Dynamic Debt Maturity

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Abstract

We study a dynamic setting where the firm chooses its debt maturity structure endogenously over time in response to observable firm fundamentals. In our model, the firm has to keep its promised outstanding bond face-values constant, but can control the firm’s maturity structure via the fraction of newly issued short-term bonds when refinancing its matured bonds. As a baseline, we show that when the firm’s cash-flows are constant then it is impossible to have the “shortening-to-death” equilibrium where the firm keeps issuing short-term bonds and default consequently. Instead, when the cash-flows deteriorate over time so that the debt recovery value is affected by the endogenous default timing, then the shortening equilibrium emerges. We give conditions under which the shortening equilibrium is inefficient relative to the equilibrium with issuing long-term bonds always.

Keywords: Maturity Structure, Dynamic Structural Models, Endogenous Default, Debt Rollover.

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1 Introduction

The 2007/08 financial crisis has put debt maturity structure of financial institutions squarely in the focus in policy discussions as well as in popular press. However, dynamic models of debt maturity structure are difficult to analyze, and hence academics are lagging behind in offering some tractable framework where the debt maturity structure follows some endogenous dynamics. In fact, the widely used framework for analyzing debt maturity structure is based on Leland (1994b, 1998) and Leland and Toft (1996) who, for tractability, take the frequency of refinancing/rollover as a fixed parameter. In that framework, equity holders are essentially committing to a policy of constant debt maturity structure until default. This stringent assumption is at odds with mounting empirical evidence that firms are having time-varying debt maturity structure.

This paper develops a new model to relax this constant debt maturity structure assumption, which allows us to rigorously analyze how equity holders adjust the firm’s debt maturity structure facing time-varying firm fundamentals, and importantly, how debt holders respond in equilibrium. In our model the firm has two kinds of debt, long and short term debt, that mature with constant but different intensities. To focus on debt maturity only, we assume that the firm can commit to a constant policy of an aggregate face-value for outstanding debt. Equity holders are unable to change the maturity terms of already issued debt. Instead, they control the firm’s debt maturity structure by changing the maturity composition of currently issued debt, i.e., the reissuance of just-matured debt. When equity holders replace the just-matured long-term debt by currently-issued short-term debt, the firm’s debt maturity structure shortens.

What is the trade-off involved when shortening the maturity structure? Going shorter increases the mass of debt to be rolled over in the future, and thus increases the exposure of equity holders to rollover risk, an effect that first emerged in a variant of the classic Leland model that involved finite maturity debt. As is standard, rollover risk may trigger early default from equity holders, and it is the debt holders who are bearing those bankruptcy losses. However, in the presence of default, going short offers higher issuance proceeds as short-term debt fetches higher valuations as it has
a higher likelihood of maturing before the default event. This higher issuance price of short-term debt dilutes long-term debt holders, as it increases the firm’s rollover risk after the long-term debt holders have committed. Of course, in equilibrium long-term debt holders take into account that equity holders may go short in the future and thus require an appropriate rate of return. The key research question is: can situations arise in which, even though going shorter maturity hurts the social value as it hastens default, in equilibrium equity holders go short because they are unable to commit?

We show two main results. First, as a benchmark, we show that when the firm’s current cash flows remain constant, there is never any slow drift towards inefficient default via shortening the firm’s maturity structure. This result can even be generalized to the setting where cash-flows are subject to only Poisson jumps: either the firm defaults immediately after negative Poisson shocks, or the firm lengthens its debt maturity structure by issuing long-term debt and thus never defaults. In this regard, the model is in direct contrast to Brunnermeier and Oehmke (2013) in which the firm can privately renegotiate the maturity terms as well as the face value of debt with each counterparty.

Second, we show that for firms that are “sinking ships”, i.e., when cash flows are deteriorating over time, it is possible to construct an equilibrium where equity holders shorten the firm’s debt maturity structure and the firm drifts slowly towards inefficient early default. There is a crucial difference between the setting with deteriorating cash flows versus the one with constant cash flows. For firms whose cash flows are deteriorating over time, all else equal debt values may be higher under an earlier inefficient default because of a higher debt recovery value. This force, which is absent in the setting with constant cash flows, entices equity holders to shorten the firm’s debt maturity structure ex post, although committing to long debt maturity ex ante maximizes the total welfare.

We construct a stylized model that highlights the endogenous dynamic maturity choice, and analyzing the situations in which equity holders go short, hastening default. The model is build on several simplifying assumptions that allows us to make sharp analytical statement; for instance, we rule out Brownian cash-flow shocks in our analysis. More importantly, we place two important
restrictions on the firm’s strategy space. First, the firm controls its maturity structure only through changing the maturity composition of currently issued debt, which is the reissuance of just-matured debt; it is unable to change the maturity terms of already issued debt. Second, the firm cannot change the aggregate amount of face-value outstanding, which rules out explicit dilution of existing bond holders by promising higher face value to new incoming bond holders. It is the second assumption that drives the difference between our paper and Brunnermeier and Oehmke (2013) who show in certain scenarios the firm always wants to privately renegotiate the bond maturity down (toward zero) with each individual bond investors.

Literature review, to be added.

2 The Setting

2.1 Firm and Asset

The economy is risk-neutral with a constant discount rate $r$. The firm has assets-in-place which produce cash flows at the rate of $y_t$, whose evolution will be specified later. There will be a Poisson event arriving with a constant intensity $\zeta$ so that the assets-in-place pay off a sufficiently large constant $X > 0$, at which point the bond holders and equity holders are paid off by $D^{rf}$ and $E^{rf} \equiv X - D^{rf} > 0$, respectively. This event can also be interpreted as the realization of growth options, and throughout we call it “upside event.”

Given the process $y_t$ to be specified in details below, the unlettered firm value, which is also called asset value, is given by

$$A(y) = E \left[ \int_0^T e^{-rt} y_t dt + 1_{\{T_{\zeta} < T_a\}} e^{-rT_{\zeta}} X \right],$$

(1)

where $1_{\{F\}}$ is the indicator function for event $F$ throughout the paper. Because $y_t$ can take negative values, the optimal project life $T$ in (1) incorporates the real option of terminating the asset, i.e., $T = \min(T_a, T_{\zeta})$ where $T_a$ is the optimal abandonment time and $T_{\zeta}$ is the Poisson event with
intensity $\zeta$.

Throughout we focus on the case where default is socially inefficient. When equity holders default, debt holders take over the firm with some exogenous bankruptcy cost (to be specified later), so that total bankruptcy value is $B(y)$ satisfying $B(y) < A(y)$. We also require that $B'(y) > 0$, i.e., the recovery value of debt holders is increasing in the project’s cash-flows.

### 2.2 Dynamic Maturity Structure and Debt Rollover

The firm is debt and equity financed, with debt being bonds of a certain face-value in place. Our focus will be on the dynamic choice of the maturity of those bonds. To this end, following Leland (1994b, 1998) we assume the firm has two kind of bonds outstanding: long-term bonds whose time-to-maturity follows an exponential distribution with mean $1/\delta_L$, and short-term bonds whose time-to-maturity follows an exponential distribution with mean $1/\delta_S$, where $\delta_S > \delta_L$. Another equivalent interpretation is that bonds are matured with Poisson intensity $\delta_i$, with $i \in (S, L)$.

Maturity is the only characteristic that differs across these two bonds. To highlight the mechanism, in the main analysis both bonds have the same coupon rate, same principal, and same seniority during bankruptcy. To avoid arbitrary valuation difference between long-term and short-term bonds, we assume bonds are with coupon rate $c = r$, so that without default both bonds have a unit price, i.e., $D_L^{r_f} = D_S^{r_f} = 1$. Once bankruptcy occurs, both bond holders receive, per unit of face-value, $B(y)$ as the asset’s liquidation value in bankruptcy. The liquidation value $B(y)$, which is strictly dominated by the asset value $A(y)$, can be endogenized by inefficient management by bond holders as considered later. Throughout we assume that at default bond investors suffer a loss, i.e.

$$B(y) < D^{r_f} = 1. \tag{2}$$

This condition plays a crucial role in later analysis. We will also discuss the role of the equal seniority assumption later.

To focus on maturity only, throughout the analysis we assume that the firm rolls over its bonds
in a way so that the total face-value of these two bonds is a constant, which is normalized to 1. The constant promised face value is a canonical assumption in the Leland setting (Leland and Toft, 1996; Leland, 1998), and can be justified by bond covenants which aim to alleviate the dilution effect caused by future net debt issuance. This is an important assumption, because our results will be in sharp contrast to Brunnermeier and Ohmke (2013) who allow for private negotiation of individual bond face value.

At any point of time, the total face value of short-term bonds, denoted by $\phi_t \in [0, 1]$, gives the fraction of short-term bonds outstanding. We call $\phi_t$ the current maturity structure of the firm. Given the current maturity structure $\phi_t$, during $[t, t + dt]$ there are $m(\phi_t) dt$ dollars of bonds maturing, where

$$m(\phi_t) \equiv \phi_t \delta_S + (1 - \phi_t) \delta_L. \quad (3)$$

The more short-term the maturity structure is, the more debt is rolled over each instant, as we have

$$m'(\phi) = \delta_S - \delta_L > 0.$$

Equity holders only adjust the debt maturity structure of the firm through the issuance policy when rolling over the maturing bonds. More precisely, let $f_t \in [0, 1]$ be the proportion of short-term debt of newly issued bonds which replace maturing bonds. The dynamics of maturity structure $\phi_t$ are then given by:

$$\frac{d\phi_t}{dt} = -\phi_t \delta_S + \underbrace{m(\phi_t) f_t}_{\text{Short-term maturing}} \quad \underbrace{m(\phi_t) f_t}_{\text{Newly issued short-term}}. \quad (4)$$

Conditional on the new issuance policy $f_t$, the drift of maturity structure $\phi$, which we denote by $\mu_\phi(\phi_t | f_t)$, is thus

$$\mu_\phi(\phi_t | f_t) \equiv -\phi_t \delta_S + \left[\phi_t \delta_S + (1 - \phi_t) \delta_L\right] f_t. \quad (5)$$

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1We assume that there is no debt buybacks, call provisions do not exist, and maturity of debt contracts cannot be changed once issued. We discuss the robustness of our result with respect to this assumption in Section 3.4.4.
One particular issuance policy that is relevant to our later analysis is a constant policy, i.e., \( f_t = \tilde{f} \). Given this constant issuance policy, we can solve equation (4) with any initial value \( \phi_0 \):

\[
\phi_t = \left[ \phi_0 - \phi_{ss}(\tilde{f}) \right] e^{-\left[\delta_S-(\delta_S-\delta_L)\tilde{f}\right]t} + \phi_{ss}(\tilde{f}),
\]

where \( \phi_{ss}(\tilde{f}) \equiv \frac{\delta_L \tilde{f}}{\delta_S-(\delta_S-\delta_L)\tilde{f}} \) is the steady-state maturity structure with \( \phi_{ss}(0) = 0 \) and \( \phi_{ss}(1) = 1 \).

Intuitively, if the firm always (never) issues short-term bonds so that \( \tilde{f} = 1 \) (\( \tilde{f} = 0 \)), then the highest (lowest) maturity structure that can go is 1 (0). Further, given the shortening policy \( f = 1 \), then for any initial \( \phi_0 \in (0,1) \) we have \( \phi_t \) drifting monotonically towards \( \phi_{ss}(\tilde{f} = 1) = 1 \), i.e. \( \mu_\phi(\phi|f=1) > 0 \). Under this policy, the firm keeps issuing short-term bonds and consequently its maturity structure increases over time toward 1.

### 2.3 Rollover Losses and Default

Given equilibrium default policy \( T_b \) (if \( T_b = \infty \) then default is off equilibrium), competitive bond investors price long-term and short-term bonds at \( D_S(y_t, \phi_t) \) and \( D_L(y_t, \phi_t) \) respectively.

