} a_x + V_{wx} F_{aw} + V_w F_{ax} + V_w F_{aw} a_x]$$
$$+ u_{xx} + u_{xa} a_x + V_{xw} F_{aw} + V_u D_x F_{ax} + 2 V_{xx} F_{xx} + V_x F_{xx}$$

This is a bit of a complicated expression. Let us note it can be written

$$(\rho - 2 F_{xx}) (V_{xx}^s - V_{xx}^r) = a_x A + a_x B a_x + C$$

with $B = u_{aa} + V_w F_{aw}$.

We use the following elementary Lemma:

**Lemma 11.1** Let $f(a) = Aa + a'Ba + C$, for $B$ symmetric negative definite. Let $a^* = \arg \max_a f(a)$, so $a^* = -\frac{1}{2} B^{-1} A$. Then, for any $a$,

$$f(a) - f(a^*) = (a - a^*) B (a - a^*) .$$

Let’s compare $V_{xx}$ under the sparse vs rational model: the difference is just in the $D_x^s$ vs $D_x^r$ term. Indeed,

$$D_x^s - D_x^r = (a^s_x - a^r_x) \partial_x$$

so, using the previous Lemma,

$$V_{xx}^s - V_{xx}^r = (\rho - 2 F_{xx}^r)^{-1} (a^s_x - a^r_x) (u_{aa} + V_w F_{aw}^r) (a^s_x - a^r_x)$$

We gather the results.
Proposition 11.2 (What are the losses from a suboptimal policy?) Consider the value function $V^r$ under the optimal policy and $V^s$ under a potentially suboptimal policy, and $V_\delta (w, x) = V^s (w, x) - V^r (w, x)$. Then, evaluating at $x = 0$, we have:

$$V^s = 0, V^\delta_0 = 0, V^\delta_{ww} = 0, V^\delta_x = 0, V^\delta_{wx} = 0$$  \hspace{1cm} (107)

and

$$V^\delta_{xx} = (\rho - 2F^x_x)^{-1} (a^x_\alpha - a^r_\alpha) (u_\alpha + V^\delta_w F^w_{aa}) (a^s_\alpha - a^r_\alpha)$$  \hspace{1cm} (108)

Equation (108) has an intuitive interpretation. At a point in time, as a function of $a$, present and continuation utility is $v (a) = u (a, w_t) dt + (1 - \rho dt) V (w_t + F^w (w_t, a_t) dt)$. Hence (omitting for the $dt$ to remove the notational clutter), $v' (a) = u_\alpha + V^\delta_w F^w_{aa}$ and $v'' (a) = u_\alpha + V^\delta_w F^w_{aa}$. Hence, reacting imperfectly to a small $x_t$ (with $a^\delta_\alpha = a^s_\alpha - a^r_\alpha$) creates an instantaneous utility loss of $\Lambda_t = - \frac{1}{2} x_t a^\delta_\alpha v_\alpha a^\delta_x x_t$. The full utility loss is the present discounted value of that, i.e.

$$2\Lambda = \int_0^\infty e^{-\rho t} 2\Lambda_t dt = - \int_0^\infty e^{-\rho t} x_t a^\delta_\alpha v_\alpha a^\delta_x x_t \text{ with } x_t = e^{-\phi t} x_0$$

$$= - \int_0^\infty e^{-\rho t} e^{-2\phi t} x_0 a^\delta_\alpha v_\alpha a^\delta_x x_0 = \frac{1}{\rho + 2\phi} x_0 v_\alpha a^\delta_x x_0$$

$$= - x_0 (\rho - 2F^x_x)^{-1} a^\delta_\alpha (u_\alpha + V^\delta_w F^w_{aa}) a^\delta_x x_0 \text{ as } F^x = - \phi$$

$$= - x_0 V^\delta_{xx} x_0.$$  

It is easy to study the “static” utility losses to derive the dynamic utility losses. This proposition 11.2 is a dynamic application of the Proposition 26 in Gabaix (2014, online appendix) regarding losses from a suboptimal policy. For convenience, we restate this Proposition here. With static problem max $u (a, x)$ s.t. $b (a, x) \geq 0$, and a Lagrangian $L (a, x) = u (a, x) + \lambda b (a, x)$, the losses from a suboptimal policy $a^\delta = a - a^r$ (where $a^r$ is the optimal policy) are to the leading order: $\frac{1}{2} a^\delta L_{aa} a^\delta$.

