Controlling a Distribution of Heterogeneous Agents

Galo Nuño
Banco de España

Benjamin Moll*
Princeton University

June 2016

Abstract

This paper analyzes the problem of a benevolent planner who controls a population of heterogeneous agents subject to idiosyncratic shocks. This is equivalent to a deterministic control problem in which the state variable is the cross-sectional distribution. We show how, in continuous time, this problem can be broken down into a dynamic programming equation plus the law of motion of the distribution and we introduce a new numerical algorithm to solve it. As an application, we analyze the constrained-efficient allocation of an Aiyagari economy with a fat-tailed wealth distribution. We find that the constrained-efficient allocation features more wealth inequality than the competitive equilibrium.

Keywords: Kolmogorov forward equation, wealth distribution, social welfare function, Gateaux derivative, mean field control.

JEL codes: C6, D3, D5, E2

*The views expressed in this manuscript are those of the authors and do not necessarily represent the views of the European Central Bank or the Bank of Spain. This paper supersedes a previous version entitled “Optimal Control with Heterogeneous Agents in Continuous Time.” The authors are very grateful to Fernando Álvarez, Chris Clayton, Jim Costain, Luca Dedola, Maurizio Falcone, Martin Holm, Francesco Lippi, Claudio Michelacci, Alessio Moro, Giulio Nicoletti, Facundo Piguillem, Carlos Thomas, Oreste Tristani, Thomas Weber, seminar participants at the Einaudi Institute for Economics and Finance, ECB, La Sapienza and conference participants at the workshop "New Perspectives in Optimal Control and Games" in Rome, CEF 2014 and Barcelona GSE Summer Forum 2016 for helpful comments and suggestions. All remaining errors are ours.
1 Introduction

Optimal control is an essential tool in economics and finance. In optimal control, a planner determines the evolution of a vector of control variables in order to maximize a certain optimality criterion. The state of the system is typically characterized by a finite number of state variables.\footnote{See Bertsekas (2012) or Fleming and Soner (2006) for an introduction to optimal control problems.} Some systems of interest are nevertheless composed of a very large number of heterogeneous agents; an economy, for example, is composed of millions of different households and firms and a network may contain thousands of nodes. In these cases, assuming a continuous distribution of state variables seems to be a reasonable approximation to the real problem under consideration.

The aim of this paper is to analyze optimal control problems in which there is a continuum of heterogeneous agents. The state of each of these agents is characterized by a finite set of variables which follow a controllable stochastic process, that is, there exists a vector of controls that allows the planner to modify the individual states. The state dynamics also depend on the evolution of a vector of aggregate variables. We consider a benevolent social planner who maximizes an aggregate welfare criterion.

We focus on the continuous time version of the problem. The key feature of working in continuous time is that the evolution of the state distribution across agents can be characterized by a partial differential equation (PDE) known as the Kolmogorov forward (KF) equation.\footnote{This equation is also known as the Fokker-Planck equation.} Despite the random evolution of each individual state, the dynamics of the distribution are deterministic due to the Law of Large Numbers. Thanks to this, the control of an infinite number of agents subject to idiosyncratic shocks can be analyzed as the control of a deterministic distribution that evolves according to the KF equation, subject to the aggregate—or market clearing—conditions.

The main contribution of the paper is to present the necessary conditions for a solution to this problem. These conditions are characterized by a system of two coupled PDEs, the planner’s Hamilton-Jacobi-Bellman (HJB) equation and the KF equation, plus the market clearing conditions. The planner’s HJB equation is a PDE that determines the marginal social value of an agent being in a certain state and the KF equation describes the distributional dynamics. This characterization of the problem allows the comparison with the decentralized solution or competitive equilibrium, in which each atomistic agent chooses her own controls without taking into account their impact on the other agents. The competitive equilibrium is also composed by a system of coupled PDEs: the individual HJB and the KF equation. The difference is that the planner’s HJB equation includes a term reflecting the impact of individual policies on the aggregate distribution, whereas the individual HJB does not. By comparing the two HJB equations we provide a condition to verify whether the competitive allocation equals the planned economy.

The second contribution of the paper is to introduce a numerical algorithm to solve optimal
control problems with heterogeneous agents. Many continuous-time models with heterogeneous-agents can be solved explicitly due to the particular set of assumptions that they make.\(^3\) In contrast, Achdou et al. (2015) provide an efficient numerical strategy based on finite difference methods in order to solve more general problems.\(^4\) Here we extend this methodology to the case of optimal control. Our algorithm solves the planner’s HJB, the KF equation and finds the Lagrange multipliers associated with the market clearing conditions.

The methodology presented here allows analyzing a variety of problems in economics. As an application, we analyze the constrained efficient allocation in an incomplete-market economy à la Aiyagari (1994) with stochastic-life agents. Recent research by Piketty and Saez (2003), Atkinson, Piketty and Saez (2011) and others has documented how since the 1970s there has been a progressive rise in top wealth inequality and in the stock of capital in several advanced economies, which in cases such as the Unites States may be close to the historical maximum. This has led to researchers such as Piketty (2014) or Atkinson (2015) to propose redistributive policies such as wealth taxation as a way to reduce wealth inequality. Notwithstanding, the increase in wealth inequality observed in the data is not necessarily negative from a social welfare perspective. It depends on which is the optimal amount of capital in the economy and on its distribution. It is not obvious that the society as a whole is better-off in a low-capital more equal situation compared to a high-capital more unequal one, for example.

We apply the optimal control techniques discussed above to try to shed light on this issue by studying the optimal constrained-efficient wealth distribution in an Aiyagari economy with stochastic-lifetimes as in the “perpetual youth” models of Yaari (1965) and Blanchard (1985). The introduction of stochastic lifetimes generates a stationary wealth distribution with an upper tail following a power law, a well-known empirical fact.\(^5\) The degree of wealth inequality, defined by the right tail exponent, depends on the difference between the return on capital and the long-run growth rate, \((r - \gamma)\), as discussed by Piketty (2014).

The constrained efficient allocation is defined as the one in which a benevolent social planner chooses the individual levels of consumption, while respecting all budget constraints. As discussed by Diamond (1967) and Davila et al. (2012), this is a notion of efficiency that does not allow the planner to directly overcome the friction implied by missing markets.\(^6\) This concept is related to

---

\(^3\)Some examples are Jovanovic (1979), Luttmer (2007) or Alvarez and Shimer (2011). See Achdou et al. (2014) for a recent survey of continuous-time models in macroeconomics.

\(^4\)Also see Rocheteau et al (2015) who propose a related incomplete-markets framework and are able to obtain a number of qualitative results.

\(^5\)The emergence of a power-law in the wealth distribution has been analyzed in a number of previous papers, such as Wold and Whittle (1957), Benhabib and Bisin (2007), Benhabib, Bisin and Zhu (2011, 2015a, 2015b), Piketty and Zucman (2015), Jones (2015) or Acemoglu and Robinson (2015), among others. Here the power law is due to the combination of random exponential lifetimes and a lower bound to the wealth distribution. A non-microfounded approach to power laws using this mechanism is discussed in Gabaix et al. (2015).

\(^6\)If the planner was able to fully redistribute across agents, the first-best allocation would be degenerated as a
that of a pecuniary externality. In the model agents do not internalize the effect of their individual saving decisions on interest rates and wages. We show how the model is constrained inefficient: the market economy is undercapitalized compared to the social optimum. The optimal allocation also features more wealth inequality than the market economy as the reduction in interest rates due to higher capital in the social optimum is not enough to compensate for the increase in aggregate savings by wealthy households.

Related literature. Our paper relates to the recent discrete-time literature, such as Davila et al. (2012) and Acikgoz (2014), analyzing problems in which a planner has to choose the controls to be applied to a continuous population of heterogeneous agents. Davila et al. (2012) analyze the constrained efficient allocation in an Aiyagari economy with infinite lifetimes using calculus of variations. Acikgoz (2014) builds on this approach in order to solve the Ramsey problem in a similar model. The continuous-time approach presented here differs from those papers in two main aspects. The first is that we characterize the problem in terms of the planner HJB instead of the Euler equation. More precisely, we show how the planner’s problem can be broken into individual HJB equations in which the value function for each person is her marginal social value under an optimal plan. The HJB equation for this marginal social value can then be compared with the HJB equation in the competitive equilibrium, thereby obtaining an easily interpretable formula that precisely characterizes the pecuniary externality causing the planner’s allocation to differ from the equilibrium one. The second lies in the approach to compute the evolution of the cross-sectional distribution. Traditional discrete-time methods either simulate a large number of agents by Montecarlo methods or discretize the state-space. In contrast, in continuous time the distributional dynamics are characterized by a partial differential equation: the KF equation. We take advantage of this fact and develop an efficient and flexible computational algorithm using finite-difference methods that applies to a wide class of planning problems in which a distribution is the relevant state variable.

Our paper is also linked to a couple of recent papers that analyze optimal control problems with heterogeneous agents in continuous time. Lucas and Moll (2014) analyze an optimal planning problem subject to the law of motion of the aggregate distribution. Their formulation nevertheless does not consider the possibility of including aggregate constraints, such as market clearing conditions, which are prevalent in most economic problems. Here instead we analyze the general problem. This requires the use of functional analysis, in particular of optimization techniques in infinite-dimensional Hilbert spaces, in order to derive the necessary conditions for a solution.

7In addition, it relates to the emerging literature in mathematics analyzing mean field control problems. The name is borrowed from the mean-field approximation in statistical physics, in which the effect on any given individual of all the other individuals is approximated by a single averaged effect. A survey may be found in Bensoussan, Frehse and Yam (2013).
Another continuous-time paper that analyzes the optimal allocation with heterogeneous agents is Afonso and Lagos (2015). They assume that their state variables can only take a finite number of values, in contrast to a continuum, and thus they can avoid the problem of optimization in infinite-dimensional spaces that we analyze here. In many applications it is more natural to work with continuous state variables, for example in models of wealth distribution like the one analyzed in the present paper.\footnote{The paper is also related to the literature that employs infinite dimensional analysis in dynamic programming problems, such as Gozzi and Faggian (2004), Fabbri and Gozzi (2008), Boucekkine, Camacho and Fabbri (2013) or Fabbri, Faggian and Freni (2015). These papers analyze in detail the mathematical properties of particular economic models in which analytical solutions can be obtained or at least tightly characterized, such as vintage capital, forestry management, population dynamics or spatial AK models. In contrast to this literature, here we provide a general methodology and a numerical strategy to solve optimal control problems in models with heterogeneous agents in which analytical solutions may not be available.}

The structure of the paper is as follows. In section 2 we discuss the problem of computing the constrained-efficient allocation in an incomplete-market economy with stochastic lifetimes as a motivation. In section 3 we analyze the general case and present the main results. In section 4 we numerically solve the problem posed in section 2. Finally, in section 5 we conclude.