#### 2.3.1 Rollover losses

In the traditional Leland setting (1994b, 1998), equity holders refinance, or roll over, the firm’s maturing bonds by issuing the same bonds. In our model, the firm can choose the fraction of short-term bonds in their newly issued debt. This implies that per dollar of face value paid out to maturing bond holders, the net refinancing cash-flows for equity holders who are issuing \( f_t \) fraction of short-term bonds are

\[
f_t D_S(y_t, \phi_t) + (1 - f_t) D_L(y_t, \phi_t) - 1.
\]

This term is so-called “rollover losses.”\(^2\) Each instant there are \( m(\phi_t) dt \) units of face value to be rolled over, hence conditional on no upside event the instantaneous cash flows to equity holders are

\(^2\)Since we set the coupon to be the discount rate \( c = r \), bond prices are always below 1 and hence this term is always negative, i.e., equity holders are always facing rollover losses. When \( c > r \), rollover gains occur for safe firms who are far from default. As emphasized in He and Xiong (2012), since rollover risk kicks in only when the firm is close to default, it is without loss of generality to focus on rollover loss only.
(recall $E^{rf} = X - D^{rf} = X - 1 > 0$ is the equity’s payoff in upside event)

$$y_t - c_{	ext{coupon}} + \zeta E^{rf} + m(\phi_t) [f_t D_S(y_t, \phi_t) + (1 - f_t) D_L(y_t, \phi_t) - 1]. \quad (6)$$

Equity holders absorb the operating gains/losses together with rollover losses to keep the firm “alive,” until they default and debt holders take over then firm.\(^3\) More specifically, the above cash flows in (6), when negative, are covered by the firm through issuing additional equity, which dilutes the value of existing shares. Equity holders are willing to buy more shares and bail out the maturing debt holders as long as the equity value is still positive (i.e. the option value of keeping the firm alive justifies absorbing these losses). When equity holders—protected by limited liability—declare default, equity value drops to zero, and creditors receive the firm’s liquidation value $B(y_t)$.

There are two distinct mechanisms that expose equity holders to greater cash flow losses, hence leading to default. The first channel through the exogenous cash flow $y_t$ has been explored in the literature. When $y_t$ deteriorates (say, $y_t$ turns negative), equity holders are absorbing operating losses (the first term in (6)). Also, a lower $y_t$ also indicates default is more imminent (say, equity holders default when $y_t$ hits some lower boundary), then bond prices $D_S$ and $D_L$ drop as well, leading to a greater rollover losses in the third term in (6).

The second channel through endogenous maturity structure $\phi_t$ is novel. A greater $\phi_t$ implies a higher rollover frequency $m(\phi_t)$ and thus more severe rollover losses in the third term in (6). This suggests that equity holders default when $\phi_t$ reaches some upper threshold, all else equal. More importantly, equity holders are controlling $f_t$ in the third term in (6), which affects the future maturity structure $\{\phi_s : s > t\}$ via equation (4).

\(^3\)This assumption highlights the so-called “endogenous” default in that equity defaults when they decided not to absorb the loss, rather than cannot absorb the loss. The underlying assumption is that either equity holders have a deep pocket or the firm faces a frictionless equity market.
2.3.2 Default boundary

As we will verify later, the above discussion suggests that there exists a default curve \((\Phi(y), y)\), where \(\Phi(\cdot)\) is an increasing function and gives the threshold fraction of short-term bonds given the current cash flow \(y\). In equilibrium, the firm defaults whenever the state lies to the north-west side of \((y, \Phi(y))\), i.e., the default states are

\[
\mathcal{B} = \{ (\phi, y) \text{ such that } \phi \geq \Phi(y) \}.
\]

Consistent with this observation, throughout we make the following assumption on off-equilibrium beliefs. When the firm stays alive at time \(t\) even though creditors expect it to be in default, new bond investors expect the firm to default as long as the \((y_s, \phi_s) \in \mathcal{B}\) for \(s > t\). This implies that if the next instant \((y_{t+dt}, \phi_{t+dt}) \in \mathcal{B}\), either because cash flow \(y_t\) is decreasing over time or the firm keeps issuing short-term debt so that \(\phi_{t+dt} > \phi_t\), then bond investors apply the lowest possible bond value of \(D_L = D_S = B(y_{t+dt})\).

2.3.3 Time-to-default \(\tau\)

For some of later analysis, illustration is more straightforward based on the dynamics of the firm’s time-to-default \(\tau\). It also implies a change of state space from \((\phi, y)\) to \((\tau, y_0)\), and along the equilibrium path it is simply a change of coordinates: \(\tau \equiv T_b - t\) is “reversed time”, i.e., time-to-default — recall \(T_b\) is the default time of the firm — and \(y_0\) is the cash-flow at the time of default. Thus, \(d\tau = -dt\) and we define \(\phi_\tau\) as the maturity structure with \(\tau\) periods left until default. Then, \(\phi_0\) is the maturity structure at default, and we have

\[
d\phi_\tau = -\mu_\phi(\phi|f) \, d\tau.
\]
3 Baseline Model: Constant Cash-Flows

We study the model with constant cash flows in this section, and prove a negative result: There does not exist an equilibrium path in which equity holders keep shortening the firm’s debt maturity structure and eventually defaults in the face of larger and larger rollover losses. This surprising result is in sharp contrast to Brunnermeier and Ohmke who show that contracting externality in a maturity rat race drives equity holders to shorten its debt maturity toward zero.

3.1 Setting and Preliminary Analysis

Consider a simple setup in which a company receives a constant cash-flow $y$, i.e., $y_\tau = y$; recall $\tau$ is the “reverse” time to default. As we will have time-varying cash flow rate $y_\tau$ later, in this section with constant $y$ we still explicitly write $y$ into the bond and equity valuations to emphasize their dependence on the firm cash flow $y$. More specifically, we use $D_S(\phi_\tau; y)$, $D_L(\phi_\tau; y)$ and $E(\phi_\tau; y)$ to denote the short-term bond, long-term bond, and equity value, respectively.

Given maturity structure $\phi_\tau$ and issuance policy, equity holders’ cash flows are given by

$$y - c + \zeta E^{rf} + m(\phi_\tau) \left[ f_\tau D_S(\phi_\tau; y) + (1 - f_\tau) D_L(\phi_\tau; y) - 1 \right].$$

(7)

Suppose our state is initialized at $(y, \phi)$. Can immediate or eventual default arise? We have the following lemma which characterizes two polar cases.

**Lemma 1** Default occurs immediately if (7) is always negative, which is implied by

$$y - c + \zeta E^{rf} < 0.$$  

On the other hand, equity never defaults if (7) is always positive, which holds when

$$y - c + \zeta E^{rf} + \delta S [D_b(y) - 1] \geq 0.$$
Proof. We use the fact that $B(y) \leq D_S \leq D^{rf} = 1$ and $B(y) \leq D_L \leq D^{rf} = 1$ to bound the rollover term in (7):

$$0 \geq m(\phi_r) [f_r D_S(\phi_r; y) + (1 - f_r) D_L(\phi_r; y) - 1]$$
$$\geq m(\phi_r) [f_r B(y) + (1 - f_r) B(y) - 1] = m(\phi_r) [B(y) - 1]$$
$$\geq \delta_S [B(y) - 1],$$

where we use as $\delta_S = \max_{\phi \in [0,1]} m(\phi)$. Hence if $y - c + \zeta E^{rf} < 0$ then the cash flows to equity are always negative, leading to immediate default. On the other hand, if $y - c + \zeta E^{rf} + \delta_S [B(y) - 1] > 0$, then even under the most pessimistic beliefs equity holders never make losses and thus never default.

Q.E.D. ■

Lemma 1 implies that the set on which dominance does not rule out multiple equilibria is given by

$$\mathcal{M} = \left\{ y \in \mathbb{R} : 0 \leq y - c < \zeta E^{rf} < \delta_S [1 - B(y)] \right\}$$

On $\mathcal{M}$, if creditors expect default, then indeed equity defaults, whereas if creditors expect bonds to be risk-free, then equity will not default.\(^4\)

3.2 “Shortening to Death” Equilibrium

On the state space $\mathcal{M}$, we are interested in the following conjectured “shortening to death” equilibrium. More specifically, do there exist equilibria in which equity holders setting $f = 1$ (i.e., issuing short-term debt) from then on, so that $\phi$ increases over time and the firm eventually defaults in the

\(^4\)We need more arguments to use the current rollover flow $m(\phi)$ such that the set becomes the two-dimensional set

$$\mathcal{M} = \left\{ (y, \phi) \in \mathbb{R} \times [0,1] : m(\phi) \left[ 1 - D_L^f \right] < y - c < m(\phi) \left[ 1 - D_h(y) \right] \right\}$$

On $\mathcal{M}$, if creditors expect default, then indeed equity defaults, whereas if creditors expect bonds to be risk-free, then equity will not default. For $D_L^f < 1$, we can equivalently write this is

$$\mathcal{M} = \left\{ (y, \phi) \in \mathbb{R} \times [0,1] : \frac{y - c}{1 - D_h(y)} < m(\phi) < \frac{y - c}{1 - D_S^f} \right\}$$

Note that $\phi \in [0,1]$ implies that $m(\phi) \in [\delta_L, \delta_S]$. For $f_L > 0$ and $f_H < 1$, the restriction leads to $\phi \in [\phi_H, \phi_L]$ with less easily interpretable rollover frequency.

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face of larger and larger rollover losses?

3.2.1 Debt Valuations

Under our assumption of “shortening to death”, bond holders who are taking equity holders’ policy $f = 1$ as given. Thus, the share of short-term debt has an evolution

$$\frac{d\phi}{d\tau} = -\mu_{\phi}(\phi|0) = -(1 - \phi_t)\delta_L, \quad (8)$$

which is (4) when $f = 1$.

It turns out that it is more transparent to keep track of the maturity structure itself $\phi$ as the state variable in the constant cash-flow case. Over $[t, t + dt]$ the maturity structure $\phi$ changes by

$$\frac{d\phi}{dt} = -\frac{d\phi}{d\tau} = (1 - \phi_t)\delta_L.$$

Hence, for bond $i \in \{S, L\}$ with maturing intensity $\delta_i$, the valuation equation for any bond follows

$$rD_i(\phi; y) = c + \delta_i[1 - D_i(\phi; y)] + \zeta[1 - D_L(\phi; y)] + (1 - \phi)\delta_L D_L'(\phi; y) \quad (9)$$

with the boundary condition

$$D(\Phi(y); y) = B(y). \quad (10)$$

Note in (9) bond holders get paid $D^{r,f} = 1$ in both the bond maturing event (occurring with intensity $\delta$) and upside option event (occurring with intensity $\zeta$).

Our later analysis requires some knowledge of the price wedge $\Delta(\phi)$ between short-term bonds and long-term bonds, which is defined as:

$$\Delta(\phi; y) \equiv D_S(\phi; y) - D_L(\phi; y).$$
Applying $\delta_S$ and $\delta_L$ to (9) and taking difference, we obtain

$$(r + \delta_L + \zeta) \Delta (\phi) = (\delta_S - \delta_L) [1 - D_S (\phi)] + (1 - \phi) \delta_L \Delta' (\phi)$$

(11)

with boundary condition

$$\Delta (\Phi (y); y) = 0.$$

(12)

In the presence of default, we have $1 - D (\phi; y) > 0$, and one can easily check that

$$\Delta (\phi) > 0 \text{ for } \phi < \Phi (y).$$

(13)

This is an important result, as it says that as long as default is possible, short-term bonds are always more valuable than long-term bonds before default $t < T_b$. This gives an advantage of short-term bonds if equity holders simply want to minimize the rollover losses. Intuitively, short-term bonds have higher value than long-term bonds because short-term bonds are paid back sooner. In this model, even though the time to default is (outside of sunspot equilibria) deterministic, the maturity of the bonds is not, and thus there is a pricing differential between $D_S$ and $D_L$.

### 3.2.2 Equity valuation and optimal issuance policy

Equity holders are not only minimizing the firm’s rollover losses; they also take into account any long-run effect that are brought on by short-term bonds. More specifically, although issuing short-term bonds lowers today’s rollover losses, but it shortens the firm’s future maturity structure and hence hurts the firm’s future rollover losses. We now show that the negative impact on future rollover losses always dominate the gain from today.

Formally, equity holders are controlling the firm’s dynamic maturity structure as in (4). The standard Hamilton-Jacobi-Bellman (HJB) equation for equity, with optimal control in $f$, can be
written as

\[
\underbrace{rE(\phi; y)}_{\text{required return}} = y - c + \zeta \left[ E^{rf} - E(\phi; y) \right] + \underbrace{E'f - E(\phi; y)}_{\text{upside event}} + m(\phi) \left\{ \frac{fD_S(\phi; y)+(1-f)D_L(\phi; y)-1}{\text{rollover losses}} \right\} + \left[ -\phi\delta_S + m(\phi) f \right] E'(\phi; y) \quad (14)
\]

In (14), in choosing \( f \), which is the fraction of newly issued short-term bonds, equity holders are minimizing “today’s rollover losses,” but taking into account of the “impact of maturity shortening” on the future equity value (potentially caused by future rollover losses).