Here the Lagrangian is $L = \int e^{-\rho t} \left[ u (a_t, z_t) + \lambda_t (\dot{z}_t + F^z (a_t, z_t)) \right] dt$, where $z_t = (w_t, x_t)$ is the state vector. Hence, the loss $\Lambda$ is expressed by (to the leading order)

$$2\Lambda = a' L_{aa} a = \int a^\delta_t L_{a_t a_t} a^\delta_t = \int e^{-\rho t} a^\delta_t [u_{a_t a_t} + \lambda_t F_{a_t a_t}] a^\delta_t dt$$

Suppose that we can linearize, $a^\delta_t = A x_t$, we have

$$2\Lambda = \int e^{-\rho t} x_t' A^\delta [u_{a_t a_t} + \lambda_t F_{a_t a_t}] A^\delta x_t dt$$

Consider the ergodic limit, where $x_t$ has a distribution independent of $t$. Recall that

$$\mathbb{E} x_t' B x_t = \mathbb{E} \sum_{i, j} x_i B_{ij} x_j = \sum_{i, j} B_{ij} \mathbb{E} [x_i x_j] = \text{Trace} (B \mathbb{E} [x x'])$$
Hence,

\[ 2\lambda = \frac{1}{\rho} \text{Trace} \left( B \mathbb{E} [x] \right) \]

\[ B = A^\delta [u_{a_{i}a_{t}} + \lambda_{t}F_{a_{i}a_{t}}] A^\delta = A^\delta L_{a_{i}a_{t}} A^\delta \]

12 Appendix: Proofs

The online appendix contains further proofs.

**Proof of Proposition 2.2** The rational reaction function satisfies:

\[ a^r (x) = a^d + \sum b_i x_i + \lambda (x) \]

for a function \( \lambda (x) = O \left( \|x\|^2 \right) \).

So, \( \partial a / \partial x_i = b_i \) and:

\[ m_i^s = \tau \left( 1, \frac{\kappa_a}{\sigma_i \cdot \partial a / \partial x_i} \right) = \tau \left( 1, \frac{\kappa_a}{\sigma_i \cdot b_i} \right) \]

We shall use the notation \( \lambda (x) := \lambda ((m_i^s x_i)_{i=1...n}) \), which also satisfies \( \lambda (x) = O \left( \|x\|^2 \right) \).

The sparse reaction function is:

\[ a^s (x) = \arg \max_a u \left( a, m_i^s x_i, ..., m_n^s x_n \right) = a^s (m_i^s x_1, ..., m_n^s x_n) \]

\[ = a^d + \sum b_i m_i^s x_i + \lambda ((m_i^s x_i)_{i=1...n}) = a^d + \sum b_i \tau \left( 1, \frac{\kappa_a}{\sigma_i \cdot b_i} \right) x_i + \lambda (x) \]

\[ = a^d + \sum \tau \left( \frac{b_i}{\sigma_i} \right) x_i + \lambda (x) = a^d + \sum \tau \left( \frac{b_i}{\sigma_i} \right) x_i + O \left( \|x\|^2 \right) . \]

**Proof of Proposition 2.2** Let us consider two functions \( U \) and \( u^s \)

\[ U^* (a, w, x) := u \left( a, w, x \right) + \beta \mathbb{E} \left( V^s \left( F^w \left( w, x, a \right), F^x \left( w, x, a \right) \right) \right) \]

\[ U^{**} (a, w, x) := u \left( a, w, x \right) + \beta \mathbb{E} V^s \left( F^w \left( w, x, a \right), F^x \left( w, x, a \right) \right) \]

and define the associated optimal actions:

\[ a^* (w, x) := \arg \max_a U^* \left( a, w, x \right) , \quad a^{**} (w, x) := \arg \max_a U^{**} \left( a, w, x \right) \]

In \( U^{**} \), there is no inattention: however, the continuation policy \( V^s \) is used: the agent will be inattentive in the future.