2 The constrained-efficient allocation in an Aiyagari economy with stochastic lifetimes

We introduce here a model that extends the incomplete-market economy à la Aiyagari (1994) with stochastic-life agents as in the “perpetual youth” models of Yaari (1965) and Blanchard (1985). Our aim is to analyze the optimal constrained efficient allocation in the sense of Davila et al. (2012), as it is explained below.

2.1 Firms

Let \((\Sigma, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) be a filtered probability space. Time is continuous. There is a representative firm with a constant returns to scale production function \(Y = F(K, L) = K^\alpha (ZL)^{1-\alpha}\), where \(K\) is the aggregate capital, \(L\) is aggregate labor and \(Z\) is TFP. The latter evolves deterministically according to

\[Z_t = e^{\gamma t},\]
where $\gamma$ is the constant long-run growth rate of the economy. Capital depreciates at rate $\delta_K$. Since factor markets are competitive, the interest rate and wage are given by

$$
\begin{align*}
    r_t &= \frac{\partial F (K_t, 1)}{\partial K} - \delta_K = \alpha \frac{Y_t}{K_t} - \delta_K = \alpha \frac{y_t}{k_t} - \delta_K, \\
    W_t &= \frac{\partial F (K_t, 1)}{\partial L} = (1 - \alpha) \frac{Y_t}{L_t} \implies w_t = (1 - \alpha) \frac{y_t}{L_t},
\end{align*}
$$

where $w_t := W_t e^{-\gamma t}$, $y_t := Y_t e^{-\gamma t}$ and $k_t := K_t e^{-\gamma t}$.

### 2.2 Individuals

There is a continuum of mass unity of agents that are heterogeneous in their wealth $A$ and labor productivity $z$. The duration of an agent’s life is uncertain. Lifetimes $\tau$ are stochastic and governed by an exponential random variable with mean $1/\eta$. At the time of death each agent is replaced by a single child so that the size of the population is constant.

Agents have standard preferences over utility flows from future consumption $C_t$ discounted at rate $\rho > 0$. We assume CRRA preferences, such that $u(C) = \frac{C^{1-\chi}}{1-\chi}$. The expected discounted utility is

$$
U = \mathbb{E}_0 \left[ \int_0^\tau e^{-\rho t} u(C_t) dt \right] = \mathbb{E}_0 \left[ \int_0^\infty e^{-(\hat{\rho} + \eta)t} \frac{C_t^{1-\chi}}{1-\chi} dt \right],
$$

where $c_t := C_t e^{-\gamma t}$ and $\hat{\rho} = \rho - (1 - \chi) \gamma$.

Individuals have no intergenerational altruism. They purchase an annuity in a perfectly competitive insurance market that pays them a flow $\eta A_t$ in exchange of taking control of all the assets when the agent dies.\(^9\) Each agent supplies $z_t$ efficiency units of labor to the labor market and these get valued at wage $W_t$. An agent’s wealth evolves according to

$$
\begin{align*}
    dA_t &= (W_t z_t + (r_t + \eta) A_t - C_t) dt, \\
    da_t &= [w_t z_t + (r_t - \gamma + \eta) a_t - c_t] dt = s(a_t, z_t, w_t, r_t, c_t) dt,
\end{align*}
$$

where $s(a_t, z_t, w_t, r_t, c_t)$ is the drift of the detrended wealth process.

Individual labor productivity evolves stochastically over time on a bounded interval $[\underline{z}, \overline{z}]$ with

\(^9\)The amount of resources collected from expired agents is $\eta K_t$, where $K_t$ is the aggregate wealth, which equals the flow of payments so that insurance companies make no profits.
\[ dz_t = \theta(\hat{z} - z_t)dt + \sigma dB_t, \]

where \( B_t \) is a \( \mathcal{F}_t \)-adapted idiosyncratic Brownian motion and \( \theta, \hat{z} \) and \( \sigma \) are positive constants.

Agents also face a borrowing limit,

\[ A_t = -\phi e^{\gamma t} \iff a_t = -\phi, \tag{5} \]

where \( \phi \geq 0 \) is a constant such that the "natural borrowing limit" is not binding:

\[ -\phi > -z \int_0^\infty e^{-\int_t^\infty r_s^t dr} w_s ds, \quad \forall t \geq 0. \]

The optimal value function results in

\[ V(t, a, z) = \max_{c_t \in \mathcal{C}(a, z)} U(t, a, z, c_t), \tag{6} \]

subject to evolution of individual wealth (4), where

\[ \mathcal{C}(a, z) := \{ c : [0, \infty) \times \mathbb{R} \times [\hat{z}, \bar{z}] \to \mathbb{R}, \ s.t. \ a_t = -\phi, \ \forall t \geq 0 \}, \]

restricts the admissible consumption policies to those ones that do not violate the borrowing constraint.

The Hamilton-Jacobi-Bellman (HJB) equation of the individual problem is

\[ (\rho + \eta) V(t, a, z) = \max_{c_t \in \mathcal{C}(a, z)} \frac{c_t^{1-\chi}}{1-\chi} + s(a, z, w(t), r(t), c) \frac{\partial V}{\partial a} + \theta(\hat{z} - z) \frac{\partial V}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial z^2} + \frac{\partial V}{\partial t}. \tag{7} \]

### 2.3 Competitive equilibrium

As described above, agents leave no bequest. New agents begin with an initial debt \(-\phi e^{\gamma t}\) as we assume that they had to borrow in order to finance their education.\(^{11}\) They are also born with a labor productivity level of \( \hat{z} \). The state of the economy is the joint distribution of wealth and labor,

\(^{10}\)This is the continuous-time counterpart of the AR(1).

\(^{11}\)We could have assumed as well that newborns have zero initial assets. It would make no qualitative difference and only a minor quantitative difference for the numerical results presented below.
The dynamics of the distribution are given by the Kolmogorov Forward (KF) equation

\[
\frac{\partial g}{\partial t} = -\frac{\partial}{\partial a} (s(a, z, w(t), r(t), c) g(t, a, z)) \\
- \frac{\partial}{\partial z} (\theta(\ddot{z} - z)g(t, a, z)) + \frac{1}{2} \frac{\partial^2}{\partial z^2} (\sigma^2 g) - \eta g(t, a, z) + \eta \delta_0.
\] (8)

The term \(-\eta g(t, a, z)\) is the outflow of agents due to death and the term \(\eta \delta_0 = \eta \delta (a + \phi) \delta (z - z)\) is the inflow of newborn agents with detrended wealth \(-\phi\) and productivity \(z\).\(^{12}\) The distribution should satisfy the normalization

\[
\int_{-\phi}^{\phi} \int_{z}^{\infty} g(t, a, z) dz da = 1.
\]

The total amount of capital supplied in the economy equals the total amount of wealth

\[
k_t = \int_{-\phi}^{\phi} \int_{z}^{\infty} ag(t, a, z) dz da,
\] (9)

and the total amount of labor supplied in the economy equals one,

\[
L_t = 1.
\]

We define a competitive equilibrium in this economy.

**Definition 1 (Competitive equilibrium)** A competitive equilibrium is composed by a pair of factor prices \(w(t), r(t)\), an aggregate capital stock \(k(t)\), a value function \(V(t, a, z)\), a consumption policy \(c(t, A, z)\) and a distribution \(g(t, a, z)\) such that

1. Given \(w, r\) and \(g\), \(V\) is the solution of the individual’s problem (6) and the optimal control is \(c\).

2. Given \(k\), firms maximize their profits and prices are given by (1).

\(^{12}\)\(\delta (\cdot)\) is the Dirac delta, not to confound with the depreciation rate. The Dirac delta is a distribution or generalized function such that

\[
\delta [f] = \int_{-\varepsilon}^{\varepsilon} f(x) \delta (x) dx = f(0), \quad \forall \varepsilon > 0, \quad f \in L^1 (-\varepsilon, \varepsilon).
\]

A heuristic characterization is that

\[
\int_{-\infty}^{\infty} \delta (x) dx = 1, \quad \delta (x) = \begin{cases} \infty, & x = 0, \\
0, & x \neq 0.
\end{cases}
\]
3. Given \( w, r \) and \( c \), \( g \) is the solution of the KF equation (8).

4. Given \( g \) and \( k \), the capital market (9) clears.

2.4 Stationary distribution

Next we analyze the ergodic case in which the aggregate distribution is time-independent. In this case, aggregate variables such as capital, wages, wealth or consumption grow at a rate \( \gamma \). The proposition below shows how the Aiyagari model with exogenous deaths features a fat-tail ergodic wealth distribution.

**Proposition 1 (Power-law)** Provided that \( \rho + \chi \eta > (1 - \chi) r \) and \( r > r^* := \rho + \chi \gamma \), the stationary wealth distribution is characterized by an asymptotic power law, \( g(a) \sim a^{-(1+\zeta)} \), with tail exponent

\[
\zeta = \frac{\eta \chi}{r - r^*} = \frac{\eta \chi}{(r - \gamma) - (\rho - (1 - \chi) \gamma)}.
\]

**Proof.** See Appendix A. ■

Benhabib and Bisin (2007) analyze how the combination of a Blanchard-Yaari setting with an optimal consumption problem produces a power-law distribution. Our result differs from theirs in three regards. First, we analyze a general-equilibrium incomplete-market model whereas Benhabib and Bisin work in a complete-market environment where interest rates and wages are exogenous. Second, we assume that agents only care about their own utility and hence we do not consider any “joy of giving” preferences for bequests. Finally, we consider CRRA preferences instead of the more restrictive case of log-utility.

2.5 Constrained efficiency

We investigate the optimal allocation of wealth in this economy. In contrast to models with a representative agent, our optimality criterion requires some degree of interpersonal utility comparison. In line with most of the literature, we consider a utilitarian social welfare function (SWF) so that the objective of the social planner is ex-ante expected utility. This amounts to a probability-weighted average: the planner is “behind the veil of ignorance.” In addition, it is necessary to specify which degree of redistribution is possible for the planner. If the planner was able to fully redistribute across agents, the first-best allocation would be degenerated as a utilitarian planner would provide the same consumption level to every agent irrespective of her assets. This allocation does not seem too interesting as a practical benchmark. Instead, we follow Davila et al. (2012) and focus on the study of the constrained-efficient allocation. In this case the planner is constrained to
consider allocations with zero net transfers across individuals. The question is whether the planner can improve on the market allocation by simply commanding different levels of consumption, while respecting all individual budget constraints. This issue is closely related to the existence of a pecuniary externality, typically present in this kind of models: individual agents do not internalize that their saving decisions affect the aggregate amount of capital, which affects the rest of agents through wages and interest rates. The planner does take this effect into account when computing the optimal individual saving decision and thus the optimal wealth allocation.