Due to linearity, the optimal issuance policy \( f^* \) admits a corner solution and is given by

\[
f^* = \begin{cases} 
1 & \text{if } \Delta(\phi; y) + E'(\phi; y) > 0 \\
0 & \text{if } \Delta(\phi; y) + E'(\phi; y) < 0 \\
[0, 1] & \text{if } \Delta(\phi; y) + E'(\phi; y) = 0 
\end{cases} \quad (15)
\]

The optimal issuance policy in (15) illustrates the trade-off faced by equity holders, and later we call \( \Delta(\phi; y) + E'(\phi; y) > 0 \) incentive compatibility condition for equity. Issuing more short-term bonds lowers the firm’s rollover losses at today, because they have higher market prices than long-term bonds, i.e., \( \Delta(\phi; y) > 0 \) in (13). However, issuing more short-term bonds today (higher \( f \)) pushes the firm’s future maturity structure toward short-term (higher \( \phi \)) and thus greater future rollover losses. This hurts equity holders’ continuation value, i.e., \( E'(\phi; y) < 0 \), because default occurs when the firm’s maturity structure \( \phi \) reaches some upper threshold \( \Phi(y) \).
3.2.3 Endogenous default

Equity also endogenously choose when to default. Since we are working with $\phi$ as the state variable, at default boundary $\Phi$ we have

\begin{align*}
E(\Phi; y) &= 0; \\
E'(\Phi; y) &= 0.
\end{align*}

The second "smooth-pasting" condition in (17) reflects the optimality of the default decision. Intuitively, since equity holders can postpone default, the optimal default must occur when the change of equity value (with respect to time) is zero.\footnote{The second order condition $E_{\tau\tau} > 0$ follows from differentiating the ODE w.r.t. $\tau$ which gives}

\begin{align*}
E &> 0 \\
E'( \phi) &= \frac{m'(\phi)}{m(\phi)} \frac{\phi'}{\phi} \frac{[D_b(y_0) - 1]}{\mu \phi(\phi|f) [1 - D_b(y_0)]} + m(\phi) f \frac{\Delta}{\phi} \\
&= (\delta - \delta_L) \phi \delta L [1 - D_b(y_0)]
\end{align*}

which is non-negative.

Based on equity valuation equation (14), conditions (16) and (17) imply that, at the optimal default boundary chosen by equity holders, their instantaneous expected flow payoff (including possible Poisson jumps) equals to zero:

\begin{equation}
y - c + \zeta E^{rf} + \max_{f \in [0,1]} m(\phi) [f D_S(\phi; y) + (1 - f) D_L(\phi; y) - 1] = 0.
\end{equation}

In other words, as there is no Brownian terms, there is no option value outside the Poisson jumps and thus equity holders will default as soon as the flow turns negative.

The conditions in equation (10) and equation (16) allow us to pin down the default boundary $\Phi(y)$. First, because at default bond values $D_S(\phi; y)$ and $D_L(\phi; y)$ are equal to $B(y)$ always irrespective of their maturity, the rollover term in equation (18) equals $m(\phi) [B(y) - 1]$, irrespective of the optimal choice $f$ at default. Recall $m(\phi) = \phi \delta_S + (1 - \phi) \delta_L$ in 3, which implies that the...
The default boundary is defined by

$$\Phi(y) = \frac{1}{\delta_S - \delta_L} \left[ \frac{y - c + \zeta E^*f}{1 - B(y)} - \delta_L \right].$$  

Because the recovery value $B(y)$ is increasing in $y$, one can easily verify that the default boundary $\Phi(y)$ is increasing in $y$, as conjectured in Section 2.3.2, and it is easy to check that $E''(\Phi; y) < 0$.

### 3.3 Impossibility of “Shortening to Death” Equilibrium

The “shortening to death” equilibrium is characterized by a path of $\{\phi \to \Phi\}$ where (15) holds; (14) holds when $f = 1$, i.e.,

$$rE(\phi; y) = y - c + \zeta \left[ E^*f - E(\phi; y) \right] + [\phi\delta_S + (1 - \phi) \delta_L] [D_S(\phi; y) - 1] + (1 - \phi) \delta_L E'(\phi; y)$$  

with boundary conditions $E(\Phi; y) = E'(\Phi; y) = 0$; and bond valuations follow (9) with boundary conditions $D_S(\Phi; y) = D_L(\Phi; y) = B(y)$. Finally, the price wedge $\Delta(\phi; y)$ satisfies (11).

To rule out the shortening-to-death equilibrium, it is sufficient to analyze the equilibrium behavior immediately before default, i.e., $\phi = \Phi - \epsilon$ with sufficiently small $\epsilon > 0$. In light of (15), we need to show that $\Delta(\Phi; y) + E'(\Phi; y) < 0$ there. Since at default we have $\Delta(\Phi; y) = 0$ in (12) and $E'(\Phi; y) = 0$ in (17), the incentive compatibility condition holds exactly at zero. The following lemma goes one order higher to sign the incentive compatibility condition at the vicinity of default boundary $\Phi$.

**Lemma 2** It is never optimal to choose $f = 1$ right before default at $\phi = \Phi - \epsilon$ if

$$\Delta'(\Phi; y) + E''(\Phi; y) > 0.$$  

**Proof.** We have shown that at default point $\Phi$ the incentive compatibility condition (15) just holds with equality. If (21) holds, then $\Delta(\Phi - \epsilon; y) + E'(\Phi - \epsilon; y)$ is strictly negative. According to (15), it is never optimal to choose $f = 1$ right before default at $\phi = \Phi - \epsilon$. Q.E.D. ■
We first analyze the benefit of shortening $\Delta' (\Phi; y)$ in (21). From (11) we know that

$$\Delta' (\Phi; y) = -\frac{(\delta_S - \delta_L) [1 - B (y)]}{(1 - \Phi) \delta_L} < 0,$$

which implies that $\Delta (\Phi - \epsilon; y) > 0$. Intuitively, when we are a bit away from default, short-term bonds have the advantage of maturing before default, leading to a strictly higher value than long-term bonds. This is the benefit of issuing short-term bonds.

Equity holders have to balance this benefit with the cost of more imminent default, which is captured by the second term $E'' (\Phi; y)$ in (21). This term is always positive, given the optimality of equity holders' endogenous default decision. To evaluate this option value at default, we take the derivative with respect to $\phi$ of the equity valuation equation (20):

$$(r + \zeta) E' (\phi; y) = (\delta_S - \delta_L) [D_S (\phi; y) - 1] + \phi \delta_S + (1 - \phi) \delta_L D'_S (\phi; y) - \delta_L E' (\phi; y) + (1 - \phi) \delta_L E'' (\phi; y).$$

Evaluating this equation at the default boundary $\Phi$, together with $E' (\Phi; y) = 0$ and $D_S (\Phi; y) = B (y)$, we have

$$E'' (\Phi) = \frac{(\delta_S - \delta_L) [1 - B (y)]}{(1 - \Phi) \delta_L} - \frac{[\Phi \delta_S + (1 - \Phi) \delta_L]}{(1 - \Phi) \delta_L} D'_S (\Phi; y).$$

(23)

However, the assumption that default leading to value losses for bond holders in (2) imply that $D'_S (\Phi; y) < 0$. In words, the shorter the debt maturity, the closer the default, and hence the lower
the bond value.\footnote{From (9) with $\delta = \delta_S$ we can derive}

\[\Delta'(\Phi; y) + E''(\Phi; y) = -\frac{[\Phi \delta_S + (1 - \Phi) \delta_L]}{(1 - \Phi) \delta_L} D'_S(\Phi; y) > 0.\]

which shows our claim. The following proposition summarizes the main result in this section. For better comparison to later results, we take the negative sign of the incentive compatibility condition.

**Proposition 1** Suppose that $y_t = y$. Right before default, we have

\[\text{sign}(-\Delta'(\Phi; y) - E''(\Phi; y)) = \text{sign}\left(D'_S(\Phi; y)\right) < 0,\]  

which is negative always under the condition that default leads to losses for bond holders. As a result, there does not exist equilibria where equity holders keep issuing short-term bonds and then default at some finite future time.

The result in (24) implies that the incentive compatibility condition for issuing short-term bonds is captured by the valuation impact of raising maturity structure $\phi$ on bond values (i.e., $D'_S(\Phi; y)$), potentially via the default effect. A similar expression as in (24) holds even when we introduce deteriorating cash flow state $y_t$ in Section 4. However, there, the fact that cash flow state $y$ being active has profound implications, which may change the sign of $D'_S(\Phi; y)$ even under inefficient default.

Combining Proposition 1 with Lemma 1, we reach the conclusion that for the case of constant cash-flow, either the firm default immediately, or the firm keeps issuing long-term bonds and never default.
3.4 Robustness of Proposition 1

Before we move on to the next section which shows that “shortening to death” equilibrium does emerge when $y_t$ is slowly deteriorating, we demonstrate that “shortening to death” equilibrium cannot hold even if we introduce Markov switching cash-flows or allow for exogenous default.

3.4.1 Exogenous default boundary

We so far follow the Leland tradition by assuming that equity holders have deep pocket. This implies that the default boundary is determined endogenously when equity’s option value of staying alive is zero, and this so-called endogenous default mechanism implies the smooth-pasting condition $E'(\Phi) = 0$. As a result, at the default boundary the incentive compatibility condition $\Delta (\Phi; y) + E'(\Phi; y) = 0$, and thus we need the help of Lemma 2 by going one order of derivative higher.

Suppose that instead, equity holders are forced to default before they are willing to; this can happen for liquidity reasons if equity holders do not have deep pocket or equity market becomes illiquid due to information-driven problems. Say the default boundary is $\hat{\Phi}$; we must have $E'(\hat{\Phi}) < 0$. It is because equity holders have the option to default always; the fact that they hang on during the process $\phi \uparrow \hat{\Phi}$ implies that $E(\phi) > E(\hat{\Phi})$ for $\phi < \hat{\Phi}$. On the other hand, same seniority implies that a zero debt valuation wedge $\Delta (\hat{\Phi}) = 0$. As a result, $\Delta (\hat{\Phi}) + E'(\hat{\Phi}) < 0$ right before the default, which rules out the possibility of shortening-to-death equilibria.

3.4.2 Exogenous Poisson default event

In the baseline model analyzed above, the only way to generate a positive price wedge between short-term and long-term bonds is the endogenous default taken by equity holders. However, a positive bond price wedge can exist if the firm experiences some exogenous default events. To show that our main theoretical result in Proposition 1 is robust to this possibility, we introduce exogenous default so that the firm might be liquidated at a value of $B(y)$ after some independent Poisson shock with intensity $\xi > 0$. 
The introduction of downward negative liquidation shock also shows that the force that drives our result is not the particular information setting in which “no news is bad news.”

Most of derivation changes slightly; for instance, for bond valuations, in contrast to (9) we have

\[
\text{required return} = \text{coup} + \delta_i [1 - D_i (\phi; y)] + \zeta [1 - D_i (\phi; y)] + \xi [B (y) - D_i (\phi; y)] + (1 - \phi) \delta_L D_i' (\phi; y)
\]

Corollary 1 Even with exogenous liquidation shocks \( \xi > 0 \), there does not exist equilibria in which equity holders keep issuing short-term bonds and then default endogenously at some finite future time.

3.4.3 Markov switching cash-flows

We have so far taken the random upside event exogenously as “the end of dynamic game” at which point bonds are valued in a risk-free fashion and hence maturity structure becomes irrelevant. In other words, the upside event can be interpreted as an upper absorbing state in which the firm never comes back. What if the upside event is not absorbing and the firm may transit back to the normal state? Will our main result in Proposition 1 change if we endogenize the maturity structure decision in the non-absorbing upper state (upside event)?

More specifically, suppose that the firm’s cash-flows follow a Markov switching process \( y_t \in \{y_H, y_L\} \), with switching intensity \( \zeta_{ij} \) from state \( i \) to state \( j \) where \( i, j \in \{H, L\} \). As before, every instant equity holders refinance the firm’s matured short-term and long-term bonds at their respective market prices, and control the fraction of newly issued short-term bonds. The following proposition implies that the result in Proposition 1 is robust to the endogenous debt maturity structure in the upper state.

Corollary 2 The firm may default immediately right after the cash-flow state jumps downward. However, there does not exist equilibria in which, without a state jump, equity holders keep issuing short-term bonds and then default at some finite future time.
3.4.4 What if the firm facing different reissuing strategy space?

The key incentive compatibility condition compares the pricing wedge to the long-run impact of shortening maturity to equity, which should involve the valuation of long-term bonds. Somewhat surprisingly, in Proposition 1 we show that at the default boundary, the comparison boils down to the property of short-term bond valuation only. Partly, this is due to the assumption that in refinancing the firm’s matured bonds, the proportion of short-term bonds of newly issued bonds has to lie within the interval $[0, 1]$; thus in the shortening-to-death equilibrium the firm keeps issuing short-term debt only.

