We analyze equilibrium leverage dynamics in a dynamic tradeoff model when the firm is unable to commit to a leverage policy ex ante. We develop a methodology to characterize equilibrium equity and debt prices in a general jump-diffusion framework, and apply our approach to the standard Leland (1998) setting. Absent commitment, the leverage ratchet effect (Admati et al. 2015) causes firms to take on debt gradually over time (rather than adjust immediately to a “target” leverage ratio). Firm leverage may go down via asset growth and debt maturity, but equity holders never reduce debt voluntarily. While equilibrium equity values match those in a model where the firm commits not to issue new debt, bond investors anticipate future leverage increases and hence price bonds lower uniformly, with significant credit spreads even when the distance to default is large. Ultimately, lower bond prices dissipate the tax benefits from future debt increases.

In our model, current leverage is dependent on the full history of the firm’s earnings. Despite the absence of transactions costs, an increase in profitability causes leverage to decline in the short-run, but the rate of new debt issuance endogenously increases so that leverage ultimately mean-reverts. Our model highlights the flexibility of short-term debt, in that firms with short-term debt can swiftly adjust their debt issuance to counter-balance profitability shocks. Tax rules that limit the debt tax shield when debt is high may lead firms to issue debt more aggressively and hence have higher leverage.

* DeMarzo: Stanford University, Graduate School of Business, and NBER. He: University of Chicago, Booth School of Business, and NBER. We thank Anat Admati, Douglas Diamond, Konstantin Milbradt, Paul Pfleiderer, and seminar participants at the London School of Economics, MIT, NYU, Princeton, for helpful comments.
1. Introduction

Understanding the determinants of a firm’s capital structure, and how its leverage is likely to evolve over time, is one of the central questions in corporate finance. Leverage and its expected dynamics are crucial to valuing the firm, assessing its credit risk, and pricing its financial claims. The optimal response of leverage to shocks, such as the 2007-2008 financial crisis, are important to forecasting the likely consequences of the crisis and its aftermath, and to evaluate alternative policy responses.

Despite its importance, a fully satisfactory theory of leverage dynamics has yet to be found. Many models assume the absolute level of debt is fixed; for example, in the traditional framework of Merton (1974), as well as Leland (1994, 1998), the firm is committed not to change its outstanding debt before maturity, irrespective of the evolution of the firm’s fundamentals. As a result, the dynamics of firm leverage is driven solely by the stochastic growth in value of the firm’s assets-in-place. More recent work that allows the firm to restructure its debt over time generally assumes that all existing debt must be retired before any new debt can be put in place.\(^1\) Neither of these assumptions is consistent with practice, where firms often borrow incrementally over time.

In contrast, we study a model in which equity holders lack any ability to commit to their future leverage choices, and can issue or buyback debt at the current market price at any time. Aside from corporate taxes and bankruptcy costs, there are no other frictions or transactions costs in our model. Nonetheless, as emphasized by Admati et al. (2015), an important agency friction emerges with regard to future leverage choices, as equity holders adjust leverage to maximize the current share price rather than total firm value. They demonstrate a “leverage ratchet” effect, in which equity holders are never willing to voluntarily reduce leverage, but always have an incentive to borrow more. While the effect is quite general, they calculate numerically a dynamic equilibrium only for a specialized model in which debt is perpetual and the firm does not grow but is subject to Poisson shocks.

Solving the dynamic tradeoff model without commitment is challenging because of the dynamic interdependence of competitive debt prices today and equity’s equilibrium

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leverage/default policies in the future. In this paper we develop a methodology to solve for such an equilibrium in a general setting that allows for finite maturity debt, and can allow for asset growth, investment, and both Brownian and Poisson shocks. In this equilibrium, equity holders increase debt gradually over time, at a rate which increases with current profitability of the firm. While equity holders never voluntarily reduce leverage, they allow leverage to decline passively via debt maturity in response to negative fundamental shocks.

Thus, in our equilibrium, though equity holders keep issuing debt to exploit tax benefits, due to lack of commitment they continue to issue debt even if the firm’s leverage passes above the “optimal” level with commitment, causing excessive inefficient default. Interestingly, we show that equity holders obtain the same value in equilibrium as if they commit not to issue any debt in the future, but over-borrowing raises the probability of default and so lowers the price of debt. This low debt price offsets the tax advantage of leverage sufficiently so that, on the margin, equity holders are indifferent to leverage increases. In other words, the extra tax shield benefits that tempt equity holders are exactly dissipated by the bankruptcy cost caused by excessive leverage.