The problem of the planner is to choose individual consumption $c(\cdot)$ across agents in order to maximize the discounted aggregate utility

$$J(g(0,\cdot)) = \max_{c(\cdot) \in C(a,z)} \int_0^\infty e^{-\rho t} \int u(c) g(t,a,z) \, da \, dz \, dt,$$  

subject to the law of motion of the aggregate distribution (8), to the factor prices (1) and to the market clearing condition (9). Here $J(g(0,\cdot))$ is the optimal value functional, as its state variable is a distribution $g(0,\cdot)$.

Notice that the planner gives the same weight to every agent irrespective of its age. This contrasts with the SWF chosen in Benhabib and Bisin (2007) which only considers the welfare of the agents alive at an arbitrary time. Notice also that the planner discounts future aggregate utility flows at the same rate of individual agents $\rho$, not at rate $(\rho + \eta)$ as it also gives a positive weight to unborn agents. The theoretical and numerical approach to solve problem (11) will be described in the next two sections.

3 General approach

In this section we analyze a general optimal control problem with heterogeneous agents and provide the necessary conditions for a solution.

3.1 Competitive equilibrium

3.1.1 Individual problem.

State. We consider a continuous-time infinite-horizon economy. Let $(\Sigma, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space. There is a continuum of unit mass of ex-ante identical agents indexed by $j \in [0,1]$. The duration of an agent’s life is uncertain. Death is governed by a Poisson random variable with rate $\eta$. At the time of death each agent is replaced by a single child so that the size of the population is constant.

Let $B^j_t$ be a $n$-dimensional $\mathcal{F}_t$-adapted Brownian motion and $X^j_t \in \mathbb{R}^n$ denote the state of an
agent $j$ at time $t \in [0, \infty)$. The state evolves according to a multidimensional Itô process of the form
\[
dX^j_t = b \left( X^j_t, \mu(t, X^j_t), Z_t \right) dt + \sigma \left( X^j_t \right) dB^j_t,
\]
where $\mu \in \mathbb{R}^m$ is a $m$–dimensional vector of control variables and $Z_t \in \mathbb{R}^p$ is a deterministic $p$-dimensional vector of aggregate variables. Here the instantaneous drift $b(\cdot)$ and volatility $\sigma(\cdot)$ are measurable functions. In the Appendix B we provide some technical assumptions to ensure the existence of a solution of the stochastic differential equation (12).

The control vector $\mu$ is an $\mathcal{F}_t$- adapted Markov control. The control strategy is the same for every agent, but it depends on time and on the specific state of the agent. The control $\mu(t, x)$ is admissible if for any initial point $(t, x)$ such that $X^j_t = x$ the stochastic differential equation (12) has a unique solution. We allow for state constraints in which the state $X$ is restricted to a compact subset $\Omega \subset \mathbb{R}^n$. We denote $\mathcal{M}(x)$ as the space of all admissible controls contained in the set of all Markov controls:

\[
\mathcal{M}(x) := \{ \mu : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m \ s.t. \ X(t) \in \Omega, \ \forall t \geq 0 \}
\]

Preferences. Agents maximize their discounted utility
\[
U = \mathbb{E}_t \left[ \int_t^\infty e^{-(\rho+\eta)s-t} u(X(t), \mu) ds \bigg| X_t = x \right],
\]
where utility $u(x, \mu)$ is strictly increasing and strictly concave and $\rho > 0$ is a constant. The optimal value function $V(t, x)$ is defined as
\[
V(t, x) = \max_{\mu(\cdot) \in \mathcal{M}(x)} U(t, x, \mu(\cdot)),
\]
subject to (12).\footnote{We have dropped the superindex $j$ as there is no possibility of confusion.} The transversality condition is
\[
\lim_{t \to \infty} e^{-\rho t} V(t, x) = 0.
\]

Hamilton-Jacobi-Bellman equation. The solution to this problem is given by a value function $V(t, x)$ and a control strategy $\mu(t, x)$ that satisfy the HJB equation
\[
\rho V(t, x) = \frac{\partial V}{\partial t} + \max_{\mu \in \mathcal{M}(x)} \{u(x, \mu) + AV\},
\]
subject to (12).
where $\mathcal{A}$ is the *infinitesimal generator* of process (12):

$$
\mathcal{A}V = \sum_{i=1}^{n} b_i(x, \mu, Z) \frac{\partial V}{\partial x_i} + \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{(\sigma(x)\sigma(x)^\top)_{i,k}}{2} \frac{\partial^2 V}{\partial x_i \partial x_k} - \eta V(t, x).
$$

The infinitesimal generator of a stochastic process is a partial differential operator that encodes the main information about the process. It is defined as

$$
\mathcal{A} f(x) = \lim_{t \to 0} \frac{\mathbb{E}_0 [ f(X_t)|X_0 = x] - f(x)}{t}.
$$

### 3.1.2 Aggregate distribution and aggregate variables.

**Kolmogorov forward equation.** We assume that new cohorts of agents are born with an initial state $x_0$ drawn from a distribution $\psi(x)$. Assume that the transition measure of $X_t$ with initial value $x_0$ has a transition density $g(t, x; x_0)$, i.e., that $\forall \varphi \in C^2(\mathbb{R}^n)$:

$$
\mathbb{E}_0 [ \varphi(X_t)|X_0 = x_0] = \int \varphi(x) g(t, x; x_0) dx.
$$

The initial distribution of $X_t$ at time $t = 0$ is $g(0, x) = g_0(x)$. The dynamics of the distribution of agents $g(t, x) = \int g(t, x; x_0) g_0(x_0) dx$ are given by the Kolmogorov Forward (KF) equation

$$
\frac{\partial g}{\partial t} = \mathcal{A}^* g + \eta \psi,
$$

(17)

$$
\int g(t, x) dx = 1,
$$

(18)

where $\mathcal{A}^*$ is the *adjoint operator* of $\mathcal{A}$.\(^{15}\)

$$
\mathcal{A}^* g = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [b_i(x, \mu, Z) g(t, x)] + \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial^2}{\partial x_i \partial x_k} \left[(\sigma(x)\sigma(x)^\top)_{i,k} g(t, x)\right] - \eta g(t, x).
$$

(19)

**Market clearing.** The vector of aggregate variables is determined by a system of $p$ equations:

$$
Z_k(t) = \int f_k(x, \mu) g(t, x) dx, \quad k = 1, \ldots, p.
$$

(20)

These equations are typically the *market clearing* conditions.

We may define a competitive equilibrium in this economy.

**Definition 2 (Competitive equilibrium)** A *competitive equilibrium* is a vector of aggregate variables $\mathbf{z} = \{Z_1, \ldots, Z_p\}$ that satisfies the market clearing conditions.\(^{14}\)

\(^{14}\)The adjoint operator generalizes the concept of matrix transpose for infinite-dimensional operators.

\(^{15}\)The adjoint operator generalizes the concept of matrix transpose for infinite-dimensional operators.
variables $Z(t)$, a value function $V(t, x)$, a control $\mu(t, x)$ and a distribution $g(t, x)$ such that

1. Given $Z_t$ and $g(t, x)$, $V(t, x)$ is the solution of the HJB equation (15) and the optimal control is $\mu(t, x)$.

2. Given $\mu(t, x)$ and $Z_t$, $g(t, x)$ is the solution of the KF equation (17, 18).

3. Given $\mu(t, x)$ and $g(t, x)$, the aggregate variables $Z_t$ satisfy the market clearing conditions (20).

Example 1 (Section 2) Using this notation, the state in the example of section 2 is $X_t = [a_t, z_t]'$, the individual control is $\mu(t, x) = c(t, a, z)$, the aggregate variable $Z_t = k_t$, the drift and volatility are

$$
b(x, \mu, Z) = \begin{bmatrix}
(1 - \alpha) k^\alpha z + [(\alpha k^{\alpha-1} - \delta_\alpha) - \gamma + \eta] a - c \\
\theta(\tilde{z} - z)
\end{bmatrix},
$$

$$
\sigma(x) = \begin{bmatrix}
0 & 0 \\
0 & \sigma z
\end{bmatrix},
$$

the instantaneous utility is $u(x, \mu) = \frac{e^{z-x}}{1-\chi}$, the exogenous newborn distribution $\psi(x) = \delta(a + \phi) \delta(z - \tilde{z})$ and the market clearing condition is such that $f(x, \mu) = a$. The domain is $\mathbb{R}^2$. Finally, the set of admissible controls is

$$
\mathcal{M}(x) = \tilde{C}(a, z) := \{c : [0, \infty) \times \mathbb{R} \times [\tilde{z}, \bar{z}] \to \mathbb{R}, \ s.t. \ a_t \in [-\phi, a^{\text{max}}], \ \forall t \geq 0\},
$$

where $a^{\text{max}}$ is a (very large) upper bound introduced in order to make the subset $\Omega = [-\phi, a^{\text{max}}] \times [\tilde{z}, \bar{z}]$ compact.

3.2 Planner’s problem

Social welfare. We now study the allocation of a planner who chooses a vector of control variables $\mu(t, x)$ to be applied to every agent $j \in [0, 1]$ with state dynamics (12). The planner also chooses the vector of aggregate variables $Z(t)$ given the constraints (20). The planner chooses the controls and the aggregate variables in order to maximize the discounted aggregate utility

$$
U^{\text{planner}} = \int_0^\infty e^{-\rho t} \int \omega(t, x) u(x, \mu) g(t, x) dx dt.
$$

The planner’s optimal value functional is
\[ J(g(0, \cdot)) := \max_{Z(\cdot), \mu(\cdot) \in \mathcal{M}(x)} U_{\text{planner}}(g(0, \cdot), \mu(\cdot), Z(\cdot)), \] (21)

subject to law of motion of the distribution (17, 18) and to the market clearing conditions (20). Notice the inclusion of the state-dependent Pareto weights \( \omega(t, x) \). If \( \omega(t, x) = 1 \) then we have a purely utilitarian social welfare function. Notice also that the planner discounts future utility flows at rate \( \rho \), not at rate \( (\rho + \eta) \).

Notice that \( J \) is the optimal value functional as it maps from the space of initial densities \( g(0, \cdot) \) into the real numbers. The planner’s problem with heterogeneous agents is an extension of the classical optimal control problem to an infinite dimensional setting, in which the state is the whole distribution of individual states \( g(t, x) \). The problem is deterministic, as so is the KF equation.