In practice, the assumption of $f \in [0, 1]$ might fail. For instance, firms might be able to repurchase (long-term) bonds, or might face the covenants which restrict the firm to reissue certain long-term bonds at minimum, implying that in the shortening-to-death equilibrium the firm might issue some mixture of short-term and long-term bonds. To accommodate these possibilities, we modify the allowable set for the fraction of newly short-term bonds to be $f \in [f_l, f_h]$ so that in equilibrium the firm takes the highest fraction $f_h$. We still require the firm’s maturity structure is shortening at the hypothetical default point $\Phi$, i.e., using (5) we impose

$$
\mu_\Phi(\Phi) = -\Phi \delta_S + (\Phi \delta_S + (1 - \Phi) \delta_L) f_h > 0.
$$

In Appendix XX we show that our result in Proposition 1 holds in this relaxed setting.

4 Maturity Shortening with Time-Decreasing Cash-Flows

“Shortening to death” equilibrium exists when the firm’s cash-flows are deteriorating slowly over time. We show that multiplicity of equilibria plays a key role in the analysis. Even though lengthening the firm’s debt maturity structure can be the desirable equilibrium from the social perspective, maturity shortening and inefficient early endogenous default might constitute another equilibrium.
4.1 Setting and Valuations

We now introduce a time-dependent cash-flow $y_\tau$ which has some local drift:

$$dy_\tau = \mu_y(y) \, d\tau.$$ 

We focus on the case where $\mu_y(y) < 0$ so that $y_\tau$ is decreasing over time. We further assume that the drift $\mu_y(y)$ is independent of calendar time $t$ and hence time-to-default $\tau = T_b - t$.

4.1.1 Valuations and equity’s incentive compatibility condition

The problem in hand now involves a two-dimensional analysis, with cash-flow state $y_\tau$ and debt maturity $\phi_\tau$ as independent state variables. Bond values solve now the following ODE, where $i \in \{S, L\}$:

$$rD_i(\phi, y) = c + \delta[1 - D_i(\phi, y)] + \zeta[1 - D_i(\phi, y)] + \mu_\phi(\phi|f) \frac{\partial}{\partial \phi} D_i(\phi, y) + \mu_y(y) \frac{\partial}{\partial y} D_i(\phi, y),$$

and equity value solves the following ODE

$$rE(\phi, y) = y - c + \zeta[E_{\text{rf}} - E(\phi, y)] + \mu_y(y) \frac{\partial}{\partial y} E(\phi, y) +$$

$$\max_{f \in [0,1]} \left\{ m(\phi) \left[ f D_S(\phi, y) + (1 - f) D_L(\phi, y) - 1 \right] + \mu_\phi(\phi|f) \frac{\partial}{\partial \phi} E(\phi, y) \right\}$$

The same argument as Section 3.2.2 leads to the same incentive compatibility condition (15) for equity holders, with a necessary modification of partial derivative with respect to $\phi$ due to two-
dimensional states:

\[
f^* = \begin{cases} 
1 & \text{if } E_\phi(\phi, y) + \Delta(\phi, y) > 0 \\
[0, 1] & \text{if } E_\phi(\phi, y) + \Delta(\phi, y) = 0 \\
0 & \text{otherwise}
\end{cases}
\]  

(28)

where throughout we use the notation \( E_\phi(\phi, y) = \frac{\partial}{\partial \phi} E(\phi, y) \).

4.1.2 Time-to-default and valuations

It is useful to note that given initial state the equilibrium path \((\phi_\tau, y_\tau)\) is essentially one-dimensional, which is indexed by the time-to-default \(\tau\). Hence given equilibrium we can rewrite the bond and equity values by \(D_S(\tau; y_0), D_L(\tau; y_0)\), and \(E(\tau; y_0)\) respectively as a function of \(\tau\) only, while essentially treating the defaulting cash-flow state \(y_0\) as a parameter. More specifically, focusing on the shortening-to-death equilibrium with \(f^*_\tau = 1\), we can write the above two PDEs in their ODE forms:

\[
rD_i(\tau, y_0) = c + \delta_i [1 - D_i(\tau, y_0)] + \zeta [1 - D_i(\tau, y_0)] - \frac{\partial}{\partial \tau} D_i(\tau, y_0), \text{ for } i \in \{S, L\}, \tag{29}
\]

\[
rE(\tau, y) = y_\tau - c + \zeta \left[ E^f - E(\tau, y_0) \right] + m(\phi_\tau) \left[ D_S(\tau, y_0) - 1 \right] - \frac{\partial}{\partial \tau} E(\tau, y_0). \tag{30}
\]

For bond values, we have the following simple closed-form solution (recall \(c = r\))

\[
D_i(\tau, y_0) = 1 + e^{-(r+\delta_i+\zeta)\tau} [B(y_0) - 1], \text{ for } i \in \{S, L\}. \tag{31}
\]

We give the solution to equity equation along the equilibrium path in the Appendix.

Finally, one can translate \(D_i(\tau, y_0)\) and \(E(\tau, y_0)\) back to the form of \(D_i(\phi, y)\) and \(E(\phi, y)\) by solving for \(\tau\) as a function of \((\phi, y)\). Of course, the expression of \(\tau\) depends on the functional form assumption of drift \(\mu(y)\). More importantly, the time-to-default \(\tau\) also depends on the endogenous default boundary \(\Phi(y_0)\), to which we turn next.
4.2 Default Boundary

As in the base case, the solution will again involve the endogenous default boundary \( \Phi(y) \). Recall that we denote by \( y_0 \) the cash-flow level at default \( \tau = 0 \); hence the endogenous default boundary in the \((y, \phi)\) space is \( \Phi(y_0) \).

We will in turn analyze the conditions needed for pure strategy (deterministic) equilibria along any possible paths, ignoring the so-called sun-spot equilibria. We will then point out regions of “no return”, in which we can rule out all but one pure-strategy equilibrium.

Similar to the discussion in Section 3.2.3, at the optimal default boundary equity holders’ instantaneous expected flow payoff (including possible Poisson jumps) equals to zero, leading to the same expression of default boundary given in (19):

\[
\Phi(y_0) = \frac{1}{\delta_S - \delta_L} \left[ \frac{y_0 - c + \zeta E^f}{1 - B(y_0)} - \delta_L \right], \text{ with } \Phi'(y_0) > 0.
\]

Because \( \phi \in [0, 1] \), we can derive bounds for the defaulting cash-flow state, i.e., \( y_0 \in [y_0^{\text{min}}, y_0^{\text{max}}] \).

Here, the bounds \( y_0^{\text{min}} \) and \( y_0^{\text{max}} \) are explicitly defined by the following two equations

\[
\frac{y_0^{\text{max}} - (1 - \pi) c + \zeta E^f}{1 - B(y_0^{\text{max}})} = \delta_S \quad \text{and} \quad \frac{y_0^{\text{min}} - (1 - \pi) c + \zeta E^f}{1 - B(y_0^{\text{min}})} = \delta_L.
\]

The following lemma gives the property of the equity’s value function at the default boundary.

**Lemma 3** At the endogenous default boundary we have value matching condition

\[
E(\Phi(y_0), y_0) = 0,
\]

and two smooth-pasting conditions on each dimension

\[
E_\phi(\Phi(y_0), y_0) = 0 \text{ and } E_y(\Phi(y_0), y_0) = 0.
\]

**Proof.** \( E(\Phi(y_0), y_0) = 0 \) at default is obvious. Given our deterministic setting, equity defaults
when their cash-flows turn exactly zero. Plugging in $E(\Phi(y_0), y_0) = 0$ into the ODE for equity valuation, we see that $E_\tau(\tau, y_0)|_{\tau=0} = 0$. Change the coordinates of the state space to $(\phi, y)$, we have

$$E_\tau(\tau, y_0)|_{\tau=0} = E_\phi(\Phi(y_0), y_0) \frac{\partial \phi}{\partial \tau} + E_y(\Phi(y_0), y_0) \frac{\partial y}{\partial \tau} = 0. \quad (32)$$

Also, both $E_\phi(\Phi(y_0), y_0)$ and $E_y(\Phi(y_0), y_0)$ exist as we can write them in terms of sum of partial derivatives of $\tau$ and $y_0$, which are analytical.

Now we show that $E_\phi(\Phi(y_0), y_0) = E_y(\Phi(y_0), y_0) = 0$. On the state space of $(\phi, y)$ around vicinity of $(\Phi(y_0), y_0)$, consider two points $(\Phi(y_0) - \epsilon, y_0)$ and $(\Phi(y_0), y_0 + \epsilon)$. Because the default boundary $\Phi(y_0)$ upward sloping, these two points are lying on the alive region, implying that

$$E(\Phi(y_0) - \epsilon, y_0) \geq 0 \text{ and } E(\Phi(y_0), y_0 + \epsilon) \geq 0.$$

As a result,

$$E_\phi(\Phi(y_0), y_0) \leq 0 \text{ and } E_y(\Phi(y_0), y_0) \geq 0.$$

Because $\frac{\partial \phi}{\partial \tau} > 0$ and $\frac{\partial y}{\partial \tau} < 0$, both terms in (32) have the same sign. Since they sum to zero, each term has to be zero. ■

### 4.3 Can Shortening-to-Death Equilibrium Exist?

We revisit the shortening-to-death equilibrium in this section. As one of the main results in the paper, we show that due to endogenous defaulting cash-flows in the setting with deteriorating fundamentals, shortening-to-death path might occur in equilibrium.

#### 4.3.1 Incentive compatibility condition right before default

Similar to the analysis in Section 3.3 in the base model, we postulate a shortening-to-death equilibrium, and then evaluate the equity holders’ incentive compatibility condition 28 at the default boundary $\Phi(y_0)$. Again as before, on $\Phi(y_0)$ the equity holders’ optimal default decision implies
$E_{\phi} (\Phi (y_0), y_0) = 0$, and due to equal seniority assumption there is no price wedge between short-term and long-term bonds, i.e., $\Delta (\Phi (y_0), y_0) = 0$. We will thus need to go to some higher-order derivatives.

We analyze the sign of $E_{\phi} (\phi, y) + \Delta (\phi, y)$ along the path of $(\phi_\tau, y_\tau)$ slightly away from $\tau = 0$, i.e., the path via which the firm is landing at the default state $(\Phi (y_0), y_0)$. More specifically, we differentiate the incentive compatibility condition $E_{\phi} (\phi, y) + \Delta (\phi, y)$ respect to $\tau$ and evaluate the sign of

$$E_{\phi \tau} (y_0, \Phi (y_0)) + \Delta_\tau (y_0, \Phi (y_0))|_{\tau=0}.$$  \hfill (33)

If (33) is strictly positive, then $E_{\phi} (\phi, y) + \Delta (\phi, y) > 0$ for $\tau > 0$ right before default, implying that equity holders indeed find it incentive compatible to issue short-term bonds only. What we show in Proposition 1 is that in the baseline constant cash-flow case, $E_{\phi} (\phi, y) + \Delta (\phi, y) < 0$ always holds right before default, in contradiction with the incentive compatibility of equity holders if shortening-to-death equilibria ever exist.

\textbf{Proposition 2} When $f = 1$ around the vicinity of $\tau = 0$, (33) is strictly positive if and only if

$$\frac{\partial}{\partial \phi} D_S (\Phi (y_0), y_0) > 0,$$  \hfill (34)

This implies that a necessary condition for “shortening-to-death” equilibrium to exist is that shortening debt maturity has a strictly positive partial impact on the value of short-term debt around the vicinity of default.