We apply our methodology to the special case of geometric Brownian motion and show that we can solve for the equilibrium debt price and issuance policy in closed form. The equilibrium speed of debt issuance is endogenously increasing with profitability. When profits decline, debt repayments exceed new issuance and the debt level falls. We show explicitly that the firm’s outstanding debt at any point of time can be expressed as a weighted average of the firm’s earnings history. The endogenous adjustment of leverage leads the firm’s interest coverage ratio to mean revert gradually in equilibrium, with the speed of adjustment decreasing with debt maturity and asset volatility. These dynamics differ from abrupt adjustment to some “target” leverage, a common implication from models with an exogenous adjustment cost (for instance, Fischer, Heinkel, Zechner, 1989; Goldstein, Ju, Leland, 2001; and Strebulaev, 2007; etc).

We compare our model without commitment to two benchmarks with full commitment. In the first case equity holders commit not to issue any debt in the future, and in the second case

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2 Because we assume the firm has zero recovery in default, we rule out the motive of directly diluting existing debt, simply because newly issued pari passu debt does not dilute the recovery value of existing debt in bankruptcy. Nevertheless, the existing debt value gets hurt via the endogenous default policy.
they commit to maintain a constant outstanding debt obligation (i.e., always issue the same amount of new debt to replace debt that is maturing) as in Leland (1998). The central difference between our model and these two benchmark models is the endogenous mean-reverting firm leverage and its implications on equilibrium debt prices, as bond investors anticipate future leverage changes. Recall that in response to positive profitability shocks the firm issues more debt, pulling the firm back toward default. In fact, the mean-reversion is sufficiently strong, that credit spreads remain strictly bounded away from zero even when the firm’s interest-coverage-ratio is arbitrarily large. In contrast, in the other two benchmark models including Leland (1998), the firm’s interest-coverage-ratio follows a geometric Brownian motion and hence implied credit spreads vanish for sufficiently large interest-coverage-ratio.

We also study the optimal debt maturity structure, which we model in terms of a constant required repayment rate for existing debt. Our model without commitment to future leverage policies provides a fresh perspective on this question. We first demonstrate that equity holders are indifferent to the maturity structure of the firm’s future debt issuance. Hence, the choice of debt maturity structure becomes relevant only when the firm must borrow a fixed amount upfront. Indeed, this question has been studied in the Leland (1998) setting, and often long-term debt, which minimizes rollover risk, is preferred (He and Xiong, 2012; Diamond and He, 2014). In contrast, firms in our model without commitment prefer short-term debt. Firms with short-term debt that matures rather quickly can swiftly adjust their debt issuance policy, and this flexibility allows them to remain highly levered but yet only suffer moderate default costs.

We further consider the extension with realistic tax codes in which firms with negative earnings lose their tax subsidy. Besides the smooth equilibrium, there exists another class of equilibria in which the firm’s equilibrium interest-coverage-ratio always lies below one. In this equilibrium, the firm maintains high enough leverage to avoid all taxes, issuing debt as needed to keep its interest-coverage-ratio from exceeding one. This class of equilibria, which features an aggressive leverage policy, tends to exist when firms are less concerned about default risk (e.g., when volatility is low). As a result, tax rules that limit the debt tax shield when debt is high may lead firms to issue debt more aggressively and hence have higher leverage.

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3 When the cash-flow volatility is sufficiently low, the smooth issuance debt policies may fail to be globally optimal—and hence no longer constitute an equilibrium. In this case, the tax savings from maintaining high leverage

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The other extension with endogenous investment and hence debt overhang shows that our methodology and results are robust to various economic forces that affects firm growth endogenously.

**Literature review. To be added.** Brunnermeier and Yogo (2009)

### 2. General Model

We begin by outlining our model and equilibrium determination in a general jump-diffusion framework. After developing our methodology, we consider in subsequent sections more specialized examples with analytically tractable solutions in order to highlight the main implications of our framework.

#### 2.1. The Setting

All agents are risk neutral with an exogenous discount rate of \( r > 0 \). A firm’s assets-in-place generates operating cash flow at the rate of \( Y_t \) which evolves according to

\[
 dY_t = \mu(Y_t) dt + \sigma(Y_t) dZ_t + \zeta(Y_t) dN_t, \tag{1}
\]

where the drift \( \mu(Y_t) \) and the volatility \( \sigma(Y_t) \) are general functions that satisfy regularity conditions; \( dZ_t \) is the increment of standard Brownian motion; \( dN_t \) is Poisson increment with intensity of \( \lambda(Y_t) > 0 \); and \( \zeta(Y_t) \) is the jump size given the Poisson event.\(^4\)

Denote by \( F_t \) the aggregate face value of outstanding debt. The constant coupon rate of the debt is \( c > 0 \), so that over \([t, t + dt]\) debt holders receive coupon payments of \( cF_t dt \) in total. Equity holders pay tax \( \pi(Y_t - cF_t) dt \), where \( \pi \) is a nondecreasing function of the firm’s profit net of interest. When the marginal tax rate is positive, \( \pi' > 0 \), the net after tax cost to the firm of the marginal coupon payment is \( (1 - \pi') \), reflecting the debt tax shield subsidy.