First, we will prove:
Lemma 12.1 Suppose that $F^x_a = 0$. We have, at $x = 0$, \[ \frac{\partial a^*(w, x)}{\partial x} = \frac{\partial a^{**}(w, x)}{\partial x} \]

**Proof.** The key fact comes from Proposition 2.1, and is:

\[
\begin{align*}
V_w(w, 0) &= V^s_w(w, 0) \\
V_{ww}(w, 0) &= V^s_{ww}(w, 0) \\
V_x(w, x)|_{x=0} &= V^s_x(w, x)|_{x=0} \\
V_{wx}(w, x)|_{x=0} &= V^s_{wx}(w, x)|_{x=0}
\end{align*}
\]

and

\[
U^*_a = u_a(a, w, x) + \beta \mathbb{E}[V^w_w(w, x, a) + V^*_a F^x_a(w, x, a)]
\]

\[
U^{**}_{ax} = u_{ax} + \beta \mathbb{E}[F^w_w \cdot V^s_{ww} F^w_a + V^s_w F^w_{aw}]
\]

\[
+ \beta \mathbb{E}[V^s_x F^x_{ax} + F^x_w V^s_{xx} F^x_a]
\]

Likewise, for $U^{**}$,

\[
U^{**}_{ax} = u_{ax} + \beta \mathbb{E}[F^w_w \cdot V^s_{ww} F^w_a + V^s_w F^w_{aw}]
\]

\[
+ \beta \mathbb{E}[V^s_x F^x_{ax} + F^x_w V^s_{xx} F^x_a]
\]

Hence, we have

\[
U^{**}_{ax} = U^*_a \text{ at } x = 0
\]

Note that we used $F^x_a = 0$. This is necessary, because in general $V_{xx} \neq V^s_{xx}$.

Likewise,

\[
U^{**}_{aa} = u_{aa}(a, w, x) + \beta \mathbb{E}[F^w_w(w, x, a) \cdot V^s_{ww} F^w_a(w, x, a) + V^s_w F^w_{aa}(w, x, a)]
\]

\[
+ 2 \beta \mathbb{E}[F^x_a(w, x, a) \cdot V^s_{xx} F^x_a(w, x, a)]
\]

\[
+ \beta \mathbb{E}[F^x_a(w, x, a) \cdot V^s_{xx} F^x_a(w, x, a) + V^s_x F^x_{aa} + V^s_{xx} F^x_{aa}]
\]

and a similar expression for $U^{**}_{aa}$, which leads to:

\[
U^{**}_{aa} = U^{**}_{aa} \text{ at } x = 0
\]

Finally, we have:

\[
\frac{\partial a^{**}(w, x)}{\partial x} \big|_{x=0} = -U^{**}_{aa} \cdot U^{**}_{ax}|_{x=0} = -U^{**}_{aa} \cdot U^{**}_{ax}|_{x=0}
\]

\[
\frac{\partial a^*(w, x)}{\partial x} \big|_{x=0}. \]

Given $a^c(w, x) = a^d(w) + \sum_i b_i(w) x_i + O(x^2)$, we have

\[
\frac{\partial a^c(w, x)}{\partial x_i} = b_i(w)
\]
Hence, the lemma gives:
\[
\frac{\partial a^{**}(w,x)}{\partial x_i} = b_i(w)
\]
so
\[
a^{**}(w,x) = a^d(w) + \sum_i b_i(w)x_i + O(x^2)
\]
Finally,
\[
a^a(x) = a^{**}(m^*_i x_i)
= a^d(w) + \sum_i b_i(w)m^*_i x_i + O(x^2)
= a^d(w) + \sum_i b_i(w)\tau\left(1, \frac{\kappa_a}{b_i(w)}\sigma_i\right) x_i + O(x^2)
= a^d(w) + \sum_i \tau\left(b_i(w), \frac{\kappa_a}{\sigma_i}\right) x_i + O(x^2).
\]