\textbf{Proof.} Fix $f = 1$ in the ODE for equity (30). Differentiate this ODE with respect to $\phi$ and evaluate this equation at $\tau = 0$, and we have

$$(r + \zeta) E_{\phi} (\tau = 0, y_0) = m' (\Phi (y_0)) [B (y_0) - 1] + m (\phi_0) \cdot \frac{\partial}{\partial \phi} D_S (\tau = 0; y_0) - E_{\phi \tau} (\tau = 0, y_0)$$
Imposing \( E_\phi = 0 \), we get

\[
E_{\phi \tau} (\tau = 0; y_0) = (\delta_S - \delta_L) [B(y_0) - 1] + m(\Phi(y_0)) \frac{\partial}{\partial \phi} D_S (\tau = 0; y_0).
\]

(35)

For bond price wedge between short-term and long-term bonds, setting \( \tau = 0 \) in (29) we have

\[
\frac{d}{d\tau} D (\tau = 0, y_0) = c + \delta + \zeta D_{rf} - (r + \delta + \zeta) B(y_0),
\]

which implies that

\[
\Delta (0, y_0) = \frac{d}{d\tau} D_S (\tau = 0, y_0) - \frac{d}{d\tau} D_L (\tau = 0, y_0) = (\delta_S - \delta_L) [1 - B(y_0)].
\]

(36)

Combining (35) and (36) we have

\[
\frac{d}{d\tau} [E_\phi (\tau, y_0) + \Delta (\tau, y_0)] \bigg|_{\tau=0} = m(\Phi(y_0)) \frac{\partial}{\partial \phi} D_S (\tau = 0; y_0) = m(\Phi(y_0)) \frac{\partial}{\partial \phi} D_S (\Phi(y_0), y_0),
\]

where in the last equation we rewrite \( D_S (\tau; y_0) \) to \( D_S (\phi, y) \). Since \( m(\phi) \geq \delta_L \) is strictly positive, we proved our claim. Q.E.D. ■

4.3.2 Why condition (34) differs from (24)?

The condition (34) in Proposition 2 is almost identical to the condition (24) in Proposition 1, with one crucial difference. Although both conditions involve taking derivative with respect to the debt maturity state \( \phi \), in (24) the derivative is “total” while in (34) the derivative is “partial.” This difference is due to the introduction of time-deteriorating cash-flows, which becomes another state variable.

As the discussion after Proposition 2 suggests, the condition (24) is equivalent to the condition that bond holders do not recover full in bankruptcy, and we have imposed this empirically relevant condition throughout the paper. This is because default is triggered whenever debt maturity \( \phi \) hits
the threshold $\Phi$, so $D'_S(\Phi; y) > 0$ essentially says investors value the bond more if the firm is a bit away from default. Interestingly, in the setting with time-deteriorating cash-flows, that “bond holders recover partially in bankruptcy” is no longer equivalent to $\frac{\partial}{\partial \phi} D_S(\Phi(y_0), y_0) > 0$ in (34). As we show, it is because of the endogenous default boundary in this setting becomes a curve $\Phi(y_0)$, which makes the bond recovery value $B(y_0)$ endogenously determined. This force is absent in the setting with constant cash-flows.

What is the marginal impact of shortening debt maturity on bond values? Writing $D_S(\phi, y_0)$ in the space of $D_S(\tau; y_0)$. We will further write it as $D_S(\tau, y_0)$ to emphasize that $y_0$ is also endogenous. Taking derivatives, we have

$$\frac{\partial}{\partial \phi} D_S(\Phi(y_0), y_0) = \frac{\partial}{\partial \phi} D(\tau, y_0) \bigg|_{\tau=0} = \frac{\partial D_S(\tau, y_0)}{\partial \tau} \frac{\partial \tau}{\partial \phi} + \frac{\partial D_S(\tau, y_0)}{\partial y_0} \frac{\partial y_0}{\partial \phi} \bigg|_{\tau=0}. \quad (37)$$

We can derive the signs of $\frac{d\tau}{d\phi}$ and $\frac{dy_0}{d\phi}$, the impact of shortening maturity on the time-to-default and the defaulting cash-flows, respectively:

$$\left. \frac{\partial \tau}{\partial \phi} \right|_{\tau=0} < 0, \text{ and } \left. \frac{\partial y_0}{\partial \phi} \right|_{\tau=0} > 0. \quad (38)$$

Both conditions are easy to understand. The first result simply says that fixing current cash-flow state, shortening debt maturity increases the firm’s rollover losses and thus reduces its time-to-default $\tau$. This force is there in the base setting with constant cash-flows. The second result is new: because cash-flows are decreasing over time, the reduction of time-to-maturity increases the cash-flows at default.

We then use the bond valuation given in (31) to derive $\frac{\partial D_S(\tau, y_0)}{\partial \tau}$, which is the impact of time-to-default on the bond value:

$$\left. \frac{\partial D_S(\tau, y_0)}{\partial \tau} \right|_{\tau=0} = -(r + \delta_S + \zeta) e^{-(r + \delta_S + \zeta)\tau} [B(y_0) - 1] \bigg|_{\tau=0}$$

$$= (r + \delta_S + \zeta) [1 - B(y_0)] > 0;$$
The last inequality follows from the prevailing assumption that bonds recover only partially in bankruptcy. Again together with \( \frac{\partial \tau}{\partial \phi} < 0 \) in (38), this force says shortening maturity brings firms closer to default and thus hurt bond value, and is going against condition (34). Not surprisingly, this negative force is present in the constant cash-flow case.

Differently from the constant cash-flow case, there is a second term when cash-flows are time varying. We derive \( \frac{\partial D_S(\tau, y_0)}{\partial y_0} \) using (31), which is the impact of defaulting cash-flows on the bond value:

\[
\left. \frac{\partial D_S(\tau, y_0)}{\partial y_0} \right|_{\tau=0} = e^{-(r+\delta_S+\zeta)\tau} B'(y_0) \left|_{\tau=0} = B'(y_0) > 0, \right.
\]

where the last inequality generally holds as firm’s liquidation value should be increasing in its current profitability. Now, because \( \frac{dy_0}{d\phi} > 0 \), the second term in (37) is positive. Intuitively, by bringing the firm closer to default, shortening debt maturity raises the profitability when bond holders take over the defaulted firm, which increases the bond value. It is this positive force that may dominate the negative first term and make (34) to hold.

### 4.4 Equilibrium Behavior Away From Default

For simplicity, we focus on equilibria where equity holders are taking “deterministic” and “cornered” issuance strategies, which are defined as follows.

**Definition 1** The equilibria are “deterministic” if the firm’s issuance policy \( f_\tau \) is deterministic function of time-to-default. The equilibria are “deterministic” and “cornered” if the firm’s deterministic issuance policy takes corner solution either \( f_\tau = 1 \) or \( f_\tau = 0 \).

Because cash-flows depend on time-to-default deterministically and there are no other payoff-relevant shocks in the model (other than the upside event shock \( \zeta \)), focusing on “deterministic” issuance policies essentially rule out sun-spot-type public randomization schemes. Cornered strategies are in general optimal for risk-neutral equity holders who are solving an linear problem, and note that the class of “deterministic” and “cornered” equilibria have not ruled out time-varying issuance
policies. Somewhat surprisingly, we can show that within the class of deterministic and cornered equilibria, there is never switching of issuance strategies in equilibrium, either from shortening $f = 1$ to lengthening $f = 0$ or vice versa.

4.4.1 No switching in equilibrium

Let us first focus on the shortening-to-death equilibrium. Suppose that we are at the point $(\Phi(y_0), y_0)$ on the default boundary that satisfies (34). For $\tau > 0$ so that the firm is away from default boundary, equity holders are shortening the firm’s maturity by taking $f_\tau = 1$ if $E_\phi(\phi_\tau, y_\tau) + \Delta(\phi_\tau, y_\tau) > 0$ on the path $(\phi_\tau, y_\tau)$ toward the default point $(\Phi(y_0), y_0)$, where $E_\phi(\phi, y)$ can be calculated as

$$
\frac{\partial}{\partial \phi} E(\phi, y) = \frac{\partial}{\partial \phi} E(\tau, y_0) = E_\tau(\tau, y_0) \frac{\partial \tau}{\partial \phi} + E_{y_0}(\tau, y_0) \frac{\partial y_0}{\partial \phi}.
$$

(39)

As before, the first term captures the effect of time-to-default $\tau$, while the second term captures the effect of defaulting cash-flows $y_0$.

Suppose that there exists some time-to-default $\hat{\tau} > 0$ (if there exists multiple, take the infimum) such that the equilibrium issuance strategies switch, i.e.,

$$
f_{\hat{\tau}^-} = 1 \text{ while } f_{\hat{\tau}^+} = 0.
$$

We can easily prove that equity values and debt values (and thus the bond value wedge $\Delta$) are continuous across $\hat{\tau}$ along the path $(\phi_\tau, y_\tau)$. However, the equity value’s derivative with respect to $\tau$, i.e., $E_\tau$, displays a discontinuity at the policy switching point $\hat{\tau}$. The appendix shows that we can relate the jump of $E_\tau$ to the jump in $f$, so that

$$
E_{\hat{\tau}^-} - E_{\hat{\tau}^+} = m(\phi) \Delta \cdot (f_{\hat{\tau}^-} - f_{\hat{\tau}^+}) = m(\phi) \Delta.
$$

(40)

However, cornered strategies indeed impose restrictions on the set of equilibria. For instance, an interior issuance policy say $\tilde{f} \in (0, 1)$ which affects bond valuations can make equity holders indifferent between shortening ($f = 1$) or lengthening ($f = 0$), which in turn implies the optimality of an interior policy $\tilde{f}$. We are currently working on this possibility.
Since \( m(\phi) \Delta > 0 \), it implies that when equity switches to issuing short-term bonds at \( \hat{\tau} \), the equity value’s derivative with respect to \( \tau \) goes up, i.e., the benefit of surviving longer goes up.

In the original \((\phi, y)\) state space, denote the corresponding switching points to be \((\hat{\phi}_-, \hat{y}_-\)) and \((\hat{\phi}_+, \hat{y}_+)\). The equity’s incentive compatibility condition depends on \( \frac{\partial}{\partial \phi} E_\phi (\phi, y) \) at these two points. One can show that in (39), both the \( \frac{\partial \tau}{\partial \phi} \) in the first term, and the entire second term related to \( y_0 \), i.e., \( E_{y_0} (\tau, y_0) \frac{\partial y_0}{\partial \phi} \), are continuous at the switching point. Hence, equation (40) implies that

\[
E_\phi (\hat{\phi}_-, \hat{y}_-) - E_\phi (\hat{\phi}_+, \hat{y}_+) = (E_{\tau-} - E_{\tau+}) \frac{\partial \tau}{\partial \phi} = m(\phi) \Delta (f_{\tau-} - f_{\tau+}) \frac{\partial \tau}{\partial \phi}.
\]

We have shown that \( \frac{\partial \tau}{\partial \phi} < 0 \), i.e., shortening maturity gives rise to a shorter time-to-default. Following the intuition right after (40), when equity switches to issuing short-term bonds, the benefit of surviving longer going up implies that marginal negative impact of shortening maturity is more severe.

If the marginal negative impact of maturity shortening is more severe, while the wedge in bond valuations is the same (\( \Delta \) is continuous at \( \hat{\tau} \)), why would equity ever want to switch from lengthening to shortening? We see precisely this logic when we sign the direction of jump of the equity’s incentive compatibility condition:

\[
\text{sign} \left( E_\phi \left( \hat{\phi}_-, \hat{y}_-\right) + \Delta \left( \hat{\phi}_-, \hat{y}_-\right) - \left[ E_\phi \left( \hat{\phi}_+, \hat{y}_+\right) + \Delta \left( \hat{\phi}_+, \hat{y}_+\right) \right] \right) = \text{sign} \left( E_\phi \left( \hat{\phi}_-, \hat{y}_-\right) - E_\phi \left( \hat{\phi}_+, \hat{y}_+\right) \right) m(\phi) \Delta \frac{\partial \tau}{\partial \phi} < 0
\]

\[
\text{sign}(f_{\tau+} - f_{\tau-}) < 0 \text{ (41)}
\]

The optimality of switching from lengthening to shortening implies that the above sign should be positive, i.e., the net benefit of shortening should go up from \( \hat{\tau}+ \) to \( \hat{\tau}− \). This contradicts with result in (41). The next proposition summarizes our result.

**Proposition 3** Focus on equilibria with deterministic cornered issuance policies. Given any point on the default boundary \((\Phi (y_0), y_0)\) which satisfies (34), the equilibrium path that leads to this default
point is unique with a constant issuance policy \( f_\tau = \bar{f} = 1 \).

**Proof.** As the key argument is given in the main text, we only need to show the “continuity” properties used in the main text. Write debt and equity as integrals, that is \( D = \int_0^{\tau} e^{-(r+\zeta)s} (c + \zeta) \cdot ds + e^{-r\tau}B(y_0) \), as well as \( E = \int_0^{\tau} e^{-(r+\zeta)s}\text{cashflow}_E(s) \ ds + e^{-(r+\zeta)\tau} \cdot 0 \). Note that expected cash-flows to debt and equity include the upper event (which occurs with intensity \( \zeta \)). We can immediately see that both \( D \) and \( E \) are continuous with respect to \( \tau \) and \( y_0 \). For the first-doer derivatives with respect to \( y_0 \) for a fixed \( \tau \), \( \frac{\partial D}{\partial y_0} \) is continuous in its argument because \( B'(y_0) \) is continuous. For \( \frac{\partial E}{\partial y_0} \), it is continuous in its arguments as well because

\[
\frac{\partial \text{cashflow}_E(s)}{\partial y_0} = \frac{\partial [y_\tau - c + \zeta E^{r\tau} + m(\phi_\tau) [D_S(\tau; y_0) - 1]]}{\partial y_0} = 1 + \frac{\partial [m(\phi_\tau) [D_S(\tau; y_0) - 1]]}{\partial y_0}
\]

is continuous. We also see that the derivative of \( E \) w.r.t. \( \tau \) for a fixed \( y_0 \) exhibits a jump as \( \text{cashflow}_E(s) \) changes due to a change in issuance policy.