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\(^4\) Here we have assumed the jump size \( \zeta(Y_t) \) conditional on cash flow \( Y_t \) is deterministic. Our analysis goes through in a further generalized setting with random jump size \( \tilde{\zeta}(Y_t) \), as long as the law of \( \tilde{\zeta}(Y_t) \) depends on \( Y_t \) only.
For simplicity we assume that debt takes the form of exponentially maturing coupon bonds with a constant maturation rate $\xi$. More specifically, each instant there are $\xi F_t dt$ units of required principal repayments from maturing bonds, corresponding to an average bond maturity of $1/\xi$. Together with the aggregate coupon of $cF_t dt$, over $[t, t + dt]$ equity holders are required to pay debt holders the flow payment of $(c + \xi) F_t dt$ in order to avoid default.

In the main analysis we assume investors recover zero value from the assets-in-place when equity holders default. The key implication of this assumption, which will simplify our analysis, is that debt seniority becomes irrelevant. In addition, because there are no claims to divide in default, old debt holders do not get “diluted” by new debt holders in bankruptcy even if the new debt has equal (or higher) priority. We make the zero recovery value assumption to emphasize that our results are not driven by the “dilution” effect which often arises when issuing pari-passu debt (e.g., Brunnermeier and Ohmkhe, 2014). Dropping this assumption, and allowing firms to only issue new debt which is junior to existing claims, would not change our qualitative results.\(^5\)

Equity holders control the outstanding debt $F_t$ through endogenous issuance/repurchase policy $d\Gamma_t$, where $\Gamma_t$ represents the cumulative debt issuance over time. We focus our main analysis on a “smooth” equilibrium in which along the equilibrium path equity holders find it optimal to adjust the firm’s outstanding debt smoothly with order $dt$. More specifically, we conjecture and verify later that, at each instant the adjustment to existing debt is $d\Gamma_t = G_t dt$, where $G_t$ specifies the rate of issuance at date $t$. From now on, we simply call $G_t$ the equity holders’ issuance policy, which could be issuing new debt if $G_t > 0$ or repurchasing existing debt in which case $G_t < 0$. Given our debt maturity assumption, the evolution of outstanding face value of debt $F_t$ is given by

$$dF_t = (G_t - \xi F_t) dt.$$  
\(^2\)

\(^5\) This alternative setting would add some complexity, as debt securities issued at different times would have distinct prices. In contrast, given zero recovery value, newly issued debt is identical to all existing debt, independent of its seniority (or timing of issuance).
Thus, the face value of debt will grow only if the rate of issuance more than offsets the contractual retirement rate. To highlight the economic forces at play, and in contrast to the bulk of the literature, we assume zero transaction costs in issuing or repurchasing debt.\(^6\)

Given the equity holders’ expected issuance/repurchase policy \(G_t\), debt holders price the newly issued or repurchased debt in a competitive market. Denote by \(p_t\) the endogenous debt price per unit of promised face value. Then over \([t, t + dt]\) the net cash flows to equity holders are equal to

\[
\begin{align*}
&\int_t^{t+dt} \left( Y_t - \pi(Y_t - cF_t) - (c + \xi)F_t + G_t p_t \right) dt.
\end{align*}
\]

The firm continues to operate until the operating cash flow \(Y_t\) drops below some sufficiently low level, at which point equity holders find it optimal to default on their contractual payment to debt holders (we will derive the optimal default boundary shortly). After default, debt holders take over the firm but recover zero by assumption.

2.2. General Analysis

We have two state variables in this model: the firm’s \textit{exogenous} operating cash flow \(Y_t\), and the outstanding aggregate debt face value \(F_t\), which is an \textit{endogenous} state variable. We will analyze the equity’s value function \(V(Y_t, F_t)\) and the debt price \(p(Y_t, F_t)\). Denote by \(\tau_B\) the equilibrium default time; presumably, this is the first time that the state pair \((Y_t, F_t)\) falls into the endogenous default region, which we denote by \(\mathcal{B}\). For the value of equity, given future issuance and debt prices \(\{(G_s, p_s) : s > t\}\), when the firm is in the survival region, i.e., \((Y_t, F_t) \notin \mathcal{B}\), we have

\[
V(Y_t, F_t) = E_t \left[ \int_t^{\tau_B} e^{-r(s-t)} \left[ Y_s - \pi(Y_s - cF_s) - (c + \xi)F_s + G_s p_s \right] ds \big| Y_t = Y, F_t = F \right].
\]

\(^6\) It is common in the dynamic capital structure literature, e.g. Fischer, Heinkel, and Zechner (1989) and Leland, Goldstein, and Ju (2000), to assume that firms—in order to adjust their capital structure—have to buy back all of their existing debt and then reissue new debt; and that there is a positive adjustment cost associated with this transaction. We eliminate this artificial transaction friction to highlight the equity holders’ intrinsic incentives to adjust leverage at any time.
For debt prices, similarly we have

\[ p(Y, F) = E_t \left[ \int_t^\infty e^{-(r+\xi)(t-s)} (c + \xi) dt \mid Y_t = Y, F_t = F \right]. \]  

(5)

Clearly, the equity value \( V(Y, F) \) equals zero in the default region, as does the debt price \( p(Y, F) \) given our zero recovery assumption.