**Proof of Proposition 3.1** *Exact result.* The problem is
\[
\max_{(c_t)_{t \geq 0}} \sum_{t \geq 0} \beta_t c_t^{1-\gamma} \quad \text{s.t.} \quad \sum_{t \geq 0} p_t c_t \leq \Omega_0
\]
where \(p_t = \frac{1}{(1+r_1) \cdots (1+r_T)}\) is the time-0 Arrow-Debreu price of a dollar received at \(t\), and \(\beta_t\) is the discount factor (which is not necessarily of the form \(\beta^t\) here), and \(\Omega_0 := w_0 + \sum p_t y_t\) is the full wealth. Forming the Lagrangian,
\[
L = \sum_{t \geq 0} \beta_t c_t^{1-\gamma} + \lambda \left(\Omega_0 - \sum_{t \geq 0} p_t c_t\right)
\]
we have \(\beta_t c_t^{-\gamma} = \lambda p_t\), i.e. (with \(\psi = \frac{1}{\gamma}\)), \(c_t = c_0 \left(\frac{\beta_t}{p_t}\right)^\psi\) for some \(c_0\). The budget constraint gives:
\[
\Omega_0 = \sum_{t \geq 0} p_t c_t = c_0 \sum_{t \geq 0} \beta_t^{\psi} p_t^{1-\psi}
\]
i.e.
\[
c_0 = \mu \Omega_0 \quad \text{and} \quad \mu := \frac{1}{\sum_{t \geq 0} \beta_t^{\psi} p_t^{1-\psi}}
\]
Given \( V'(\Omega_0) = u'(c_0) = u'(\mu \Omega_0) \), we have (as the function is also homogeneous of degree \( 1 - \gamma \))
\[
V(\Omega_0) = \frac{1}{\mu} u(\mu \Omega_0)
\] (111)

Suppose now that \( \beta_t = \beta^t \) and \( \beta R = 1 \). Then the interest rate is constant, \( p_t = R^{-t} \), and
\[
\mu = \frac{1}{\sum_{t \geq 0} R^{-t}} = \frac{1}{1 - \frac{1}{R}} = \frac{r}{R}.
\]

Taylor expansion.

When the interest rate is not constant, consider the impact of a change \( dr_\tau \), for just one date \( \tau \). Then,
\[
dp_t = \frac{1}{R^{t+1} dr_\tau 1_{t \geq \tau}},
\]
\[
d\mu = -\mu^2 (1 - \psi) \sum_{t \geq 0} \beta_t^\psi P_t^\psi dp_t = -\left( \frac{r}{R} \right)^2 (1 - \psi) \sum_{t \geq \tau} \frac{-1}{R^{t+1}} dr_\tau
\]
\[
= (1 - \psi) \left( \frac{r}{R} \right)^2 \frac{R}{r} \frac{dr_\tau}{R^{t+1}} = \mu (1 - \psi) \frac{1}{R} \frac{dr_\tau}{R^r}
\]
Also,
\[
d\Omega_0 = \hat{y} \sum_{t \geq 1} dp_t = \hat{y} \sum_{t \geq \tau} \frac{-1}{R^{t+1}} dr_\tau 1_{t \geq \tau} = -\frac{\hat{y}}{r} \frac{dr_\tau}{R^r}
\]
\[
d\Omega_0 = \frac{-\hat{y}}{r \Omega_0} \frac{dr_\tau}{R^r}
\]
Recall that at the default, \( c_0 = \mu \Omega_0 = \frac{r}{R} (w_0 + \frac{\hat{y}}{r}) = \frac{r w_0 + \hat{y}}{R} \). In general, \( c_0 = \mu \Omega_0 \) gives:
\[
dc_0 = \mu \Omega_0 \frac{d\mu}{\mu} + \mu d\Omega_0 = c_0 (1 - \psi) \frac{1}{R} \frac{dr_\tau}{R^r} + \frac{r}{R} \frac{-\hat{y}}{r} \frac{dr_\tau}{R^r}
\]
\[
= \left( -\psi c_0 + \frac{r w_0 + \hat{y}}{R} - \frac{\hat{y}}{R} \right) \frac{dr_\tau}{R^{t+1}}
\]
\[
= \left( -\psi c_0 + \frac{r (w_0 - \hat{y})}{R} \right) \frac{dr_\tau}{R^{t+1}}
\]