We also claim that both the \( \frac{\partial \tau}{\partial y_0} \) and \( \frac{\partial y_0}{\partial y_0} \) are continuous at the switching point ??

\[\blacksquare\]

**4.4.2 Maturity-lengthening equilibrium**

So far we have focused on equilibria where equity holders are shortening the firm’s maturity. Because of deteriorating cash-flows, equity will default at some point even if the firm keeps lengthening its debt maturity, i.e. only issuing new long-term bonds with \( f = 0 \). Almost exactly the same analysis as before applies to this case. The following Corollary gives the counterpart of Proposition 2 and Proposition 3. In words, lengthening-to-death equilibrium holds, if at the default boundary \((\Phi(y_0), y_0)\) the value of long-term bond gets hurt by shortening the maturity structure.

**Corollary 3** Around the vicinity of \( \tau = 0 \), the optimality of issuing only long-term bonds requires (33) to be strictly positive, which holds if and only if \( \frac{\partial}{\partial \phi} D_L(\Phi(y_0), y_0) < 0 \). The equilibrium path
that leads to this default point is unique with a constant issuance policy \( f_T = \bar{f} = 0 \).

Naturally, Corollary 3 and previous results suggest that multiple equilibria, i.e., shortening equilibrium or lengthening equilibrium, might exist given some initial condition. This multiplicity of equilibria emerges without too much surprise, as the intuition is similar to the notion self-enforcing default in the literature of sovereign debt. More specifically, if bond investors expect equity holders to keep shortening the firm’s maturity structure in the future, then bond investors price this expectation in the bond’s market valuation, which can self-enforce the optimality of issuing short-term bonds only by equity holders. Similarly, the belief of issuing long-term bonds always can be self-enforcing as well.

The dynamics embedded in our model allows us to say a bit more. The existence of multiple equilibria is not guaranteed, and for some initial state, either the shortening equilibrium or the lengthening equilibrium becomes the unique equilibrium within the class of deterministic cornered equilibria. Intuitively, if the firm has already been extremely close to those default boundary points satisfying (34) in Proposition 2, then starting from there the only equilibrium path is indeed the shortening equilibrium, as a benign expectation of lengthening maturity in the future is “too late” to save the firm. Apparently, this result hinges on the assumption of a bounded issuance strategy space (here, \( f \in [0, 1] \)), for otherwise the firm can change its maturity structure instantaneously.

**Proposition 4** Given the initial starting value \((\phi_T, y_T)\), there might exist two deterministic and cornered equilibria: one with shortening always \( f_T = 1 \), and the other with lengthening always \( f_T = 0 \). However, there exists a no-returning region with positive measure, in which starting from there shortening equilibrium is the unique equilibrium.

### 4.5 An Example with Constant Negative Drift

We consider the special case of the cash-flow drift being a negative constant, i.e., \( \mu_y(y) = -\mu_y \) where with a bit abuse of notation \( \mu_y \) is a positive constant.
4.5.1 Liquidation value $B(y)$

Some extra care needs to be taken in deriving the firm’s liquidation value $B(y)$. Consistent with the notion of bankruptcy cost, we assume that relative to original equity holders, debt holders are less inefficient in running the liquidated firm. More specifically, recall that the asset generates an upside event with payoff $X > 0$ with intensity $\zeta$; we assume that under the debt holders’ management the growth option becomes $\alpha X > 0$ where the constant $\alpha X \in (0, 1)$. And, given the current cash-flow $y$, we assume that the cash-flow under the management of debt holders is $\alpha y$. Since all of our numerical examples we have defaulting cash-flows $y_0 < 0$, to capture the inefficiency due to bankruptcy we set $\alpha > 1$.

The liquidated firm is assumed to be unlettered and hence debt maturity plays no role. The liquidation value, as a function of cash-flow state $y$, satisfies the following ODE

$$rB(y) = \alpha y + \zeta [\alpha X - B(y)] - \mu_B(y);$$

the last term captures the drift of state variable $y$. Also, debt holders have the option to terminate the firm any time. Standard optimal control implies that debt holders should terminate the project when the expected flow payoff $\alpha y + \zeta \alpha X$ hits zero from above. This implies $B(y) = 0$ at $y = -\frac{\alpha X}{\alpha y} \zeta X$. Using this boundary condition, we can derive the liquidation value $B(y)$ in close-form:\footnote{Without stopping option, one can show that $B(y) = \frac{\zeta \alpha y + \alpha y_0}{\zeta + r} + \frac{\mu}{(\zeta + r)^2}$. Hence the exponential term captures the stopping option.}

$$B(y) = \begin{cases} \frac{\zeta \alpha X + \alpha y_0}{\zeta + r} + \frac{\exp \left[ -\frac{\zeta + r}{\mu} (\zeta \alpha X + \alpha y_0) \right] - 1}{(\zeta + r)^2} \mu & \text{for } y > -\frac{\alpha X}{\alpha y} \zeta X, \\ 0 & \text{otherwise} \end{cases}$$

(42)

4.5.2 Shortening and lengthening equilibria, and no return region

Figure XX show two equilibrium paths, one is shortening equilibrium and the other is lengthening equilibrium, together with the default boundary $\Phi(y)$. The baseline parametrization is given in the caption. Among them, the parameter that plays a key role is the bankruptcy cost coefficient
on cash-flows $\alpha_y$, which we set to be XX. In the shortening equilibrium, the firm keeps issuing short-term bonds and default at $(\phi_0^S, y_0^S)$ if the upside event fails to realize along the path. Since the cash-flow at default $y_0^S$ is negative, $\alpha_y > 1$ captures the idea that the project under bond holder’s management will experience even more negative cash-flows. The higher the $\alpha_y$, the more the recovery value is sensitive to cash-flows, and as a result the greater the $B'(y_0)$. From (37) and the discussion afterward, we know that $B'(y_0)$ contributes to the second positive term in (37) which is crucial to guarantee the equity’s incentive compatibility condition in shortening equilibrium.

As shown in Figure XX, there is another lengthening equilibrium given the same initial state $(\phi, y)$, in which equity holders find it optimal to keep issuing long-term bonds and default at $(\phi_0^L, y_0^L)$ if the upside event fails to realize along the path. The time of default, $T_b$, differ greatly across these two equilibria: $T_b^S = XX$ for shortening equilibrium while $T_b^L = XX$ for lengthening equilibrium. In the next section we consider the welfare ranking of these two equilibria by analyzing the welfare as a function of time of default $T_b$.

In Figure XX we also show the no-return region, which is highlighted as hatched/shaded area. There are multiple equilibria to the left of this region. Suppose that we are at the shortening equilibrium, i.e., creditors believe that equity holders will keep shortening the firm’s maturity structure. If the belief of creditors switches to “believe equity holders will keep issuing long-term bonds,” then we can switch to a lengthening equilibrium. However, once we are already in the no-return region, then the mere switch of belief cannot bring the path back from shortening equilibrium to lengthening equilibrium. In some sense, there is a “black hole” in the state space: the firm is absorbed into the shortening equilibrium, without any hope of returning.

4.6 Social Welfare

We study the social welfare of in this section. In the setting with time-deteriorating cash-flows, there is a natural optimal stopping time problem even for unlettered firms. The welfare analysis becomes interesting when we layer this optimal stopping problem with standard equity-debt agency frictions, in which equity is choosing the optimal debt maturity structure to maximize the equity
Throughout this section we will base our analysis on the example in Section 4.5, but we will comment on the generality of the result.

4.6.1 Time of default and firm value

As the key difference between shortening or lengthening equilibria is their respective default times, we analyze the impact of default time on firm value first.

Take any arbitrary time of default, and denote it by $T_b$. Each instant, in expectation the alive firm generates cash-flows $(y_t + \zeta X) \, dt$. And, in default, the firm recovers $B (y_T)$. Hence, the firm value, given the default time $T_b$ (we omit the initial cash-flow $y$)

$$V (T_b) = \int_0^{T_b} e^{-rt} (y_t + \zeta X) \, dt + e^{-r T_b} B (y_{T_b}).$$  \hspace{1cm} (43)

Comparing this firm value to the asset’s social value $A (y)$ in (1), we see the difference is due to the efficient default, i.e., $B (y) < A (y)$. Note that $B (\cdot)$ has taken into account the potential optimal stopping problem after default in the setting of time-decreasing cash-flows.

The optimal stopping time which maximizes (43), call it $T_b^{FB} = \arg \max_{T_b} V (T_b)$, is simple. Because of the inefficiency of “default and let debt holders run the firm,” the optimal solution should set the default timing as the optimal stopping time for unlettered firm denoted by $T_a$ (recall Section 2.1). As in expectation the firm generates cash-flows at a rate of $y_t + \zeta X$, we should shut down the firm $y_t$ hits $-\zeta X$:

$$T_b^{FB} = T_a = \inf \{ t : y_t < -\zeta X \}.$$

Although the global optimum of $V (T_b)$ as a function of $T_b$ is trivial, the local behavior of $V (T_b)$ can be more intriguing for $T_b < T_b^{FB}$, which is the relevant region in our example as shortening equilibrium leads to inefficient earlier default. As shown by Panel A in Figure XX which plots

---

9 More specifically, if $y_t = 0 = y$, the recursive form holds for the asset value, i.e., $A (y) = \int_0^{T_b} e^{-rt} (y_t + \zeta X) \, dt + e^{-r T_b} A (y_{T_b})$ for $T_b < T_a$ which is the first-best abandonment time.
\( V(T_b) \) as a function of \( T_b \), there is a local maximum for firm value with \( T_b \) far less than \( T_b^{FB} \). It implies that around this point, a faster default, potentially due to shortening of debt maturity, is actually welfare enhancing.

To better understand the mechanism, we take derivative of \( V(T_b) \) with respect to \( T_b \), which is the marginal impact of delaying default on the firm value (multiplying both sides by \( e^{rT_b} \)):

\[
e^{rT_b} \cdot V'(T_b) = yT_b + \zeta X - rB(yT_b) + B'(yT_b) \cdot \frac{dyT_b}{dT_b} = yT_b + \zeta X - rA(yT_b) + A'(yT_b) \cdot \frac{dyT_b}{dT_b} + \left[ B'(yT_b) - A'(yT_b) \right] \cdot \frac{dyT_b}{dT_b}
\]

The first term is standard in any frictionless real option setting: at \( T_b \) the expected cash-flows are \( yT_b + \zeta X \), but the opportunity cost of waiting is \( rV(yT_b) \) and worse continuation value. Because we are at the point of earlier stopping than the first-best, this term is positive.\(^\text{10}\) Without inefficient default, we have the first term only, and increasing \( T_b \) when \( T_b < T_b^{FB} \) is always welfare enhancing.

Due to inefficient default, we have another two extra terms. The second term captures inefficient default and thus is positive. The third term captures the impact of delaying default on the firm's recovery value. In our example with \( \alpha_y > 1 \), \( B'(yT_b) - A'(yT_b) > 0 \) because the worse cash-flows in default. Combining with the fact that cash-flows are deteriorating over time, the third term is negative. Panel B in Figure XX plots the three components in (44). As expected, the local maximum is due to the third negative term.

### 4.6.2 Is shortening equilibrium inefficient?

The firm is not choosing default time directly in the model. Rather, the equilibrium maturity policy affects the default time indirectly through affecting equity holders’ decision that maximizes the equity value. For the example considered in Section 4.5.2, we highlight the two equilibrium

\(^{10}\)The first-best project value \( V(y) \) satisfies the ODE \( rV(y) = y + \zeta [X - V(y)] + V'(y) \frac{dy}{dt} \), which implies that

\[
y + \zeta X - rV(y) + V'(y) \frac{dy}{dt} = \zeta V(y) > 0,
\]

where the last inequality is due to the abandonment option.
default times, $T^S_b$ for shortening equilibrium and $T^L_b$ for lengthening equilibrium, in Figure XX. In general, shortening equilibrium leads to a shorter default time, i.e., $T^S_b < T^L_b$. And, in this example, the shortening equilibrium is relatively less efficient than the lengthening equilibrium.