**Planner’s HJB.** We provide necessary conditions to the problem (21).

**Proposition 2 (Necessary conditions)** If a solution to problem (21) exists with \( e^{-\rho t} g, e^{-\rho t} \mu \in L^2([0, \infty) \times \mathbb{R}^n) \) and \( e^{-\rho t} Z \in L^2([0, \infty), \) then the optimal value functional \( J(g(0, \cdot)) \) can be expressed as

\[ J(g(0, \cdot)) = \int j(0, x)g(0, x)dx + \eta \int_0^\infty \int e^{-\rho t} j(t, x) \psi(x) dx dt, \]

where \( j(t, x) \) is the marginal social value function, which represents the social value of an agent at time \( t \) with an state \( x \).\(^{16} \) The social value function satisfies the HJB

\[ \rho j(t, x) = \max_{\mu \in \mathcal{M}(x)} \omega(t, x)u(x, \mu) + \sum_{k=1}^p \lambda_k(t) [f_k(x, \mu) - Z_k(t)] + A j + \frac{\partial j}{\partial t}, \]

where the Lagrange multipliers \( \lambda_k(t), k = 1, \ldots, p \) are given by

\[ \lambda_k(t) = -\int j(t, x) \left\{ \sum_{i=1}^n \left[ \frac{\partial^2 b_i}{\partial Z_k \partial x_i} g(t, x) + \sum_{j=1}^m \frac{\partial^2 b_i}{\partial Z_k \partial \mu_j} \frac{\partial \mu_j}{\partial x_i} g(t, x) + \frac{\partial b_i}{\partial Z_k} \frac{\partial g}{\partial x_i} \right] \right\} dx. \]

The proof can be found in the Appendix C. Here \( L^2([0, \infty) \times \mathbb{R}^n) \) is the space of functions with a square that is Lebesgue-integrable in \([0, \infty) \times \mathbb{R}^n\). This proposition is the central result of the paper. It provides a system of partial differential equations consisting of the HJB (23), the KF (17, 18) and the market clearing conditions (20) which link the dynamics of the social value function \( j \), the policies \( \mu \), the aggregate variables \( Z \) and the distribution \( g \).\(^{17} \) The Lagrange multipliers in (24) reflect the ‘shadow prices’ of the market clearing condition. They price, in utility terms, the

\(^{16}\) The term \( \eta \int_0^\infty \int e^{-\rho t} j(t, x) \psi(x) dx dt \) captures the welfare associated to newborns.

\(^{17}\) We have implicitly assumed that \( j \in C^2(\Omega) \). We do not provide any theoretical result if this is not the case although our numerical procedure is able to accommodate viscosity solutions, as described below.
deviation of an agent from the value of the aggregate variable: $f_k(x, \mu) - Z_k$.

Notice that the necessary conditions in the planner’s problem with $\omega = 1$ are the same as those in the competitive equilibrium, with the exception of the term $\sum_{k=1}^{p} \lambda_k (f_k(x, \mu) - Z_k)$ in the planner’s HJB equation (23). Therefore, it is trivial to check the following corollary.

**Corollary 1 (Constrained optimality of the competitive equilibrium)** The competitive equilibrium equals the social optimum in the utilitarian sense ($\omega = 1$) if

$$\sum_{k=1}^{p} \tilde{\lambda}_k(t) [f_k(x, \mu) - Z_k(t)] = 0,$$

where $\tilde{\lambda}_k(t)$ are given by

$$\tilde{\lambda}_k(t) = -\int V(t, x) \left\{ \sum_{i=1}^{n} \left[ \frac{\partial^2 b_i}{\partial Z_k \partial x_i} g(t, x) + \sum_{j=1}^{m} \frac{\partial^2 b_i}{\partial Z_k \partial x_i} \frac{\partial \mu_j}{\partial x_i} g + \frac{\partial b_i}{\partial Z_k} \frac{\partial g}{\partial x_i} \right] \right\} dx.$$

Notice that we have replaced $j(t, x)$ by $V(t, x)$ in (26), that is, the marginal social value equals the individual value. Therefore, it is enough to solve the competitive equilibrium and to compute (25) to check whether it is socially optimal.

**Example 2 (Section 2 revisited)** Coming back to the example of section 2, the planner’s HJB equation is

$$(\hat{\rho} + \eta) j(t, a, z) = \max_{c \in C(a, z)} \left\{ \frac{c^{1-\chi}}{1-\chi} + \lambda (a - k(t)) + [w(t) z + (r(t) - \gamma + \eta) a - c] \frac{\partial j}{\partial a} \right\}$$

$$+ \theta (z - \bar{z}) \frac{\partial j}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 j}{\partial z^2} + \frac{\partial j}{\partial t},$$

with the Lagrange multiplier

$$\lambda(t) = \frac{\alpha (1 - \alpha)}{k(t)^{\frac{\lambda}{2}} - \alpha} \int j(t, a, z) \left( g(t, a, z) + a \frac{\partial g}{\partial a} - k(t) z \frac{\partial g}{\partial a} \right) da dz.$$ 

Notice that equation (27) equals the individual HJB (7) plus $\lambda (a - k(t))$. The term $\lambda (a - K)$ reflects the correction in the social value $j$ compared to the individual value $V$ due to the difference between the agent’s wealth $a$ and the average $k$. If the Lagrange multiplier $\lambda$ is positive, then the social value of wealthy agents is higher than their private value and hence there is capital underaccumulation. If $\lambda$ is negative, then the private value is higher than the social one and there is capital overaccumulation.
4 The optimal wealth distribution with incomplete markets and stochastic lifetimes

We solve numerically the stationary problem of section 2 using the steady-state algorithm described in the Appendix D. In order to solve the HJB and the KF equations, we employ a finite difference method. As discussed in Achdou et al. (2015), the appropriate solution concept of HJB equation with state constraints is that of a “viscosity solution” (Crandall and Lions, 1983; Crandall, Ishii and Lions, 1992) and the proposed finite difference method converges to the unique viscosity solution of this problem (Barles and Souganidis, 1991). The idea of the finite difference method is to approximate the value function $V(a, z)$ and the distribution $g(a, z)$ on a finite grid with steps $\Delta a$ and $\Delta z$ and to compute derivatives as differences.

4.1 Calibration

Let the unit of time be one year, such that all rates are in annual terms. We assume a long-run growth rate of output, $\gamma$, of 1 per cent roughly close to the long-run per capita GDP growth in the US economy. We also assume a death rate, $\eta$, of 2 percent, equivalent to an average lifetime of 50 years. The capital share parameter, $\alpha$, is taken to be 0.36 and the depreciation rate of capital, $\delta_K$, is 0.10. The borrowing constraint, $\phi$, in our paper is set to 5. This value is chosen such that the right-tail exponent of the distribution $\zeta$, is roughly 1.5, a similar value to that in the United States according to Achdou et al. (2015). The mean of the income process, $\bar{z}$, is set to 1. The calibration of the rest of parameters follows Aiyagari (1994) and Davila et al. (2012), taking into account that in contrast to both papers we have stochastic lifetimes and long-run growth. The intertemporal elasticity of substitution $\frac{1}{\chi}$ is set to 0.5 so that the risk aversion is 2. The income process is calibrated to have an autocorrelation of 0.6 and a coefficient of variation, of 0.2, so that $\theta = 0.4$ and $\sigma = 0.2$. The subjective discount rate, $\rho$, is set to 0.01 such that the stationary discount rate $\hat{\rho} + \eta = (\rho + \eta) - (1 - \chi)\gamma$ is set to 0.04, which is the equivalent concept to the discount rate in the two papers. Finally, in order to solve numerically the model, we employ a grid with 500 points in wealth, ranging from $-5$ to 200, and 20 points in income, from 0.5 to 1.5. We introduce an upper bound to the wealth distribution of 200, equivalent to around 50 times the average wealth, in order to capture most of the dynamics at the right tail of the distribution.$^{18}$

4.2 Results

The first column in Table 1 displays the steady-state values for the main aggregate variables in the competitive equilibrium. Capital is two and a half times larger than output and the interest

$^{18}$Result are robust to changes in the size of the domain.
rate, \( r \), is 4.45 per cent, a value larger than \( r^* := \rho + \chi\gamma = 3\% \) in Proposition 1. Notice also that the adjusted interest rate \( (r - \gamma + \eta) = 5.45\% \) is larger than the adjusted discount rate \( \hat{\rho} = 4\% \), but it poses no problem as agents cannot accumulate wealth indefinitely due to their random deaths. Figure 1 displays the savings policy \( s(a, z) := (wz + (r - \gamma + \eta)a - c(a, z)) \) and the wealth-productivity distribution \( g(a, z) \). Notice how, in the case of the distribution, there is a large proportion of the agents at the borrowing limit of \( -\phi = -5 \) for values of \( z \) below 1. By construction the model also replicates the wealth inequality observed in the United States, with an exponent, \( \zeta \), around 1.5.\(^{19}\)

<table>
<thead>
<tr>
<th></th>
<th>Competitive equilibrium</th>
<th>Constrained optimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregate capital, ( k )</td>
<td>4.16</td>
<td>4.88</td>
</tr>
<tr>
<td>Output, ( y )</td>
<td>1.67</td>
<td>1.77</td>
</tr>
<tr>
<td>Capital-output ratio, ( k/y )</td>
<td>2.49</td>
<td>2.76</td>
</tr>
<tr>
<td>Interest rate, ( r )</td>
<td>4.45</td>
<td>3.04</td>
</tr>
<tr>
<td>Pareto exponent, ( \zeta )</td>
<td>1.53</td>
<td>0.76</td>
</tr>
</tbody>
</table>

In order to solve the planner’s problem we need to jointly solve the HJB equation (27), the KF equation (8) and the definition of the Lagrange multiplier (28), as described in Appendix D. For the calibration above, the value of \( \lambda \) results in 0.004. The positiveness of \( \lambda \) implies that the social value of individuals with large wealth is higher than their private value. The second column in Table 1 displays the results in this case. The constrained-efficient allocation features larger levels of output, capital and of the capital-output ratio than the competitive equilibrium. This means that the market economy is undercapitalized compared to the social optimum, which necessarily implies larger levels of savings and lower interest rates in the optimum. The social optimum also features a higher level of wealth inequality with a Pareto exponent of 0.76.\(^{20}\) Notice that despite the fact that the slope is smaller than one, the wealth distribution has a well-defined mean as it is bounded by a maximum \( a = 200 \).\(^{21}\)

\(^{19}\) The exponent is computed numerically for wealth levels between the first and the last decile of the wealth distribution. The wealth distribution is defined as

\[
f(a) := \int_{z}^{\xi} f(a, z) dz.
\]

\(^{20}\) The planning economy also displays a power-law distribution, but with a smaller slope than the competitive equilibrium.

\(^{21}\) This also explains why the interest rate may be larger than \( g = 1\% \).
Figure 1: Savings policy and wealth-productivity distribution in the competitive equilibrium.

Figure 2 illustrates how, despite the lower interest rates, savings $s(a, z)$ are higher in the optimal allocation than in the market economy, which explains the lower value of the tail exponent in that case. The bottom line of this analysis is that the reduction in interest rates in the social optimum is not enough to compensate for the increase in aggregate savings by wealthy households, thus resulting in an increase in wealth inequality. This is naturally linked to the fact that the competitive allocation displays less capital than the constrained-efficient allocation, and therefore a larger level of aggregate savings is welfare-improving.