### An Optimality Condition

Recall that we are interested in an equilibrium when there is no commitment of equity holders to future leverage policies. Thus at any point in time, the issuance policy \( G_s \) for \( s > t \) has to be optimal in solving the equity holders’ instantaneous maximization problem at time \( s \), given the equity’s value function and equilibrium debt prices.

In this section we consider the necessary and sufficient conditions for optimality of the equity holders’ debt issuance policy \( G_t \). The Hamilton-Jacobi-Bellman (HJB) equation for equity holders is

\[
rV(Y, F) = \max_G \left[ Y - \pi(Y - cF) - \left( c + \xi \right) F + Gp(Y, F) + (G - \xi F)V_r(Y, F) \right] + \mu(Y)V_r(Y, F) + \frac{1}{2} \sigma(Y)\tilde{V}_{yy}(Y, F) + \lambda(Y)\left[ V(Y + \zeta(Y)) - V(Y) \right] \]

(6)

In the first line, the objective is linear in \( G \) with a coefficient of \( p(Y, F) + V_r(Y, F) \) which represents a (endogenous) marginal benefit of increasing debt. If equity holders find it optimal to take an interior solution, then it must be that this coefficient equals zero everywhere, i.e.

\[ p(Y, F) + V_r(Y, F) = 0. \]  

(7)

This first-order condition (FOC) must hold for any \((Y, F)\) along the equilibrium path.\(^7\)

Moreover, this first-order condition implies global optimality, provided that \( p_r(Y, F) \leq 0 \), i.e.,

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\(^7\) This relation holds trivially in the default region \( \beta \), as for defaulted firm the debt price \( p = 0 \) and \( V(Y, F) = 0 \) implies \( V_r(Y, F) = 0 \) as well. It is worth pointing out that the zero-bankruptcy-recovery assumption is only sufficient, but not necessary, for \( p = 0 \) at the default region. Even with a strictly positive bankruptcy recovery, newly issued
the debt price is weakly decreasing in the firm’s total debt. The following proposition states this result formally.

**Proposition 1.** Suppose the debt price \( p \) is weakly decreasing in the total face value \( F \) of the firm’s debt. Then condition (7) implies that the policy \( G_t \) solves the equity holders’ instantaneous debt issuance problem in (6), and the value of equity \( V(Y, F) \) is convex in \( F \).

**Proof.** Equity holders are solving the following problem each moment along the optimal path

\[
\max_{\Delta} V(Y, F + \Delta) + \Delta p(Y, F + \Delta) - V(Y, F).
\]

(8)

For the current policy to be optimal, \( \Delta = 0 \) must be optimal in (8). This problem has the first-order condition \( V_F + p + \Delta p_F = 0 \) at \( \Delta = 0 \), which implies that \( p = -V_F \). To check for global optimality, suppose that equity holders choose any \( \Delta > 0 \). Then equity’s gain is

\[
\begin{align*}
V(Y, F + \Delta) - V(Y, F) + \Delta p(Y, F + \Delta) &= \int_0^\lambda V_F(Y, F + \delta) d\delta + \int_0^\Delta p(Y, F + \delta) d\delta \\
&\leq \int_0^\lambda V_F(Y, F + \delta) d\delta + \int_0^\Delta p(Y, F + \delta) d\delta \\
&= \int_0^\lambda \left[ V_F(Y, F + \delta) + p(Y, F + \delta) \right] d\delta = 0
\end{align*}
\]

where in the second line we have used the condition that \( p \) is weakly monotone in \( F \). The above inequalities still hold if \( \Delta < 0 \); in this case \( p(F + \Delta) \geq p(F + \delta) \) but \( d\delta < 0 \). Finally, the condition that the debt price is weakly decreasing in leverage implies that equity’s value is convex in leverage, i.e. \( p_F \leq 0 \Leftrightarrow V_{FF} \geq 0 \). ■

**Equilibrium Policies**

The First-Order Condition (FOC) in (7), which implies a zero-profit condition for equity holders in adjusting the debt burden instantaneously, has deep implications for the equilibrium in our model. Plugging condition (7) into the equity HJB equation (6), we have the following revised HJB equation for equity:

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debt has a zero price as long as we assume that the newly issued debt has to be junior to all existing debt. Then, at default, since the existing debt gets all the recovery, newly issued debt is worthless.
\[ rV(Y,F) = Y - \pi(Y - cF) - (c + \xi)F + \mu(Y)V_f(Y,F) - \xi p_f(Y,F) \]
\[ + \frac{1}{2} \sigma(Y)^2 V_{ff}(Y,F) + \lambda(Y) \left[ V(Y + \zeta(Y),F) - V(Y,F) \right]. \] (9)

This equation says that in the no-commitment equilibrium, the equity value can be solved as if there is no debt adjustment \( G_t = 0 \), except for the natural retirement at rate \( \xi \).