Proof of Proposition 5.1 Let’s first study the mechanics of accumulation. We have
\[
w_{t+1} = (R + \hat{r}_t) (w_t + y - c_t) + k^y z_{t+1}
\]
\[
z_{t+1} = A z_t
\]
and a policy:
\[
c_t = \frac{r}{R} w_t + y + B z_t
\] (112)
as the income is $k^y z_t$. This implies that next period consumption is:

$$
c_{t+1} = \frac{r}{R} ((R + \hat{r}_t) w_t + Ry - Rc_t + k^y z_{t+1}) + y + B_z z_{t+1}
$$

$$
= \frac{r}{R} \left( (R + \hat{r}_t) w_t + Ry - R \left( \frac{r}{R} w_t + y + B_z z_t \right) + k^y z_{t+1} \right) + B_z z_{t+1}
$$

$$
= \frac{r}{R} w_t + \frac{r}{R} \hat{r}_t w_t + \left( -r B_z + \left( \frac{r}{R} k^y + B_z \right) A \right) z_t + y
$$

$$
c_{t+1} = \frac{r}{R} w_t + \left( -(r - A) B_z + \frac{r}{R} k^y + \frac{r}{R} w_t k^x \right) z_t + y
$$  \hspace{1cm} (113)

as $\hat{r}_t = k^r z_t$. Let us first study the rational case.

*The rational case.* The (rational) Euler equation imposes:

$$
c_{t+1} - c_t = \psi \frac{d}{dt} \frac{\hat{r}_t}{R}
$$

i.e., using (112) and (113),

$$
-(R - A) B_z + \frac{r}{R} k^y + \frac{r}{R} w_t k^x = \psi \frac{d}{dt} \frac{k^x}{R}
$$

This gives the normatively correct sensitivity:

$$
B_z = (R - A)^{-1} \left[ (r w_t - c^d_t \psi) k^x + r k^y A \right] \frac{1}{R}
$$  \hspace{1cm} (114)

*The behavioral case.* The subjective model is:

$$
z_{t+1} = mA z_t + \varepsilon_{t+1}
$$

$$
w_{t+1} = (R + m \hat{r}_t^s) (w_t + y - c_t) + m k^y z_{t+1}
$$

$$
= (R + k^r m z_t) w_t - Rc_t + k^y m z_{t+1}
$$

Here, $\hat{r}_t^s = k^r z_t$ is a subjectively perceived interest rate, that may not be the correct one (e.g. because of inattention, or money illusion). Note that I replaced $(R + \hat{r}_t) c_t$ by $R c_t$, by Taylor expansion.

The policy has still the form (112), but with a different policy vector $B_z$. With the DP approach, the policy is as (114), replacing $A, k^y$ and $k$ by $mA, mk^y, mk^r$. Hence (114) is replaced by:

$$
B_z = (R - mA)^{-1} \left[ \left( \frac{r}{R} w_t - c^d_t \psi \right) \frac{mk^r}{R} + \frac{r}{R} k^r mA \right]
$$  \hspace{1cm} (115)

$$
= \left( \frac{R}{m} - A \right)^{-1} \left[ \left( \frac{r}{R} w_t - c^d_t \psi \right) \frac{k^r}{R} + \frac{r}{R} k^y A \right]
$$  \hspace{1cm} (116)
Using the formal value of $c_{t+1}$ in (113), with $M$ to be determined, and with $\hat{r}_t = k^r z_t$, and set $\xi = \frac{1}{M} - 1 \geq 0$, we calculate:

\[
c_{t+1} - \frac{1}{M} c_t = -\xi \frac{r}{R} w_t + \left( -\left( \frac{1}{M} + r - A \right) B_z + \frac{r}{R} k^y A + \frac{r}{R} w_t k^r \right) z_t
\]

\[
= -\xi \frac{r}{R} w_t + \left( -\left( \frac{1}{M} + r - A \right) B_z + \left( r w_t - c_t^d \psi \right) \frac{k^s}{R} + \frac{r}{R} w_t k^r \right) z_t,
\]

using (115),

\[
= \left( -\xi + (k^r - k^s) \right) \frac{r}{R} w_t + \left( -\left( \frac{1}{M} + r - \frac{R}{m} \right) B_z + c_t^d \psi \frac{k^s}{R} \right) z_t
\]

To get a clean Euler equation, we choose $M$ to satisfy:

\[
\frac{m}{R - rm}
\]