Finally, we compare aggregate welfare in both allocations. Aggregate welfare in the stationary case can be defined as

$$U = \frac{1}{(\rho + \eta)} \int_0^\infty \int_{z}^{\hat{z}} u(c) g(t, a, z) dadz.$$  (29)

In order to compare both allocations, we express the ratio of welfare in consumption equivalent terms, that is, we express it as the proportion $\Theta$ of increase in the stationary consumption $c(a, z)$ of all agents such that the welfare in both allocations (planned and competitive) is the same

$$\int_0^\infty \int_{z}^{\hat{z}} u^{\text{planner}}(c) g^{\text{planner}}(t, a, z) dadz = \int_0^\infty \int_{z}^{\hat{z}} u^{\text{ce}}(c(1 + \Theta)) g^{\text{ce}}(t, a, z) dadz,$$

and hence

$$\Theta = \left(\frac{U^{\text{planner}}}{U^{\text{ce}}}\right)^{\frac{1}{1-\gamma}} - 1 = 0.089,$$  (30)

that is, there is an average 8.9 percent gain in consumption-equivalent terms in the optimal allocation compared to the market one.
5 Conclusions

This paper analyzes the problem of a planner who controls a population of heterogeneous agents subject to idiosyncratic shocks in order to maximize an optimality criterion related to the distribution of states across agents. If the problem is analyzed in continuous time, the KF equation provides a deterministic law of motion of the entire distribution of state variables across agents. The problem can thus be analyzed as one of deterministic optimal control in which both the control and the state are distributions. If a solution exists, we show how it should satisfy a system of coupled PDEs composed by the planner’s HJB and the KF equations. We also introduce a simple criterion to check whether a competitive equilibrium is constrained efficient.

We provide a numerical algorithm in order to find the solution to the planning problem. As an application, we employ this algorithm to analyze the welfare properties of an Aiyagari economy with stochastic lifetimes. In particular, we analyze the constrained social optimum in which a social planner maximizes the aggregate welfare subject to the same equilibrium budget constraints and competitive price setting as the individual agents. We show how the social optimum features more capital than the market economy. We also show how the level of wealth inequality is higher in the social optimum as the reduction in interest rates due to higher capital is not enough to compensate for the increase in aggregate savings by wealthy households.
References


Appendix

A. Proof of Proposition 1

The proposition is similar to Proposition 6 in Achdou et al. (2015). First we show that individual consumption is asymptotically linear in $a$ as $a \to \infty$: $c \propto a$. We consider the auxiliary problem without labor income, $wz$, and without borrowing constraint ($\phi = \infty$), characterized by the HJB equation

$$(\hat{\rho} + \eta) V(a) = \max_c \frac{c^{1-\chi}}{1-\chi} + ((r - \gamma + \eta) a - c) V'(a).$$

(31)

We guess and verify a solution of the form $V(a) = \kappa^{-\chi} a^{1-\chi}$, so that the first-order condition is

$$c^{-\chi} = V'(a) = \kappa^{-\chi} a^{-\chi}.$$ 

The HJB results in

$$(\hat{\rho} + \eta) \kappa^{-\chi} a^{1-\chi} = \kappa^{1-\chi} a^{1-\chi} + ((r - \gamma + \eta) - \kappa) \kappa^{-\chi} a^{1-\chi},$$

and then

$$\kappa = \frac{\rho + \chi \eta - (1 - \chi) r}{\chi}.$$ 

The optimal consumption is

$$c = \frac{\rho + \chi \eta - (1 - \chi) r}{\chi} a,$$ 

(32)

which has a positive slope as long as $\rho + \chi \eta - (1 - \chi) r > 0$.

Second, given the HJB equation (23), for any $\xi > 0$,

$$V(a, z) = \xi^{1-\chi} V_{\xi}(a/\xi, z),$$

where $V_{\xi}(a, z)$ solves

$$(\hat{\rho} + \eta) V_{\xi}(a, z) = \max_{c_{\xi}} \frac{c_{\xi}^{1-\chi}}{1-\chi} + (w z/\xi + (r - \gamma + \eta) a_{\xi} - c_{\xi}) \frac{\partial V_{\xi}(a, z)}{\partial a}$$

$$+ \theta(\xi - z) \frac{\partial V_{\xi}(a, z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 V_{\xi}(a, z)}{\partial z^2}, \quad a \geq -\phi/\xi.$$
This can be easily verified as

\[ V(a, z) = \xi^{1-\chi} V_\xi(a/\xi, z), \quad \frac{\partial V(a, z)}{\partial a} = \xi^{-\chi} \frac{\partial V_\xi(a/\xi, z)}{\partial (a/\xi)}, \]
\[ \frac{\partial V(a, z)}{\partial z} = \xi^{1-\chi} \frac{\partial V_\xi(a/\xi, z)}{\partial z}, \quad \frac{\partial^2 V(a, z)}{\partial z^2} = \xi^{1-\chi} \frac{\partial^2 V_\xi(a, z)}{\partial z^2}. \]

Third, notice how in the asymptotic limit \( \xi \to \infty \):

\[ \lim_{\xi \to \infty} V_\xi(a, z) = V(a) \]

and

\[ \lim_{\xi \to \infty} c_\xi(a, z) = c(a), \]

the latter given by equation (32). This is equivalent so state that, for \( a \) large enough we have

\[ c(a, z) = \frac{\rho + \chi \eta - (1 - \chi) r}{\chi} a. \]

The stationary KF equation for large \( a \) then results in

\[ 0 = -\frac{d}{da} \left[ \left( r - \gamma + \eta - \frac{\rho + \chi \eta - (1 - \chi) r}{\chi} \right) ag(a) \right] - \eta g(a). \]

We may guess that \( g(a) \sim a^{-(1+\zeta)} \) and then verify

\[ \zeta = \frac{\chi \eta}{r - (\rho + \chi \gamma)}. \]

**B. Technical assumptions**

The assumptions are based on the ones in Bensoussan, Chan and Yam (2015):

1. **Lipschitz continuity.** There exists \( K > 0 \), such that

\[ |b(x, \mu, Z) - b(x', \mu', Z')| \leq K (|x - x'| + |\mu - \mu'| + |Z - Z'|), \]
\[ |\sigma(x) - \sigma(x')| \leq K |x - x'|. \]

2. **Linear growth.** There exists \( K > 0 \), such that

\[ |b(x, \mu, Z)| \leq K (1 + |x| + |\mu| + |Z|), \]
\[ |\sigma(x)| \leq K (1 + |x|). \]
3. Quadratic condition on the objective. There exists $K > 0$, such that

$$
\left| e^{-\rho t} \omega(t, x) u(t, x, \mu) g(t, x) - e^{-\rho t'} \omega(t', x') u(t', x', \mu') g(t', x') \right| \leq K \left( \frac{1 + |t| + |t'| + |x| + |x'| }{ + |\mu| + |\mu'| } \right) \left( |t - t'| + |x - x'| \right) + |\mu - \mu'| .
$$

C. Proof of Proposition 2.

The idea of the proof is to solve problem (21) using differentiation techniques in infinite-dimensional Hilbert spaces.

C.1. Mathematic preliminaries

First we need to introduce some mathematical concepts.\textsuperscript{22} Let $L^2(\Phi)$ be the space of functions with a square that is Lebesgue-integrable over $\Phi \subset \mathbb{R}^s$. It is a Banach space with the norm

$$
\| g \|_{L^2(\Phi)} = \sqrt{ \int_{\Phi} |g(x)|^2 \, dx },
$$

that is, it is a complete normed vector space. In contrast to $n$-dimensional Banach spaces such as $\mathbb{R}^n$, $L^2(\Phi)$ is infinite-dimensional.

The space of Lebesgue-integrable functions $L^2(\Phi)$ with the inner product

$$
\langle u, g \rangle_\Phi = \int_{\Phi} u g \, dx, \quad \forall u, g \in L^2(\Phi),
$$

is a Hilbert space.

An operator $T$ is a mapping from one vector space to another. For example, given the process $X_t$ described in (12), its \textit{infinitesimal generator} $\mathcal{A}$ is an operator in $L^2(\Phi)$ defined by (16). The \textit{adjoint operator} $T^*$ of a linear operator $T$ in a Hilbert space is defined by the equation

$$
\langle u, Tg \rangle_\Phi = \langle T^*u, g \rangle_\Phi .
$$

\textbf{Example 3} We may verify that $\mathcal{A}$ and $\mathcal{A}^*$, defined in (16) and (19) respectively, are adjoint

\textsuperscript{22} All the contents here are adapted from Luenberger (1969), Gelfand and Fomin (1991), Sagan (1992) and Brezis (2011).
operators in $L^2(\mathbb{R}^n)$. Given $\forall u, g \in L^2(\mathbb{R}^n)$

$$\langle g, Au \rangle = \int g A u dx = \sum_{i=1}^{n} \int g(x) b_i(x, \mu, Z) \frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{(\sigma(x)\sigma(x)^\top)_{i,k}}{2} \frac{\partial^2 u}{\partial x_i \partial x_k} - \eta u(t, x) \right) \right) dx$$

$$= \int u \left( - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( gb_i \right) dx + \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial^2}{\partial x_i \partial x_k} \left( g \frac{(\sigma(x)\sigma(x)^\top)_{i,k}}{2} \right) dx - \eta g \right) dx$$

$$= \int u A^* g dx = \langle A^* g, u \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^n)$ and we have integrated by parts.

Let $J(g) : L^2(\Phi) \to \mathbb{R}$ be a functional of $g$. There are two concepts of derivatives in Hilbert spaces.

**Definition 3 (Gateaux derivative)** Let $J(g)$ be a functional and let $h$ be arbitrary in $L^2(\Phi)$. If the limit

$$\delta J(g; h) = \lim_{\alpha \to 0} \frac{J(g + \alpha h) - J(g)}{\alpha}$$

exists, it is called the Gateaux derivative of $J$ at $g$ in the direction $h$. If the limit (33) exists for each $h \in L^2(\Phi)$, the functional $J$ is said to be Gateaux differentiable at $g$.

If the limit exists, it can be expressed as $\delta J(g; h) = \frac{d}{d\alpha} J(g + \alpha h) \vert_{\alpha=0}$. A more restricted concept is that of the Fréchet differential.