Intuitively, because equity holders gain no marginal surplus from adjustments to debt, their equilibrium payoff must be the same as if they never issue/repurchase any debt. This implies that we can solve for the equilibrium equity value \( V(Y,F) \), without the knowledge of the equilibrium debt price \( p(Y,F) \) (which does not enter equation (9)).

Given \( V \), we then invoke the FOC in (7) to obtain the equilibrium debt price \( p(Y,F) = -V_f(Y,F) \). Finally, to confirm that this outcome indeed represents an equilibrium, we must verify whether \( p(Y,F) = -V_f(Y,F) \) is weakly decreasing in \( F \), or equivalently that the equity value is convex in \( F \).

Having solved for the equilibrium value of equity and debt, we can then derive the optimal issuance policy \( G^* \) explicitly. On one hand, by differentiating the HJB equation (9) for \( V(Y,F) \) by \( F \), and using \( p = -V_f \), we obtain

\[ -rp(Y,F) = \pi'(Y - cF)c - (c + \xi) + \xi p(Y,F) + \xi p_f(Y,F) \]
\[ -\mu(Y)p_f(Y,F) - \frac{1}{2} \sigma(Y)^2 p_{ff}(Y,F) + \lambda(Y) \left[ -p(Y + \zeta(Y),F) + p(Y,F) \right]. \] (10)

Although equation (10) is written in terms of the debt price \( p \), we emphasize that it follows mechanically from the valuation equation (9) for equity, together with the FOC \( p = -V_f \) for the optimal issuance policy. On the other hand, we can also write the HJB equation for the debt price in (5) by acknowledging the true equilibrium evolution of the state variables, i.e.

\[ \frac{rp(Y,F)}{\text{required return}} = \underbrace{c + \xi(1 - p(Y,F))}_{\text{coupon}} + \underbrace{(G^* - \xi F)p_f(Y,F)}_{\text{debt retirement}} + \underbrace{\mu(Y)p_f(Y,F)}_{\text{evolution of debt } df} + \underbrace{\frac{1}{2} \sigma(Y)^2 p_{ff}(Y,F)}_{\text{evolution of cash flow } dY} + \underbrace{\lambda(Y) \left[ p(Y + \zeta(Y),F) - p(Y,F) \right]}_{\text{evolution of cash flow } dY}. \] (11)

Adding (10) to (11), we obtain a simple expression for \( G^* \):
\[ G^*(Y,F) = \frac{\pi'(Y-cF)c}{-p_F(Y,F)} = \frac{\pi'(Y-cF)c}{V_{FF}(Y,F)}. \]  

(12)

Under the condition \(-p_F(Y,F) = V_{FF}(Y,F) \geq 0\) in **PROPOSITION 1** which ensures the global second-order conditions for the interior issuance policy, it immediately follows that \(G^*\) is always positive provided a strictly positive tax benefit \(\pi' > 0\). Alternatively, if there is no tax subsidy \(\pi' = 0\), then \(G^*(Y,F) = 0\), i.e. it is always optimal not to adjust leverage, despite the fact that there are deadweight costs of bankruptcy.

**Summary**

In sum, for the general model in which equity holders are free to issue or repurchase any amount of debt at the prevailing market price, one can solve for the no commitment equilibrium as follows:

(i) Solve for the equity holder’s value function \(V(Y,F)\) assuming that \(G = 0\), i.e. equity holders commit to not issue any future debt, using (9);

(ii) Set the debt price \(p(Y,F) = -V_F(Y,F)\);

(iii) Check the global second-order condition by verifying the debt price \(p(Y,F)\) is weakly decreasing in aggregate debt \(F\), or equivalently \(V(Y,F)\) is convex in \(F\);

(iv) Finally, given \(p(Y,F)\) we can solve for the optimal time consistent issuance policy \(G^*(Y,F)\) from (12).

In the remainder of the paper we will use this methodology to analyze several specific examples.

**3. Geometric Brownian Motion**

We now apply the general methodology developed in the previous section to the widely used framework of a lognormal cash flow process.\(^8\) The results from Section 2 allow us to fully characterize an equilibrium in closed form, and evaluate the corresponding leverage dynamics.