The fact that the firm value $V(T_b)$ is downward sloping at $T_b = T^S_b$ is intriguing, as it indicates that equity holders are maximizing the whole firm value by shortening the maturity structure, conditional on only local deviations being allowed. In other words, because $\frac{dV_b}{d\phi} < 0$ so that maturity shortening leads to a faster default, we know that $V_\phi(\phi, y)$ is positive.

The Modigliani-Miller logic implies that we can decompose the total impact on the firm value into the impact on equity, long-term bonds, and short-term bonds, respectively. Because $F = E + \phi D_S + (1 - \phi) D_L$, we can decompose $F_\phi(\phi, y) > 0$ into

$$F_\phi(\phi, y) = \underbrace{E_\phi(\phi, y) + \Delta(\phi, y)}_{\text{Incentive compatibility}} + \underbrace{\phi \frac{\partial}{\partial \phi} D_S(\phi, y)}_{\text{Impact on ST bonds}} + \underbrace{(1 - \phi) \frac{\partial}{\partial \phi} D_L(\phi, y)}_{\text{Impact on LT bonds}} \quad (45)$$

Around the vicinity of the default boundary $(\phi = \Phi(y), y)$ at the shortening equilibrium, the first part is about equity’s incentive compatibility condition which has to be positive. Condition (34) in Proposition 2 implies that short-term bonds gain value by shortening maturity, leading to a positive second term. The third term is the impact on long-term bonds. Under the condition $c = r$, one can show that $\frac{\partial}{\partial \phi} D_L(\phi, y) > \frac{\partial}{\partial \phi} D_S(\phi, y)$, which says that the value of long-term bonds are more sensitive than short-term bonds. Since $\frac{\partial}{\partial \phi} D_S(\phi, y) > 0$, we have $\frac{\partial}{\partial \phi} D_L(\phi, y) > 0$, i.e., long-term bond holders also gains during maturity shortening.

The above discussion implies that, although the lengthening equilibrium can be the more efficient outcome, the shortening equilibrium has the property that maturity shortening taken by equity holders improves the firm value locally. In other words, all parties in the firm will vote against lengthening the firm’s maturity a bit. A deeper question is: can we ever have shortening equilibria in which equity-value-maximizing maturity shortening hurts the firm value?

Because the first two terms in (45) are positive in any shortening equilibria, the only possibility
is that maturity shortening hurts the value of long-term bonds in the third term. For the case where long-term bonds are coupon bonds but short-term bonds are zero-coupon, it is possible to have $\frac{\partial}{\partial \phi} D_L(\phi, y) < 0$. To see this, similar to (37) we have ($T_b$ has the same role as $\tau$ in equation (37)):

$$\frac{\partial}{\partial \phi} D_i(\phi, y) = \underbrace{\frac{\partial D_S}{\partial T_b} \cdot \frac{\partial T_b}{\partial \phi}}_{\text{negative}} + \underbrace{\frac{\partial D_S}{\partial y_0} \cdot \frac{\partial y_0}{\partial \phi}}_{\text{positive}}$$

for $i \in \{S, L\}$.

Here is the intuition. The higher the coupon, the less the dependence for the bond value on its recovery value, and hence the heavier the weight of the first negative term regarding default time $T_b$. This implies that the first negative term is relatively more important for long-term bonds, while the second positive term is more important for short-term bonds. Hence, it is possible to have $\frac{\partial D_S}{\partial \phi} > 0 > \frac{\partial D_L}{\partial \phi}$, which is our desired result.

5 Conclusion

to be added.
A Appendix

We will solve the model in terms of \((\tau, y_0)\), that is time to maturity and cash-flow at time of bankruptcy. This makes the equations all ODEs that we have to consider on the equilibrium path. We then calculate separately the IC conditions via the derivatives of \(E_\phi\) under different assumptions of the issuance strategies.

B Change of coordinates

Suppose instead that \(f\) changes to \(f^c\) at a fixed \(\tau_f\). Then, we have, with

\[\phi \equiv \frac{S}{P}\]

being the proportion of short-term debt, for \(\tau < \tau_f\)

\[\phi (\tau, \phi_0) = [\phi_0 - \phi_{as} (f)] e^{[\delta S - (\delta S - \delta L)]f^c \tau} + \phi_{as} (f)\]

and thus, for \(\tau \geq \tau_f\),

\[
\begin{align*}
\phi (\tau, \phi_0; \tau_f) &= \phi (\tau - \tau_f, \phi (\tau_f, \phi_0)) \\
&= \left[\phi_{as} (f^c) \right] e^{[\delta S - (\delta S - \delta L)]f^c \tau} + \phi_{as} (f^c) \\
&= \left[\phi_0 - \phi_{as} (f) \right] e^{[\delta S - (\delta S - \delta L)]f^c \tau} + \phi_{as} (f^c) + \phi_{as} (f^c) \\
&= \left[\phi_0 - \phi_{as} (f^c) \right] e^{[\delta S - (\delta S - \delta L)]f^c \tau} + \phi_{as} (f^c) + \phi_{as} (f^c) \\
&= \left[\phi_0 - \phi_{as} (f) \right] e^{[\delta S - (\delta S - \delta L)]f^c \tau} + \phi_{as} (f^c) + \phi_{as} (f^c)
\end{align*}
\]

The ODE is solve in terms of \(z = (\tau, y_0)\). However, the incentives of the equity holders are derived from the Markov system \(x = (\phi, y)\), as the optimal \(f\) requires the derivative \(E_\phi\). We are looking for points \(z = g (x)\) such that \(f (x, z) = f (x, g (x)) = 0\) where

\[
f (x, z) = \begin{pmatrix} f_1 (x, z) \\ f_2 (x, z) \end{pmatrix} = \begin{pmatrix} -\phi + \phi (\tau, \phi (y_0)) \\ -f_0 (y) + f_0 (y_0) \end{pmatrix} = 0
\]

where \(f_0 (x) = x\) in the linear-growth specification, and \(f_0 (x) = \log (x)\) in the exponential growth specification. Thus, we have

\[
f_1 (x, z) = \begin{cases} 
- [\phi - \phi_{as} (f)] + \phi (y_0) - \phi_{as} (f) e^{[\delta S - (\delta S - \delta L)]f^c \tau} \\
- [\phi - \phi_{as} (f^c)] + \phi (y_0) - \phi_{as} (f) e^{[\delta S - (\delta S - \delta L)]f^c \tau} + \phi_{as} (f^c) e^{[\delta S - (\delta S - \delta L)]f^c \tau}
\end{cases}, \tau \leq \tau_f (y_0)
\]

with \(\tau_f (y_0) = \frac{1}{\mu} [f_0 (y_f) - f_0 (y_0)]\)

with \(\tau_f (y_0) = - \frac{f_0 (y_0)}{\mu}\), as we assume the switch happens at a fixed position \(y_f\).\(^{11}\) We also have

\[
f_2 (x, z) = -f_0 (y) + f_0 (y_0) + \mu \tau
\]

We are looking for a function

\[g (x) = \begin{pmatrix} \tau (\phi, y) \\ y_0 (\phi, y) \end{pmatrix}\]

To calculate the derivative of \(E (\tau, y_0) = E (z)\) w.r.t. \(\phi\), we have to use

\[
\frac{\partial}{\partial \phi} E (\tau, y_0) = E_v (\tau, y_0) \frac{\partial \tau}{\partial \phi} + E_{y_0} (\tau, y_0) \frac{\partial y_0}{\partial \phi} = \left[ \frac{\partial}{\partial z} E (z) \right] \cdot \left[ \frac{\partial z}{\partial \phi} \right]
\]

\(^{11}\)The difference here is if the agent switches strategy at a fixed point \(y_f\) or at a fixed time to default \(\tau_f\). If it is a fixed point, we have to account for the fact that
The Jacobian matrix is given by
\[ J = \frac{\partial f(x, z)}{\partial z} = \begin{bmatrix} \frac{\partial f_1}{\partial \tau} & \frac{\partial f_1}{\partial \phi} \\ \frac{\partial f_2}{\partial \tau} & \frac{\partial f_2}{\partial \phi} \end{bmatrix} \]

Then, applying the chain rule when taking the derivative w.r.t. \( x_i \), \( \frac{\partial f \tau}{\partial x_i} + \frac{\partial f \phi}{\partial x_i} = 0 \), we have for \( x_i = \phi \),
\[ \frac{\partial x}{\partial \phi} = \frac{\partial g(x)}{\partial \phi} = \frac{\partial}{\partial \phi} \left( \tau(\phi, y), y_0(\phi, y) \right) = -J^{-1} \frac{\partial f}{\partial \phi}(x, z) \]

Let us calculate the different derivatives. First, we have
\[ \frac{\partial f_1}{\partial \tau} = -1 \quad \frac{\partial f_2}{\partial \phi} = 0 \]

so that \( \frac{\partial f}{\partial \phi}(x, z) = -[1, 0]^T \).

For \( \tau < \tau_f(y_0) \), we have
\[ \frac{\partial f_1}{\partial \tau} = [\delta_S - (\delta_S - \delta_L) f^c] \left( [\Phi(y_0) - \phi_{ss}(f)] e^{[\delta_S - (\delta_S - \delta_L)f]^c} + [\phi_{ss}(f) - \phi_{ss}(f^c)] e^{[\delta_S - (\delta_S - \delta_L)f^c](\tau - \tau_f(y_0))} \right) \]
\[ \frac{\partial f_1}{\partial y_0} = \Phi'(y_0) e^{[\delta_S - (\delta_S - \delta_L)f^c][\tau_f(y_0)][\delta_S - (\delta_S - \delta_L)f^c]^c} \]
\[ + \tau_f(y_0) \left( \left[ \delta_S - (\delta_S - \delta_L)(f^c - f) \right] [\Phi(y_0) - \phi_{ss}(f)] e^{[\delta_S - (\delta_S - \delta_L)(f^c - f)] \tau_f(y_0) e^{[\delta_S - (\delta_S - \delta_L)f^c](\tau - \tau_f(y_0))} - [\delta_S - (\delta_S - \delta_L)f^c] [\phi_{ss}(f) - \phi_{ss}(f^c)] e^{[\delta_S - (\delta_S - \delta_L)f^c](\tau - \tau_f(y_0))} \right) \]
\[ = \Phi'(y_0) e^{[\delta_S - (\delta_S - \delta_L)f^c][\tau_f(y_0)][\delta_S - (\delta_S - \delta_L)f^c]^c} \]
\[ + \tau_f(y_0) \left( \left[ \delta_S - (\delta_S - \delta_L)(f^c - f) \right] [\phi_{ss}(f) - \phi_{ss}(f^c)] e^{[\delta_S - (\delta_S - \delta_L)f^c](\tau - \tau_f(y_0))} \right) \]

We will be especially interested in evaluating these derivatives at \( y = y_{f+} \) or \( \tau = \tau_{f+} \).
\[ \frac{\partial f_1}{\partial \tau} \bigg|_{\tau = \tau_{f+}} = [\delta_S - (\delta_S - \delta_L) f^c] \left( [\Phi(y_0) - \phi_{ss}(f)] e^{[\delta_S - (\delta_S - \delta_L)f]^c} + [\phi_{ss}(f) - \phi_{ss}(f^c)] \right) \]
\[ \frac{\partial f_1}{\partial y_0} \bigg|_{\tau = \tau_{f+}} = \Phi'(y_0) e^{[\delta_S - (\delta_S - \delta_L)f^c]} + \tau_f(y_0) \left( \left[ \delta_S - (\delta_S - \delta_L)(f^c - f) \right] [\Phi(y_0) - \phi_{ss}(f)] e^{[\delta_S - (\delta_S - \delta_L)f^c]} \right) \]
\[ = \Phi'(y_0) e^{[\delta_S - (\delta_S - \delta_L)f^c]} \]
\[ + \tau_f(y_0) \left( \left[ \delta_S - (\delta_S - \delta_L)(f^c - f) \right] [\phi_{ss}(f) - \phi_{ss}(f^c)] \right) \]

Here, we need to use a word of caution. We note that \( \tau \geq \tau_f \geq 0 \). Thus, at \( \tau = \tau_f = 0 \) so