**Definition 4 (Fréchet derivative)** Let $J(g)$ be a functional and let $h$ be arbitrary in $L^2(\Phi)$. If there exists $\delta J(g; h)$ which is linear and continuous with respect to $h$ such that

$$\lim_{\|h\|_{L^2(\Phi)} \to 0} \frac{|J(g + h) - J(g) - \delta J(g; h)|}{\|h\|_{L^2(\Phi)}} = 0,$$

then $\delta J(g; h)$ is the Fréchet derivative of $J$ at $g$ in the direction $h$. If the limit (34) exists for each $h \in L^2(\Phi)$, the functional $J$ is said to be Frechet differentiable at $g$.

The following proposition links both concepts.

**Proposition 3** If $J$ is Fréchet differentiable at $g$, then it is Gateaux differentiable at $g$ and the Gateaux and Fréchet derivatives are equal.


The familiar technique of maximizing a function of a single variable by ordinary calculus can be extended in infinite dimensional spaces to a similar technique based on more general differentials.
We use the term extremum to refer to a maximum or a minimum over any set. A function \( g \in L^2(\Phi) \) is a maximum of \( J(g) \) if for all functions \( h \), \( \| h - g \|_{L^2(\Phi)} < \varepsilon \) then \( J(g) \geq J(h) \). The following theorem is the Fundamental Theorem of Calculus.

**Theorem 2** If \( J \) is Gateaux differentiable on \( L^2(\Phi) \), a necessary condition for \( J \) to have a local extremum at \( g \in L^2(\Phi) \) is that \( \delta J(g; h) = 0 \) for all \( h \in L^2(\Phi) \).


Finally, we can extend this result to the case of constrained optimization.

**Theorem 3 (Lagrange multipliers)** Let \( H \) be a continuously Fréchet differentiable mapping from \( L^2(\Phi) \) into \( \mathbb{R}^n \). If \( J \) is Fréchet differentiable on \( L^2(\Phi) \), a necessary condition for \( J \) to have a local extremum at \( g \in L^2(\Phi) \) under the constraint \( H(g) = 0 \) is that there exists a function \( \lambda \in L^2(\Phi) \) such that the Lagrangian functional

\[ \mathcal{L}(g) = J(g) + \langle \lambda, H(g) \rangle \Phi \]  

is stationary in \( g \), i.e., \( \delta \mathcal{L}(g; h) = 0 \), for all \( h \in L^2(\Phi) \).


We define \( \tilde{L}^2(\Phi) \) as the space of functions \( f \) such that

\[ \| f \|_{\tilde{L}^2(\Phi)}^2 = \int_{\Phi} e^{-\rho t} |f|^2 \, dx < \infty. \]

**Lemma 4** The space \( \tilde{L}^2(\Phi) \) with the inner product

\[ (f, g)_{\Phi} = \int_{\Phi} e^{-\rho t} f g \, dx \]

is a Hilbert space.

**Proof.** We need to show that \( \tilde{L}^2(\Phi) \) is complete, that is, that given a Cauchy sequence \( \{ f_n \} \) with limit \( f : \lim_{n \to \infty} f_n = f \) then \( f \in \tilde{L}^2(\Phi) \). If \( \{ f_n \} \) is a Cauchy sequence then

\[ \| f_n - f_m \|_{\tilde{L}^2(\Phi)} \to 0 \text{ as } n, m \to \infty, \]

or

\[ \| f_n - f_m \|_{\tilde{L}^2(\Phi)}^2 = \int_{\Phi} e^{-\rho t} |f_n - f_m|^2 \, dx = \left\langle e^{-\frac{\rho t}{2}} (f_n - f_m), e^{-\frac{\rho t}{2}} (f_n - f_m) \right\rangle_{\Phi} = \left\| e^{-\frac{\rho t}{2}} (f_n - f_m) \right\|_{L^2(\Phi)}^2 \to 0, \]
as \( n, m \to \infty \). This implies that \( \{ e^{-\frac{2}{e} t} f_n \} \) is a Cauchy sequence in \( L^2(\Phi) \). As \( L^2(\Phi) \) is a complete space, then there is a function \( \hat{f} \in L^2(\Phi) \) such that
\[
\lim_{n \to \infty} e^{-\frac{2}{e} t} f_n = \hat{f}
\] (36)
under the norm \( \| \cdot \|_{L^2(\Phi)}^2 \). If we define \( f = e^{\frac{2}{e} t} \hat{f} \) then
\[
\lim_{n \to \infty} f_n = f
\]
under the norm \( \| \cdot \|_{L^2(\Phi)}^2 \); for any \( \varepsilon > 0 \) there is an integer \( N \) such that
\[
\| f_n - f \|_{L^2(\Phi)}^2 = \| e^{\frac{2}{e} t} (f_n - \hat{f}) \|_{L^2(\Phi)}^2 = \| e^{\frac{2}{e} t} f_n - \hat{f} \|_{L^2(\Phi)}^2 < \varepsilon,
\]
where the last inequality is due to (36). It only remains to prove that \( f \in \tilde{L}^2(\Phi) \):
\[
\| f \|_{\tilde{L}^2(\Phi)}^2 = \int_{\Phi} e^{-\mu t} |f|^2 \, dx = \int_{\Phi} |\hat{f}|^2 \, dx < \infty.
\]

Notice that by the Riesz-Fréchet Theorem, the dual space of the a real Hilbert space is itself.\(^{23}\)

\( \) C.2. Proof of the proposition

We define the extended domain \( \Phi := [0, \infty) \times \mathbb{R}^n \). The problem of the planner is to maximize
\[
\int_{\Phi} e^{-\mu t} \omega(t,x) u(x,\mu) g(t,x) \, dtdx = \left< e^{-\mu t} \omega u, g \right>_{\Phi} = (\omega u, g)_{\Phi},
\]
subject to the KF equation (17)
\[
-\frac{\partial g}{\partial t} + A^* g + \eta \psi = 0, \ \forall (t,x) \in \Phi
\]
and the market clearing condition (20)
\[
\int (f_k(x,\mu) - Z_k(t)) g(t,x) \, dx, \ \ k = 1, \ldots, p, \ \forall t \in [0, \infty).
\] (37)

If a solution to the planner’s problem (21) exists with \( e^{-\mu t} g, e^{-\mu t} \mu \in L^2(\Phi) \) and \( e^{-\mu t} Z \in \)

we can express the Lagrangian functional (35) for this problem as

\[ \mathcal{L}(g, \mu_1, \ldots, \mu_m, Z_1, \ldots, Z_p) = \left( e^{-\rho t} \omega u, g \right)_\Phi + \left( j, -\frac{\partial g}{\partial t} + A^* g + \eta \psi \right)_\Phi + \sum_{k=1}^{p} \left( \lambda_k, (f_k - Z_k) g \right)_\Phi \]

where \( e^{-\rho t} j(t, x) \in L^2(\Phi) \) and \( e^{-\rho t} \lambda_k(t) \in L^2[0, \infty), k = 1, \ldots, p \) are the Lagrange multipliers associated to equations (17) and (20), respectively. As stated in Theorem 3, a necessary condition for \((g, \mu_1, \ldots, \mu_m, Z_1, \ldots, Z_p)\) to be a maximum of (38) is that the Gateaux derivative with respect to each of these functions equals zero in any direction.

It will prove useful to modify the second term in the Lagrangian

\[ \left\langle j, e^{-\rho t} \left( \frac{\partial g}{\partial t} + A^* g + \eta \psi \right) \right\rangle_\Phi = \int_0^\infty \int e^{-\rho t} j(t, x) \left( -\frac{\partial g}{\partial t} + A^* g + \eta \psi \right) dt dx \]

\[ = -\int e^{-\rho t} j(t, x) g(t, x) \bigg|_0^\infty dx + \int_0^\infty \int e^{-\rho t} \left( \frac{\partial j}{\partial t} - \rho j(t, x) \right) g dt dx \]

where we have integrated by parts with respect to time in the term \( \frac{\partial g}{\partial t} \) and applied the fact that \( A^* \) is the adjoint operator of \( A \) in \( \tilde{L}^2(\Phi) \).

\[ = -\lim_{T \to \infty} \int e^{-\rho T} j(T, x) g(T, x) dx + \int j(0, x) g(0, x) dx \]

\[ + \eta \int_0^\infty e^{-\rho t} j(t, x_0) dt + \left\langle e^{-\rho t} A j, g \right\rangle_\Phi, \]

where \( e^{-\rho t} Z \in L^2(0, \infty) \) we may construct an extended control \( e^{-\rho t} Z(t) \mathbf{1}_{\{x \in \Omega\}} \in L^2(\Phi) \).

Provided that \( e^{-\rho t} Z \in L^2(0, \infty) \) we may construct an extended control \( e^{-\rho t} Z(t) \mathbf{1}_{\{x \in \Omega\}} \in L^2(\Phi) \).

\[ \int_0^\infty e^{-\rho t} dt = \frac{1}{\rho} < \infty. \]
The Gateaux derivative with respect to \( g \) in the direction \( h (t, x) \in L^2 (\Phi) \):

\[
\lim_{\alpha \to 0} \frac{d}{d\alpha} \left( e^{-\rho t} \omega u, g + \alpha h \right)_\Phi + \lim_{\alpha \to 0} \frac{d}{d\alpha} \left( e^{-\rho t} \left( \frac{\partial j}{\partial t} - \rho j + \mathcal{A} j \right), g + \alpha h \right)_\Phi \\
+ \lim_{\alpha \to 0} \frac{d}{d\alpha} \sum_{k=1}^{p} \left( e^{-\rho t} \lambda_k, (f_k - Z_k) (g + \alpha h) \right)_\Phi - \lim_{\alpha \to 0} \lim_{T \to \infty} \frac{d}{d\alpha} \int e^{-\rho t} j (T, x) (g (T, x) + \alpha h (T, x)) \, dx \\
+ \lim_{\alpha \to 0} \frac{d}{d\alpha} \int j (0, x) (g (0, x) + \alpha h (0, x)) \, dx
\]

\[
= \left( e^{-\rho t} \omega u, h \right)_\Phi + \left( e^{-\rho t} \left( \frac{\partial j}{\partial t} - \rho j + \mathcal{A} j \right), h \right)_\Phi + \sum_{k=1}^{p} \left( e^{-\rho t} \lambda_k, (f_k - Z_k) h \right)_\Phi \\
- \lim_{T \to \infty} \int e^{-\rho t} j (T, x) h (T, x) \, dx,
\]

where we have applied the fact that \( h (0, \cdot) = 0 \) as the initial distribution \( g (0, \cdot) = 0 \) is given. The Gateaux derivative should be zero for any \( h(t, x) \), therefore

\[
\rho j (t, x) = \omega u + \sum_{k=1}^{p} \lambda_k (f_k - Z_k) + \mathcal{A} j + \frac{\partial j}{\partial t}, \quad \forall (t, x) \in \Phi,
\]

(43)

\[
\lim_{T \to \infty} e^{-\rho T} j (T, \cdot) = 0,
\]

(44)

which is the HJB equation of the planner (23).