\(^8\) This setting is consistent with e.g. Merton (1974), Fischer et al. (1989), Leland (1994), Leland and Toft (1996), and follows the development of starting from cash flows rather than firm value as in Goldstein et al. (2001).
We then compare the equilibrium without commitment to two benchmark cases: the first is that the firm can fully commit not to issue/repurchase any debt in the future, i.e. \( G_t = 0 \); and the second is the Leland (1998) assumption that the firm commits to keep the aggregate face value constant, i.e. \( G_t = \xi F_t \). Finally, we consider the implications of alternative debt maturities.

### 3.1. The Setting and Scale Invariance

In the special case of lognormal operating cash flow, \( Y_t \) follows a geometric Brownian motion so that

\[
\mu(Y_t) = \mu Y_t \quad \text{and} \quad \sigma(Y_t) = \sigma Y_t, \quad \text{with} \quad r > \mu. \tag{13}
\]

Given the scale invariance of the firm in this special case, it is reasonable to analyze the model using a unidimensional state variable equal to operating cash flow scaled by the outstanding face value of debt \( F_t \), i.e.

\[
y_t \equiv \frac{Y_t}{F_t}, \tag{14}
\]

As an interpretation, note that \( y_t / c \) equals the firm’s interest coverage ratio, i.e. the ratio of operating income \( Y_t \) to total interest expense \( cF_t \), a widely used measure of leverage and financial soundness. Alternatively, \( 1 / y_t \) expresses the amount of debt as a multiple of the firm’s cash flow.

To maintain homogeneity, we assume the tax expense also scales with leverage so that \( \pi(Y_t - cF_t) = \pi(y_t - c) \cdot F_t \). The homogeneity assumption for \( \pi(\cdot) \) implies that \( \pi(\cdot) \) must be linear in the scaled earnings \( y_t - c \) (and so have a constant marginal tax rate) with a possible kink at the origin. In this section we focus on the case of a constant tax rate, i.e.

\[
\pi(y_t - c) = \bar{\pi} \cdot (y - c), \tag{15}
\]

where the positive constant \( \bar{\pi} > 0 \) is the marginal corporate tax rate that applies to both losses and gains. In Section 4.1 we analyze the alternative case where the firm gets a debt tax benefit only when it is profitable, i.e., \( \pi(y - c) = \bar{\pi} \cdot \max(y - c, 0) \).
With this setting, we conjecture and verify that the equity value function \( V(Y, F) \) and debt price \( p(Y, F) \) are homogeneous so that

\[
V(Y, F) = V\left(\frac{Y}{F}, 1\right) F \equiv v(y) F \quad \text{and} \quad p(Y, F) = p\left(\frac{Y}{F}, 1\right) \equiv p(y). \tag{16}
\]

We will solve for the (scaled) equity value function \( v(y) \) and debt price \( p(y) \) in closed form.

Given the evolution of our state variables \( Y_t \) and \( F_t \):

\[
dY_t = \mu Y_t dt + \sigma Y_t dZ_t, \quad \text{and} \quad dF_t = \left( G_t - \xi F_t \right) dt, \tag{17}
\]

the scaled cash-flows evolve as

\[
\frac{dy_t}{y_t} = \left( \mu + \xi - g_t \right) dt + \sigma dZ_t, \quad \text{where} \quad g_t \equiv G_t / F_t. \tag{18}
\]

As (18) shows, the scaled cash flow growth has the same volatility as total cash flow, as the outstanding debt face value \( F_t \) in (2) grows in a locally deterministic way. The drift of the scaled cash flow growth corresponds to the growth in operating cash flows less the net growth rate of the debt \( g_t - \xi \), where \( \xi \) is the debt retirement rate and \( g_t = G_t / F_t \) is the endogenous growth rate from debt issuance. The more the firm issues new debt, the faster the scaled cash-flows shrink.

Following the Leland tradition, equity holders cannot commit ex ante to a specific default policy. When the scaled cash flow \( y_t \) falls below some endogenous default boundary \( y_b \), equity holders are no longer willing to service the debt, and therefore strategically default. At that event, equity holders walk away and debt holders recover nothing by the assumption of zero liquidation value.

### 3.2. Model Solution

Recall from Section 2 that we can solve for the equilibrium equity value as if \( g_t = 0 \) and equity holders do not actively adjust the firm’s outstanding debt \( F_t \), even though they do. Using the fact that
\[ V_r(Y, F) = v'(y), \quad V_r(Y, F) = v(y) - yv'(y), \quad \text{and} \quad FV_{yy} = v''(y), \]  

we can rewrite (9) with lognormal cash flows in terms of scaled cash-flow \( y \) as follows:

\[ (r + \xi)v(y) = (y - c - \xi) - \pi(y - c) + (\mu + \xi)yv'(y) + \frac{1}{2}\sigma^2y^2v''(y). \]  