Then:

\[
c_{t+1} - \frac{1}{M} c_t = \left( -\xi + (k^r - k^s) \right) \frac{r}{R} w_t + c_t^d \psi \frac{k^s}{R}
\]

i.e., as $\xi = \frac{1}{M} - 1$

\[
ME_t [c_{t+1}] - c_t = \left( -M \xi + M \left( k^r - k^s \right) \right) \frac{r}{R} w_t + M c_t^d \psi \frac{k^s}{R}
\]

\[
= \left( M - 1 + M \left( r_t - r^s \right) \right) \frac{r}{R} w_t + M c_t^d \psi \frac{k^s}{R}
\]

$\square$

**Proof of Proposition 4.2** Taxes lower the present value of his income by $He^{-r(T-t)}$, so the consumer’s response is:

\[
\hat{c}_t = r \hat{w}_t - \tau \left( He^{-r(T-t)}, \kappa \right)
\]

so wealth accumulation is:

\[
\frac{d}{dt} \hat{w}_t = r \hat{w}_t - \hat{c}_t = \tau \left( He^{-r(T-t)}, \kappa \right).
\]

The consumer starts thinking about it at a time $s$ s.t. $He^{-r(T-s)} = \kappa$ (assuming that the solution is in $(0, T)$), i.e.

\[
s = \max \left( 0, \min \left( T, \frac{1}{r} \ln \frac{\kappa}{He^{-rT}} \right) \right) \quad (117)
\]

First, consider the case: $s < T$.

Then, for $t \in [s, T)$,

\[
\frac{d}{dt} \hat{w}_t = \tau \left( He^{-r(T-t)}, \kappa \right) = He^{-r(T-t)} - \kappa
\]

\[
\hat{w}_t = \int_s^t \left( He^{-r(T-t')} - \kappa \right) dt'
\]

\[
= \frac{H}{r} e^{-rT} \left( e^{rt} - e^{rs} \right) - \kappa (t - s)
\]

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\begin{align*}
\hat{c}_t &= r \hat{w}_t - \tau \left( He^{-r(T-t)}, \kappa \right) \\
&= r \left( \frac{H}{r} e^{-rT} (e^{rt} - e^{rs}) - \kappa (t-s) \right) - \left( He^{-r(T-t)} - \kappa \right) \\
\hat{c}_t &= -He^{-r(T-s)} + \kappa (1 - r(t-s)) \\
&= 0
\end{align*}

So at \( t = T \)
\[ \hat{w}_T = \frac{H}{r} \left( 1 - e^{-r(T-s)} \right) - \kappa (T-t) \]

At \( T \), the tax \( H \) is enacted, so that for \( t \geq T \), the agent is aware of it. This yields:
\[ \hat{c}_t = r \hat{w}_t - H \]
\[ \frac{d}{dt} \hat{w}_t = r \hat{w}_t - H - \hat{c}_t = \text{investment income - taxes - consumption change} = 0 \]

hence for \( t > T \), \( \hat{w}_t = \hat{w}_T \), and \( \hat{c}_t = r \hat{w}_T - H \).

We conclude that consumption is:
\[ \hat{c}_t = \begin{cases} 
0 & \text{for } t < s \\
-He^{-r(T-s)} + \kappa (1 - r(t-s)) & \text{for } s \leq t < T \\
r \hat{w}_T - H & \text{for } t \geq T
\end{cases} \]

and wealth is
\[ \hat{w}_t = \begin{cases} 
0 & \text{for } t < s \\
\frac{H}{r} e^{-rT} (e^{rt} - e^{rs}) - \kappa (t-s) & \text{for } s \leq t < T \\
\frac{H}{r} \left( 1 - e^{-r(T-s)} \right) - \kappa (T-s) = \hat{w}_T & \text{for } t \geq T
\end{cases} \]