The Gateaux derivative with respect to the individual control \( \mu_j \) is

\[
\lim_{\alpha \to 0} \frac{d}{d\alpha} \left( (e^{-\rho t} \omega u (x, \mu_j + \alpha h)), g \right)_\Phi + \lim_{\alpha \to 0} \frac{d}{d\alpha} \left( e^{-\rho t} \left( \frac{\partial j}{\partial t} - \rho j + \mathcal{A} (\mu_j + \alpha h) j \right), g \right)_\Phi \\
+ \lim_{\alpha \to 0} \frac{d}{d\alpha} \sum_{k=1}^{p} \left( e^{-\rho t} \lambda_k, (f_k (x, \mu_j + \alpha h) - Z_k) g \right)_\Phi,
\]

(45)

where \( \mathcal{A} (\mu_j + \alpha h) j \) is defined as

\[
\mathcal{A} (\mu_j + \alpha h) j := \sum_{i=1}^{n} b_i (x, \mu_1, ..., \mu_j + \alpha h, ..., \mu_m, Z) \frac{\partial j}{\partial x_i} + \sum_{i=1}^{n} \sum_{k=1}^{n} \left( \sigma (x) \sigma (x)^T \right)_{i,k} \frac{\partial^2 j}{\partial x_i \partial x_k} - \eta j (t, x).
\]

Provided that \( x \in \text{int}(\Omega) \) then the condition that the Gateaux derivative is zero yields

\[
\mu = \arg \max_{\tilde{\mu}} \left\{ \omega u (x, \tilde{\mu}) + \sum_{k=1}^{p} \lambda_k f_k (x, \tilde{\mu}) + \mathcal{A} \tilde{\mu} j \right\},
\]

(46)

whereas if \( x \in \partial \Omega \) then \( \mu \) is such that \( X (t) \) does not leave the subset \( \Omega \).
The Gateaux derivative with respect to the aggregate variable \( Z_k \) is
\[
\lim_{\alpha \to 0} \frac{d}{d\alpha} \left< e^{-\rho t} j, \mathcal{A}^*_\alpha \right> + \lim_{\alpha \to 0} \frac{d}{d\alpha} \sum_{k=1}^p \left< e^{-\rho t} \lambda_k, (f_k - (Z_k + \alpha h)) g \right>,
\]
for any \( h(t) \in \tilde{L}^2[0, \infty) \). \( \mathcal{A}^*_\alpha \) is defined as
\[
\mathcal{A}^*_\alpha \left< e^{-\rho t} j, \mathcal{A}^*_\alpha \right> \quad := \quad -\sum_{i=1}^n \frac{\partial}{\partial x_i} \left[ b_i (x, \mu, Z_1, \ldots, Z_k + ah, \ldots, Z_p) g(t, x) \right] \quad + \quad \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n \frac{\partial^2}{\partial x_i \partial x_k} \left[ (\sigma(x)\sigma(x)^\top)_{i,k} g(t, x) \right] - \eta g(t, x) .
\]
This can be expressed as
\[
\lim_{\alpha \to 0} \int_0^\infty \int e^{-\rho t} j(t, x) \frac{d}{d\alpha} \left\{ -\sum_{i=1}^n \frac{\partial}{\partial x_i} \left[ b_i (x, \mu, Z_1, \ldots, Z_k + ah, \ldots, Z_p) g(t, x) \right] - \sum_{k=1}^p \lambda_k (Z_k + ah) g \right\} ,
\]
and hence
\[
\int_0^\infty e^{-\rho t} h(t) \left\{ \int j(t, x) \left( \sum_{i=1}^n \frac{\partial^2 b_i}{\partial Z_k \partial x_i} g(t, x) + \sum_{j=1}^m \frac{\partial^2 b_i}{\partial Z_k \partial \mu_j} \frac{\partial \mu_j}{\partial x_i} g(t, x) + \frac{\partial b_i}{\partial Z_k} \frac{\partial g}{\partial x_i} \right) dx + \lambda_k(t) \right\} dt = 0.
\]
As this is satisfied for any \( h(t) \in \tilde{L}^2[0, \infty) \), we obtain that
\[
\lambda_k(t) = -\int j(t, x) \left\{ \sum_{i=1}^n \frac{\partial^2 b_i}{\partial Z_k \partial x_i} g(t, x) + \sum_{j=1}^m \frac{\partial^2 b_i}{\partial Z_k \partial \mu_j} \frac{\partial \mu_j}{\partial x_i} g(t, x) + \frac{\partial b_i}{\partial Z_k} \frac{\partial g}{\partial x_i} \right\} dx \quad (47)
\]
Finally, in order to prove that the Lagrange multiplier \( j \) is indeed the social value function we multiply by \( e^{-\rho t} g(t, x) \) at both sides of the planner’s HJB equation (43) and integrate over \( \Phi \):
\[
\int_0^\infty \int \rho e^{-\rho t} j dx dt = \int_0^\infty \int e^{-\rho t} \left( \omega u + \sum_{k=1}^p \lambda_k (f_k - Z_k) + \lambda_j + \frac{\partial j}{\partial t} \right) g dx dt \quad (48)
\]
Taking into account that, due to the market clearing condition (20),
\[
\int (f_k - Z_k) f dx = 0,
\]
and, as $A^*$ is the adjoint operator of $A$,

$$
\int g A j dx = \int j A^* g dx,
$$

$$
\int_0^{\infty} e^{-\rho t} g \frac{\partial j}{\partial t} dt = e^{-\rho t} g(t, x) j(t, x)|_0^{\infty} - \int_0^{\infty} j \frac{\partial}{\partial t} \left( e^{-\rho t} g \right) dt
= -g(0, x) j(0, x) + \int_0^{\infty} e^{-\rho t} \rho j g dt - \int_0^{\infty} e^{-\rho t} j \frac{\partial g}{\partial t} dt,
$$

then (48) results in

$$
\int_0^{\infty} \int e^{-\rho t} \omega u g dx dt + \int_0^{\infty} \int e^{-\rho t} j \left( -\frac{\partial g}{\partial t} + A^* g \right) dx dt = \int g(0, x) j(0, x) dx.
$$

Given the KF equation $\frac{\partial g}{\partial t} = A^* g + \eta \psi$,

$$
\int_0^{\infty} \int e^{-\rho t} j \left( -\frac{\partial g}{\partial t} + A^* g \right) dx dt = -\eta \int_0^{\infty} \int e^{-\rho t} j (t, x) \psi (x) dx dt.
$$

The social welfare functional can then be expressed as

$$
J \left( f(0, \cdot) \right) = \int_0^{\infty} \int e^{-\rho t} \omega u f dx dt = \int f(0, x) j(0, x) dx + \eta \int_0^{\infty} \int e^{-\rho t} j (t, x) \psi (x) dx dt.
$$

D. Description of the numerical algorithm

The general idea for the solution of infinite-horizon coupled HJB-KF systems is to first solve the steady-state and then iterate backward and forward in time, as described in Achdou et al. (2015). Here we concentrate on the steady-state solution, being the extension to analyze transitional dynamics straightforward. The steady-state can be computed using a relaxation algorithm. Given a constant $\theta \in (0, 1)$, begin with an initial guess of the aggregate variables $Z^0$ and the Lagrange multipliers $\lambda^0 = 0$, set $n, m = 0$:

1. Given $Z^n$ and $\lambda^m$, solve the HJB equation (23) in the stationary case ($\frac{\partial j}{\partial t} = 0$) and obtain the social value function $j^n$ and the optimal policies $\mu^n$.

2. Given the optimal policies $\mu^n$ solve the KF equation (17) in the stationary case ($\frac{\partial g}{\partial t} = 0$) and obtain the distribution $g^n$.

3. Given the optimal policies $\mu^n$ and the distribution $g^n$, compute the aggregate variables $\tilde{Z}^{n+1}$ using the market clearing conditions (20). If $\tilde{Z}^{n+1} \neq Z^n$, set $Z^{n+1} = \theta \tilde{Z}^{n+1} + (1 - \theta) Z^n$. 

32
update $n := n + 1$ and return to step 1.

4. Compute $\tilde{\lambda}^{n+1}$ using the definition (24). If $\tilde{\lambda}^{n+1} \neq \lambda^n$, set $\lambda^{n+1}$ as discussed below, update $m := m + 1$ and return to step 1.

If the algorithm converges, it should produce the steady-state value function $j_\infty$, the optimal policies $\mu_\infty$, the aggregate variables $Z_\infty$, the Lagrange multipliers $\lambda_\infty$ and the distribution $g_\infty$.

Step 1: Solution to the Hamilton-Jacobi-Bellman equation

The HJB equation is solved by a finite difference scheme following Achdou et al. (2015). It approximates the value function $V(a, z)$ on a finite grid with steps $\Delta a$ and $\Delta z$. We use the notation $V_{i,j} := V(a_i, z_j)$, $i = 1, ..., I; j = 1, ..., J$. The derivative of $V$ with respect to $a$ can be approximated with either a forward or a backward approximation:

$$\frac{\partial V(a_i, z_j)}{\partial a} \approx \partial_{a,F} V_{i,j} := \frac{V_{i+1,j} - V_{i,j}}{\Delta a},$$

$$\frac{\partial V(a_i, z_j)}{\partial a} \approx \partial_{a,B} V_{i,j} := \frac{V_{i,j} - V_{i-1,j}}{\Delta a},$$

where the decision between one approximation or the other depends on the sign of the savings function $s_{i,j} = wz_j + ra_i - c_{i,j}$ through an “upwind scheme” described below. The derivative of $V$ with respect to $z$ is approximated using a forward approximation

$$\frac{\partial V(a_i, z_j)}{\partial z} \approx \partial_z V_{i,j} := \frac{V_{i,j+1} - V_{i,j}}{\Delta z},$$

$$\frac{\partial^2 V(a_i, z_j)}{\partial z^2} \approx \partial_{zz} V_{i,j} := \frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{(\Delta z)^2}.$$  

The HJB equation (23) is

$$(\rho + \eta) V = u(c) + (wz + (r - \gamma) a - c) \frac{\partial V}{\partial a} + \theta(\hat{z} - z) \frac{\partial V}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial z^2},$$

where

$$c = (u')^{-1} \left( \frac{\partial V}{\partial a} \right),$$

$$\hat{\rho} = \rho - (1 - \chi) \gamma$$

and $u(c) = \frac{c^{1-\lambda}}{1-\lambda}$ in the competitive equilibrium or $u(c) = \frac{c^{1-\lambda}}{1-\lambda} + \lambda (a - k)$ in the