There are two boundary conditions for the above Ordinary Differential Equation (ODE) (20). First, when \( y \to \infty \), default becomes unimportant and we can treat the debt as riskless, and hence the equity value should converge to

\[ \bar{v}(y) = \frac{y(1-\pi)}{r - \mu} + \frac{\pi c}{r + \xi} - \frac{c + \xi}{r + \pi} = \frac{y(1-\pi)}{r - \mu} - \frac{c(1-\pi) + \xi}{r + \pi}, \]  

On the other hand, when \( y = y_b \), equity is worthless so \( v(y_b) = 0 \). Solving (20) with these two boundary conditions, we obtain

**Proposition 2.** Given a constant tax rate \( \pi \), the equity value function with no debt issuance is given by

\[ v(y) = \frac{y(1-\pi)}{r - \mu} - \frac{c(1-\pi) + \xi}{r + \pi} \left(1 - \frac{1}{1+\gamma} \left(\frac{y}{y_b}\right)^{-\gamma}\right), \]  

where the constant \( \gamma \) is defined as

\[ \gamma = \left(\mu + \xi - 0.5\sigma^2\right) + \sqrt{\left(\mu + \xi - 0.5\sigma^2\right)^2 + 2\sigma^2\left(r + \xi\right)} > 0, \]  

and the optimal default boundary is given by

\[ y_b = \frac{\gamma}{1+\gamma} \left[\frac{r - \mu}{r + \xi} + \frac{c + \xi}{1-\pi}\right]. \]  

**Proof.** We can write the value function as

\[ v(y) = \overline{v}(y) + E\left[e^{-(r+\xi)\tau_a}(0-\overline{v}(y_b))\right] = \overline{v}(y) - \overline{v}(y_b)\left(\frac{y}{y_b}\right)^{-\gamma}, \]
where the expression for \( E \left[ e^{-(r+\zeta)\tau} \right] \) and \( \gamma \) follows by solving the ODE

\[
(r + \zeta)f(y) = (\mu + \xi) yf'(y) + \frac{1}{2} \sigma^2 y^2 f''(y)
\]

with boundary conditions \( f(y_b) = 1 \) and \( f(\infty) = 0 \). Finally, the optimal default boundary \( y_b \) is determined by the smooth-pasting condition, \( v'(y_b) = 0 \). □

Having solved for the value of equity, recall from (7) that we can determine the equilibrium debt price from the FOC \( p(y) = -F_p(Y, F) \). Then from (19), and using (22),

\[
p(y) = -V_F = yv'(y) - v(y) = \frac{c(1-\bar{\pi}) + \xi}{r + \zeta} \left( 1 - \left( \frac{y}{y_b} \right)^{\gamma} \right).
\] (26)

Recall we need to verify the optimality of the issuance policy by checking the monotonicity of the equilibrium debt pricing function. It is easy to see that \( p'(y) > 0 \) in (26), i.e. the greater the scaled cash flow the higher the debt price. As a result, the key condition in \textbf{PROPOSITION 1} – that the debt price decreases with total debt – follows because

\[
p_F(Y, F) = p'(y) \cdot \left( -\frac{Y}{F^2} \right) = -\frac{y^2 v''(y)}{F} < 0.
\]

Finally, we can use (12) combined with (26) to derive the equilibrium debt issuance policy:

\textbf{PROPOSITION 3.} Given a constant tax rate \( \bar{\pi} \), the equity value function and debt price are given by (25)-(26), and the equilibrium issuance policy is

\[
g^*(y) = \frac{G^*}{F} = \frac{\bar{\pi}c}{-Fp_F(Y, F)} = \frac{\bar{\pi}c}{yp'(y)} = \frac{\bar{\pi}c}{y^2 v''(y)} = \frac{(r + \zeta)\bar{\pi}c}{c(1-\bar{\pi}) + \xi} \cdot \frac{1}{y_b} \left( \frac{y}{y_b} \right)^{\gamma}.
\] (27)

In equilibrium, the firm’s new debt issuance \( g^*(y) \) is always positive, and is increasing in the coupon rate, the tax rate, and the scaled cash flow \( y \).

Based on the equilibrium values for both equity and debt, total firm value (or total enterprise value, TEV) can be expressed as a multiple of the firm’s cash flow (i.e. TEV to EBIT) as
where the first equality follows from the equilibrium condition for the debt price, and the last equality uses the expression of $y_b$ in (24). Note that the firm’s TEV multiple is strictly increasing with the scaled cash flow $y$. Consequently, holding the level of cash flows $Y$ fixed, total firm value decreases with the debt face value $F$, implying no gain to total firm value from leverage. We will discuss this implication further when we compare TEV multiples across different benchmark models with commitment.

### 3.3. Debt Dynamics

From **Proposition 3**, we see that the firm will issue debt at a faster rate when cash flows are high, and the rate of issuance slows as the firm approaches default. Figure 1 illustrates the net debt issuance rate given different debt maturities as a function of the firm’s current leverage.