Proof of Proposition 5.3  We have: \( \dot{x}_t = \xi x_t - \sigma (\mathbf{r} + \pi_t) \). To solve for the system, we note:
\[ \dot{x}_t = \xi \dot{x}_t - \sigma \dot{\pi}_t = \xi \dot{x}_t - \sigma (\rho \pi_t - \kappa x_t) = \xi \dot{x}_t + \sigma \kappa x_t - \rho \sigma \pi_t \]
\[ = \xi \dot{x}_t + \sigma \kappa x_t + \rho (\dot{\pi}_t - \xi x_t + \sigma \mathbf{T}) = (\rho + \xi) \dot{x}_t + (\sigma \kappa - \rho \xi) x_t + \rho \sigma \mathbf{T} \]

so that:
\[ \dot{x}_t - (\rho + \xi) \dot{x}_t + (\rho \xi - \sigma \kappa) x_t = \rho \sigma \mathbf{T} \]  
(119)

and the boundary conditions are: \( x_T = \pi_T = 0 \), hence (taking the left derivative):
\[ x_T = 0, \dot{x}_T = -\sigma \mathbf{T}. \]  
(120)

To analyze (119), we look for solutions of the type \( x_t = e^{\lambda t} \). Call \( \lambda \leq \lambda' \) the two roots of:
\[ \lambda^2 - (\rho + \xi) \lambda + \rho \xi - \sigma \kappa = 0 \]  
(121)
Then, with $D = \frac{\rho \sigma r}{\rho \xi - \sigma \kappa}$, the solution is:

$$x_t = D + \frac{(D \lambda - \sigma r) e^{\lambda (t-T)} - (D' \lambda - \sigma r) e^{\lambda' (t-T)}}{\lambda' - \lambda}. \tag{122}$$

In the traditional case, $\xi = 0$, so that $\lambda < 0 < \lambda'$. As $D > 0$, this implies that, as $t \to -\infty$, $x_t \to -\infty$. We obtain an unbounded large recession. This is the logic that Werning analyzes.

However, take the case where cognitive myopia is strong enough, $\xi > \frac{\sigma \kappa}{\rho}$. Then, both roots of (121) are positive. Hence, we have a bounded recession. Indeed, as $D < 0$ in that case, $x_t$ is increasing in $t$. □

**Proof of Proposition 7.1** The BR agent has value function, which satisfies

$$V^t (w_t) = u (c_t) + V^{t+1} (w_t - c_t + y + \hat{y}_t)$$

Note that the rational value function is, for $t$

$$V^{t,r} = \frac{1}{T - t} u \left( \frac{w + \sum_{s=t}^{T-1} y_s}{T - t} \right)$$

Proposition 2.1 shows that for small $\hat{y}$, this is also the Taylor expansion (up to $O (\hat{y}^2)$ terms) of the value function under a BR policy. Hence, we will study an agent who uses the simplified value function

$$V^{t,S} (w, m^V)_{m^V=0} = (T - t) u \left( \frac{w + \sum_{s=t}^{T-1} y_s}{T - t} \right)$$

Given this, consumption before retirement ($t < L$) is given by:

$$\max_{c_t; m} u (c_t) + V^{t+1} (w_t - c_t, m^V)_{m^V=0}$$

i.e.

$$\max_{c_t; m} u (c_t) + (T - t - 1) u \left( \frac{w - c_t + \sum_{s=t+1}^{T-1} y_s}{T - t - 1} \right)$$

The FOC is

$$u' (c_t) = u' \left( \frac{w - c_t + \sum_{s=t+1}^{T-1} y_s}{T - t - 1} \right)$$

so $c_t = \frac{w - c_t + \sum_{s=t+1}^{T-1} y_s}{T - t - 1}$, i.e.

$$c_t = \frac{w + \sum_{s=t+1}^{T-1} y_s}{T - t}$$
so the agent consumes the perceived permanent income.

The choice of attention comes from (assuming that \( \mu = 0 \), e.g. because \( \kappa \) is large enough)

\[
\text{smax}_{c_t;m} v^t(c_t, m) := u(c_t) + (T - t - 1) u \left( \frac{w_t - c_t + \sum_{s=t+1}^{T-1} (y + \hat{m})}{T - t - 1} \right)
\]

so we obtain

\[
c_t = \frac{w + m_t B \hat{\gamma}}{T - t} + y.
\]

We have \( v^t_{cc} (c_t, m) \big|_{m=0} = (1 + \frac{1}{T-t-1}) u'' (c) \), so \( v^t_{cc} (c_t, m) \big|_{m=0} = u'' (c) \) in the continuous time limit, so

\[
c_t = \frac{w_t}{T - t} + \tau \left( \frac{B \hat{\gamma}}{T - t}, \kappa \right) + y.
\]
References