---

26 We do not provide any proof of convergence of our numerical algorithm.

27 Notice that subindexes $i$ and $j$ have a different meaning here than in the main text.
planning economy. The HJB equation is approximated by an upwind scheme

\[
\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta} + \hat{p}V_{i,j}^{n+1} = u(c_{i,j}^n) + \partial_{a,F}V_{i,j}^{n+1}s_{i,j,F}^{n}1_{s_{i,j,F}^{n}>0} + \partial_{a,B}V_{i,j}^{n+1}s_{i,j,B}^{n}1_{s_{i,j,B}^{n}<0} + \theta(\hat{z} - z_j)\partial_z V_{i,j}^{n+1} + \frac{\sigma^2 z_j^2}{2}\partial_{zz} V_{i,j}^{n+1} - \eta V_{i,j}^{n+1},
\]

where

\[
s_{i,j,F}^{n} = wz_j + (r - \gamma) a_i - (u')^{-1}(\partial_{a,F} V_{i,j}^n),
\]

\[
s_{i,j,B}^{n} = wz_j + (r - \gamma) a_i - (u')^{-1}(\partial_{a,B} V_{i,j}^n).
\]

Moving all variables with \(n+1\) superscripts to the left hand side and those with \(n\) superscripts to the right hand side:

\[
\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta} + \hat{p}V_{i,j}^{n+1} = u(c_{i,j}^n) + V_{i-1,j}^{n+1} \xi_{i,j} + V_{i,j+1}^{n+1} \beta_{i,j} + V_{i,j}^{n+1} \beta_{i,j} + V_{i,j-1}^{n+1} \chi_{i,j} + V_{i,j}^{n+1} \eta_{i,j},
\]

where

\[
c_{i,j}^n = (u')^{-1}(\partial_{a,F} V_{i,j}^n)1_{s_{i,j,F}^{n}>0} + \partial_{a,B} V_{i,j}^{n+1}1_{s_{i,j,B}^{n}<0} + u'(wz_j + ra_i)1_{s_{i,j,F}^{n}<0,s_{i,j,B}^{n}>0},
\]

\[
\xi_{i,j} = -\frac{s_{i,j,F}^{n}1_{s_{i,j,F}^{n}>0}}{\Delta a},
\]

\[
\beta_{i,j} = -\frac{s_{i,j,B}^{n}1_{s_{i,j,B}^{n}>0} + s_{i,j,B}^{n}1_{s_{i,j,B}^{n}<0}}{\Delta a} - \frac{\theta(\hat{z} - z_j)}{\Delta z} - \frac{\sigma^2}{(\Delta z)^2} - \eta,
\]

\[
\chi_{i,j} = \frac{s_{i,j,F}^{n}1_{s_{i,j,F}^{n}>0}}{\Delta a},
\]

\[
\xi = \frac{\sigma^2}{2(\Delta z)^2},
\]

\[
\xi_j = \frac{\sigma^2}{2(\Delta z)^2} + \frac{\theta(\hat{z} - z_j)}{\Delta z}.
\]

The state constraint (5) \(a \geq -\phi\) is enforced by setting \(s_{i,j,F}^{n} = 0\). Similarly, as \(a \leq a_{max}\), \(s_{i,j,B}^{n} = 0\). Therefore, the values \(V_{i,j}^{n+1}\) and \(V_{i+1,j}^{n+1}\) are never used. The boundary conditions with respect to \(z\) are

\[
\frac{\partial V(a,z)}{\partial z} = \frac{\partial V(a,\hat{z})}{\partial z} = 0,
\]
as the process is reflected. At the boundaries in the \( j \) dimension, equation (53) becomes

\[
\begin{align*}
\frac{V_{n+1}^{i,j} - V_{n}^{i,j}}{\Delta} + \delta V_{n}^{i,j} &= u(c_{i}^{n}) + V_{i-1,j}^{n+1} \theta_{i,1} + V_{i,j}^{n+1} (\beta_{i,1} + \xi) + V_{i+1,j}^{n+1} \chi_{i,1} + V_{i,j}^{n+1} \zeta_{1}, \\
\frac{V_{n+1}^{i,j} - V_{n}^{i,j}}{\Delta} + \delta V_{n}^{i,j} &= u(c_{j}^{n}) + V_{i,j-1}^{n+1} \theta_{i,j} + V_{i,j}^{n+1} (\beta_{i,j} + \xi_{j}) + V_{i,j+1}^{n+1} \chi_{i,j} + V_{i,j}^{n+1} \zeta_{j}.
\end{align*}
\]

Equation (53) is a system of \( I \times J \) linear equations which can be written in matrix notation as:

\[
\frac{V_{n+1}^{n} - V_{n}^{n}}{\Delta} + \delta V_{n}^{n+1} = u^{n} + A^{n} V_{n}^{n+1},
\]

where the matrix \( A^{n} \) and the vectors \( V_{n}^{n+1} \) and \( u^{n} \) are defined by:

\[
A^{n} = \begin{bmatrix}
\beta_{1,1} + \xi & \chi_{1,1} & 0 & \cdots & 0 & \zeta_{1} & 0 & 0 & \cdots & 0 \\
\theta_{2,1} & \beta_{2,1} + \xi & \chi_{2,1} & 0 & \cdots & 0 & \zeta_{1} & 0 & \cdots & 0 \\
0 & \theta_{3,1} & \beta_{3,1} + \xi & \chi_{3,1} & 0 & \cdots & 0 & \zeta_{1} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \theta_{I,1} & \beta_{I,1} + \xi & \chi_{I,1} & 0 & 0 & \cdots & 0 \\
\xi & 0 & \cdots & 0 & \theta_{1,2} & \beta_{1,2} & \chi_{1,2} & 0 & \cdots & 0 \\
0 & \xi & \cdots & 0 & \theta_{2,2} & \beta_{2,2} & \chi_{2,2} & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \cdots & 0 & \theta_{I-1,J} & \beta_{I-1,J} + \zeta_{J} & \chi_{I-1,J} \\
0 & 0 & 0 & \cdots & 0 & 0 & \theta_{I,J} & \beta_{I,J} + \zeta_{J}
\end{bmatrix},
\]

\[
V_{n}^{n+1} = \begin{bmatrix}
V_{1,1}^{n+1} \\
V_{2,1}^{n+1} \\
\vdots \\
V_{1,2}^{n+1} \\
V_{2,2}^{n+1} \\
\vdots \\
V_{I-1,J}^{n+1} \\
V_{I,J}^{n+1}
\end{bmatrix}, \quad u^{n} = \begin{bmatrix}
u(c_{1,1}^{n}) \\
u(c_{2,1}^{n}) \\
\vdots \\
u(c_{1,2}^{n}) \\
u(c_{2,2}^{n}) \\
\vdots \\
u(c_{I-1,J}^{n}) \\
u(c_{I,J}^{n})
\end{bmatrix}.
\]

The system can in turn be written as

\[
B^{n} V_{n}^{n+1} = d^{n},
\]

where \( B^{n} = \left( \frac{1}{\Delta} + \delta \right) I - A^{n} \) and \( d^{n} = u^{n} + \frac{V_{n}^{n}}{\Delta} \). \( I \) is the identity matrix. Matrix \( B^{n} \) is a sparse
matrix, and the system (55) can be efficiently solved in Matlab.

The algorithm to solve the HJB equation runs as follows. Begin with an initial guess $V_{i,j}^0 = u(ra_i + wz_j)/\rho$, set $n = 0$. Then:

1. Compute $\partial_a F V_{i,j}^n, \partial_a B V_{i,j}^n, \partial_z V_{i,j}^n$ and $\partial_{zz} V_{i,j}^n$ using (49)-(52).
2. Compute $c_{i,j}^n$ using (54).
3. Find $V_{i,j}^{n+1}$ solving the linear system of equations (55).
4. If $V_{i,j}^{n+1}$ is close enough to $V_{i,j}^n$, stop. If not set $n := n + 1$ and go to step 1.

**Step 2: Solution to the Kolmogorov Forward equation**

The KF equation is also solved using an upwind finite difference scheme. The equation (8) in this case is

$$
0 = -\frac{\partial}{\partial a} [(wz + (r - \gamma) a - c) g] - \frac{\partial}{\partial z} [\theta(z - z)g] + \frac{1}{2} \frac{\partial^2}{\partial z^2} \sigma^2 g - \eta_0 + \eta_0,
$$

or equivalently

$$g_{i-1,j} = g_{i+1,j} g_{i+1,j} + g_{i,j} \beta_{i,j} + g_{i,j} + \xi_j = -\eta_0,
$$

then (58) is also a system of $I \times J$ linear equations which can be written in matrix notation as:

$$A^T g = h,
$$

where $A^T$ is the transpose of $A = \lim_{n \to \infty} A^n$ and $h$ is a vector of zeros with a $-1$ at the first position. We solve the system (59) and obtain a solution $g$. Then we renormalize as

$$g_{i,j} = \frac{g_{i,j}}{\sum_{i=1}^I \sum_{j=1}^J g_{i,j} \Delta a \Delta z}.$$
Step 3: Finding the equilibrium aggregate capital

In order to find the aggregate capital $k$, we employ a relaxation method. Given $\theta \in (0,1)$, begin with an initial guess of the aggregate capital $k^0$, set $n = 0$. Then:

1. Compute $r^n$ and $w^n$ as a function of $k^n$.
2. Given $r^n$ and $w^n$, solve the planner’s HJB equation as in Step 1 to obtain an estimate of the value function $V^n$ and of the consumption $c^n$.
3. Given $c^n$, solve the KF equation as in Step 2 and compute the aggregate distribution $g^n$.
4. Compute the aggregate capital stock $S = \sum_{i=1}^{I} \sum_{j=1}^{J} a_i g_{i,j} \Delta a \Delta z$.
5. Compute $k^{n+1} = \theta S^n + (1 - \theta) k^n$. If $k^{n+1}$ is close enough to $k^n$, stop. If not set $n := n + 1$ and go to step 1.

Step 4: Finding the Lagrange multiplier (only in the optimal allocation)

In order to find the value of the optimal Lagrange multiplier in the planning problem (38), we begin with an initial guess $\lambda^0 = 0$, then we need to find the value of $\lambda$ that satisfies

$$\lambda = \frac{\alpha (1 - \alpha)}{k^{2-\alpha}} \sum_{i=1}^{I} \sum_{j=1}^{J} \left[ g_{i,j} + a_i \frac{g_{i+1,j} - g_{i,j}}{\Delta a} - k z_j \frac{g_{i+1,j} - g_{i,j}}{\Delta a} \right] V_{i,j} \Delta a \Delta z,$$

where $g_{i,j}$, $k$, and $V_{i,j}$ are obtained by solving the planner’s problem with this value of $\lambda$ and utility $u(c) = \frac{c^{1-\chi}}{1-\chi} + \lambda (a - k).$²⁸ We employ Matlab’s routine fzero to find this value of $\lambda$.

²⁸ $V$ is the value function of the planner in this case, that we denote as $j(a,z)$ in the main text.