Also, we have
\[ \frac{\partial f_2}{\partial \tau} = \mu \quad \frac{\partial f_2}{\partial y_0} = f'_0(y_0) \]
Then, we have
\[
\frac{\partial z}{\partial \phi} = - \left[ \frac{\partial f_1}{\partial \phi} \frac{\partial f_2}{\partial \phi} \frac{\partial f_3}{\partial \phi} \right]^{-1} \left[ \frac{\partial f_1}{\partial \tau} \frac{\partial f_2}{\partial \tau} \right] = \frac{1}{f_0'(y_0) \frac{\partial f_1}{\partial \tau} - \mu \frac{\partial f_1}{\partial \phi}} \left[ \frac{f_0'(y_0)}{y_0} \right] = \frac{1}{\Phi'(y_0)} \frac{\partial f_1}{\partial \tau} - \frac{\partial f_1}{\partial \phi} \right)
\]
Plugging in the expressions, we have for \( \tau < \tau_f \),
\[
1 \quad \frac{f_0'(y_0) \frac{\partial f_1}{\partial \tau} - \mu \frac{\partial f_1}{\partial \phi}}{f_0'(y_0) \frac{\partial f_1}{\partial \tau} - \mu \frac{\partial f_1}{\partial \phi}} = \frac{f_0'(y_0) [\delta_S - (\delta_S - \delta_L) f] [\phi - \phi_{ss} (f)] - \mu \Phi'(y_0) e^{\delta_S - (\delta_S - \delta_L) f} \tau}{f_0'(y_0) [\delta_S - (\delta_S - \delta_L) f] [\phi - \phi_{ss} (f)] - \mu \Phi'(y_0) e^{\delta_S - (\delta_S - \delta_L) f} \tau}
\]
for \( \tau = \tau_f(y_0) \), we have
\[
\frac{1}{f_0'(y_0) \frac{\partial f_1}{\partial \tau} - \mu \frac{\partial f_1}{\partial \phi}} = \frac{f_0'(y_0) [\delta_S - (\delta_S - \delta_L) f] [\phi - \phi_{ss} (f)] - \mu \Phi'(y_0) e^{\delta_S - (\delta_S - \delta_L) f} \tau}{f_0'(y_0) [\delta_S - (\delta_S - \delta_L) f] [\phi - \phi_{ss} (f)] - \mu \Phi'(y_0) e^{\delta_S - (\delta_S - \delta_L) f} \tau}
\]
We realize that the coordinate derivatives are both the same.

B.1 At the boundary
Lastly, let us evaluate the derivatives at \( \tau = 0, y = y_0 \) and \( \phi = \Phi(y_0) \). Then,
\[
\frac{1}{f_0'(y_0) \frac{\partial f_1}{\partial \tau} - \mu \frac{\partial f_1}{\partial \phi}} = \frac{1}{f_0'(y_0) [\delta_S - (\delta_S - \delta_L) f] [\Phi(y_0) - \phi_{ss} (f)] - \mu \Phi'(y_0)}
\]

C Preliminary results
The default boundary is given by
\[
m(\Phi(y_0)) = \frac{y_0 - (1 - \pi) c + \zeta E^{r_f}}{1 - B(y_0)}
\]
\[
\Phi(y_0) = \frac{1}{\delta_S - \delta_L} \left[ \frac{y_0 - (1 - \pi) c + \zeta E^{r_f}}{1 - B(y_0)} - \delta_L \right]
\]
Of course, the default boundary is bounded by \( \Phi(y_0) \in [\phi_L, \phi_U] \). Thus, we have \( y_0 \in [y_0^{\min}, y_0^{\max}] \) where the bounds are defined by
\[
\delta_S = \frac{y_0^{\max} - (1 - \pi) c + \zeta E^{r_f}}{1 - B(y_0^{\max})} \frac{1}{1 - B(y_0^{\min})}
\]
\[
\delta_L = \frac{y_0^{\min} - (1 - \pi) c + \zeta E^{r_f}}{1 - B(y_0^{\min})}
\]
D Admissible paths

D.1 Equally spaced points

First, we will aim to have equally spaced points along the curve $\Phi(y_0)$. To this end, we will first calculate the total length of the arc, which is given by

$$TL = \int_{y_0}^{\max} \left[ 1 + \Phi'(y_0)^2 \right] dy_0$$

by standard formulas.\textsuperscript{12} We then calculate recursively the set of equally spaced points $\{y_0^0, y_0^1, \ldots, y_0^n, \ldots, y_0^N\}$, where $N+1$ is the number of points, by the following algorithm:

$$y_0^0 = y_{0\min}$$
$$y_0^n : \quad \frac{TL}{N} = \int_{y_0}^{y_0^n} \left[ 1 + \Phi'(y_0)^2 \right] dy_0, \quad 1 \leq n \leq N$$

D.1.1 Going long: $f = f_L$

For $f = f_L$, we will need to compare $\Phi'(y_0)$ with

$$\frac{\partial \Phi(\tau, y_0)_{\tau=0}}{\partial y} |_{\tau=0} = \frac{-\mu(y_0) f_L - \mu(y_0)}{\mu(y_0)}$$

We will check the conditions path-by-path, so that we start the system on the bankruptcy boundary, i.e., $(y_0, \Phi(y_0))$, and move backwards in time, i.e., let $\tau$ increase. Then, we should only consider the set of $y_0$ such that

$$\Phi'(y_0) \geq \text{slope}(y_0) \equiv \frac{\partial \Phi(y_0, \tau)}{\partial y} |_{\tau=0, f=f_L}$$

Note that for $f_L = 0$ and $f_H = 1$, we have

$$\text{slope} \left( y_{0\min} \right) = 0$$
$$\text{slope} \left( y_{0\max} \right) = \frac{\delta_L}{\mu(y_0)}$$

Define the set of initial $y_0$ that fulfills this condition. $[y_0^{\text{upper}}, y_0^{\text{max}}]_L$ and the in-between points $[y_0^{\text{lower}}, y_0^{\text{upper}}]$ will be inaccessible. Additionally, as everything is continuous, we have

$$\Phi'(y_0^{\text{lower}}) = \text{slope}(y_0^{\text{lower}})$$
$$\Phi'(y_0^{\text{upper}}) = \text{slope}(y_0^{\text{upper}})$$

Next, define $y_0^{\text{switch}} \in [y_0^{\text{min}}, y_0^{\text{lower}}]$ as the solution to the following system of equations:

$$\phi(\tau, \Phi(y_0)) = \Phi(y_0^{\text{upper}})$$
$$y(\tau, y_0) = y_0^{\text{upper}}$$

First, we solve for

$$\tau(\phi_0, \phi_x) = \frac{1}{\delta_S - (\delta_S - \delta_L) f_L} \log \left[ \frac{\phi_x - \phi_{sx}(f_L)}{\phi_0 - \phi_{sx}(f_L)} \right]$$

and then plug this into $y(\tau, y_0)$ to get

$$y(\tau, \Phi(y_0^{\text{upper}}), y_0) = y_0^{\text{upper}}$$

which is solved by $y_0 = y_0^{\text{switch}}$. The path originating from $(y_0^{\text{switch}}, \Phi(y_0^{\text{switch}}))$ is thus tangent to $\Phi(y_0)$ at $y_0 = y_0^{\text{upper}}$ and cuts the space above $\Phi(y_0)$ into three areas:

1. The area above and to the right of the path. Any path in this area has its origin on $[y_0^{\text{min}}, y_0^{\text{switch}}]$.

\textsuperscript{12}Alternatively we can use the integration along $\tilde{\phi}$ dimension to get $TL = \int_{\tilde{\phi}_L}^{\tilde{\phi}_H} \sqrt{1 + \frac{d\phi^{-1}(\tilde{\phi})}{d\tilde{\phi}}} d\tilde{\phi}$. 43
2. The area below the path to the left of $y_0^{\text{upper}}$. Any non-degenerate path in this area has its origin on $[y_0^{\text{switch}}, y_0^{\text{lower}}]$.$^{13}$

3. The area below the path to the right of $y_0^{\text{upper}}$. Any path in this area has its origin on $[y_0^{\text{upper}}, y_0^{\text{max}}]$.

**Point of no return** Note that whatever the strategy $\tilde{f}$, area (3.) forms a black-hole for the firm. Once inside this area, the path cannot exit the area given that $\tilde{f} \in [f_H, f_L]$. Next, numerically, we have

Even more so, suppose that there is a point $y_0$ such that $y_0 \in [y_0^{\text{min}}, y_0^{\text{max}}]$ has the IC condition

D.2 Going short $f = f_H$

As we have $\Phi'(y_0) > 0 > \frac{\rho}{\rho} [f_H D_L (\tau, y_0) + (1 - f_L) D_S (\tau, y_0)]_{\tau=0}$, all points $y_0 \in [y_0^{\text{min}}, y_0^{\text{max}}]$ are admissible.

E Incentive compatibility

Incentive compatibility in this model centers on the sign of the term $E_\phi + \Delta$

If positive, we pick $f = f_H$, shortening the maturity structure, and if negative, we pick $f = f_L$, lengthening the maturity structure.

E.0.1 Going long $\tilde{f} = f_L$

Numerical results deliver poles of the IC condition $\frac{\partial}{\partial y_{\text{left}}} [f_L D_L (\tau, y_0) + (1 - f_L) D_S (\tau, y_0)]_{\tau=0}$ at exactly $y_0^{\text{lower}}$ and $y_0^{\text{upper}}$.

E.1 Bankruptcy recovery value

E.1.1 Exponentially decaying

First, let us derive unlevered firm value. To this end, define $X = E^f + D^f$, the total value of the firm after the risk-free jump (as the firm jumps to risk-free, there is no further deadweight cost of debt, and we assume $\pi = 0$ so there is also no benefit to debt). Let $V (y)$ be the unlevered firm value. Then we have

$$r V (y) = y + \zeta [X - V (y)] - \mu y V' (y)$$

As $y > 0$ and $\zeta X > 0$, the unlevered firm will never choose to stop operating, so that

$$V (y, X) = \frac{\zeta X}{r + \zeta} + \frac{y}{r + \zeta + \mu}$$

Next, let $\alpha_y$ and $\alpha_X$ be the recovery coefficients w.r.t. to current cash-flows and future opportunities, respectively. Then, the recovery value is defined by

$$B (y) \equiv V (\alpha_y y, \alpha X)$$

Note that with $\alpha_y = \alpha_X = \alpha$, we have $B (y) = V (y, X)$ as the function is homogenous.

E.1.2 Linearly decaying

First, let us derive unlevered firm value. To this end, define $X = E^f + D^f$, the total value of the firm after the risk-free jump (as the firm jumps to risk-free, there is no further deadweight cost of debt, and we assume $\pi = 0$ so there is also no benefit to debt). Let $V (y)$ be the unlevered firm value. Then we have

$$r V (y) = y + \zeta [X - V (y)] - \mu y V' (y)$$

$^{13}$Non-degenerate here is to refer to the path that are accessible. Recall that boundary points $[y_0^{\text{lower}}, y_0^{\text{upper}}]$ are inaccessible and thus inadmissible as initial points.
As \( y_t \) can turn negative, and the flow payment \( y + \zeta X \) is not bound to stay positive. Thus, unlettered firm optimally chooses to stop operating when the flow payment hits zero, i.e., when \( y = -\zeta X \), so that \( V(-\zeta X) = V'(-\zeta X) = 0 \). Then

\[
V(y, X) = \frac{\zeta X + y}{\zeta + r} + \left[ e^{-(\zeta + r)(\zeta X + y)\mu} - 1 \right] \mu
\]

It is easy to show that the stopping option adds the exponential term to the equation.\(^{14}\)

Next, let \( \alpha_y \) and \( \alpha_X \) be the recovery coefficients w.r.t. to current cash-flows and future opportunities, respectively. Then, the recovery value is defined by

\[
B(y) \equiv V(\alpha_y y, \alpha_X X)
\]

Note that with \( \alpha_y = \alpha_X = \alpha \), we do not have \( B(y) \neq \alpha V(y, X) \) as the function is not homogenous due to the presence of the exponential term and the drift \( \mu \). To recover the homogeneity, we need to add \( \alpha \mu = \alpha \), which then gives \( B(y) = \alpha V(y, X, \mu) = V(\alpha y, \alpha X, \alpha \mu) \).

\(^{14}\)Solving for an arbitrary stopping point \( y = k \), we have

\[
V(y, k) = \frac{(r + \zeta)(y + \zeta X) - \mu}{(r + \zeta)^2} + \frac{\mu - (r + \zeta)(k + \zeta X) e^{-(r + \zeta)(y-k)\mu}}{(r + \zeta)^2}
\]

\[
V_k(y, k) = -\frac{k + \zeta X e^{-(r + \zeta)(y-k)\mu}}{\mu}
\]

and we see that \( V_k(y, k) > 0 \) for \( k < -\zeta X \) and \( V_k(y, k) < 0 \) for \( k > -\zeta X \), as well as \( \lim_{k \to \infty} V(y, k) = \frac{y + \zeta X}{(r + \zeta)} - \frac{\mu}{(r + \zeta)^2} \).