![Net Debt Issuance versus Firm Leverage for Different Debt Maturities](image)

**Figure 1: Net Debt Issuance versus Firm Leverage for Different Debt Maturities**

Parameters: $\mu = 2\%$, $\sigma = 40\%$, $\bar{\pi} = 30\%$, $c(1 - \bar{\pi}) = r_f = 5\%$)
Because the mapping from the cash flow or leverage is strictly monotonic, there is a unique level of the scaled cash flow \( \hat{y}_g \) such that new net debt issuances occur at any given rate \( g \). We can compute \( \hat{y}_g \) from (27) as follows:

\[
\hat{y}_g \equiv y_b \left( \frac{\gamma (c(1-\pi) + \xi)}{(r+\xi)\pi c} \cdot g \right)^{1/y}.
\]  

We can interpret \( \hat{y}_\xi \) (i.e., when the new debt issuance exactly balances with retiring of existing debt) as the “neutral” level of the scaled cash flow: at this point, the firm is neither accumulating nor retiring debt, corresponding to the intersection points with x-axis in Figure 1. If cash flows were fixed at this level, the firm’s debt would remain constant.\(^9\)

While the rescaling \( y = Y/F \) is convenient for the analysis, it does make it awkward to analyze the evolution of the firm’s debt starting from an initially unlevered position, \( F_0 = 0 \). Fortunately, we can also derive the evolution of the firm’s debt explicitly as a function of the firm’s initial debt position and its earnings history, as shown next.

**Proposition 4.** Given the debt issuance policy \( g^* \) and initial debt face value \( F_0 \geq 0 \), the firm’s debt on date \( t \) given the cash-flow history \( \{Y_s : 0 < s < t\} \) is

\[
F_t = F_0 \gamma e^{-\gamma \xi t} + \int_0^t \gamma \xi \left( \frac{Y_s}{\hat{y}_\xi} \right)^{\gamma} e^{\gamma (r-s)} ds
\]  

\[
(30)
\]

**Proof:** Using (27) the change in the face value of debt is (where we denote \( \dot{F} = \frac{dF}{dt} \))

\[
\dot{F} = (g^*(Y/F) - \xi)F = \left( \frac{(r+\xi)\pi c}{\gamma (c(1-\pi) + \xi)\hat{y}_\xi^\gamma} Y^\gamma \right) F^{1-\gamma} - \xi F.
\]  

\[
(31)
\]

Let \( H = F^\gamma \), then \( \dot{H} = \gamma F^{\gamma-1} \dot{F} \) and so (31) implies

\[
9 \text{ Recall though that the cash flows are expected to grow at rate } \mu, \text{ and therefore } \hat{y}_{\mu+\xi} \text{ is the scaled cash flow level at which debt is expected to grow at the same rate as the cash flows. Therefore at } \hat{y}_{\mu+\xi} \text{, absent any cash flow shocks, } y_t, \text{ as well as the firm’s interest coverage ratio } \gamma, \text{ would be expected to remain constant (as can be seen from (18)).}
\]

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\[
\dot{H} = \gamma F^{r+1} = \gamma \left( \frac{(r + \xi) \bar{c}}{\gamma (c (1 - \bar{c}) + \xi)} Y' \right) - \gamma^2 F^r = \gamma^2 \left( \frac{Y}{\bar{y}} \right)^r - \gamma^2 H.
\]

Given \( H_0 \), this equation is a linear differential equation with general solution

\[
H_t = e^{-\gamma^2 t} \left[ H_0 + \int_0^t \gamma^2 \left( \frac{Y_s}{\bar{y}} \right)^r e^{\gamma^2 y} ds \right].
\]

(30) then follows from \( F = H^{1/\gamma} \).

Equation (30) implies that the firm’s debt at today equals the sum of discounted initial debt \( F(0) \), and a sequence of properly discounted shocks that are dependent on past cash flows \( \{Y_s : 0 < s < t\} \). This point becomes transparent in the special case of \( F_0 = 0 \), i.e., when the firm starts with no debt. Setting \( F_0 = 0 \) in (30), we have

\[
F_t = \frac{1}{\bar{y}} \gamma \left( \gamma^2 \int_0^t \gamma^2(s-t) e^{\gamma^2 Y_s} ds \right)^{1/\gamma}
\]

(32)

Since \( \gamma^2 \int_0^t e^{\gamma^2(s-t)} ds = 1 - e^{-\gamma^2 t} \), (32) implies that in equilibrium debt starts at 0 and grows gradually with order of \( t^{1/\gamma} \), with the long run debt level dependent on the weighted average of the firm’s historical earnings. The weight put on recent cash flows relative to more distant ones depends on the product \( \gamma^2 \). Intuitively, faster maturity allows leverage to shrink more quickly in the face of declining cash flows. From (23) one can show that higher \( \gamma \) is associated with lower volatility, and makes the firm more aggressive about adding leverage when cash flows are high.

Figure 2 simulates the evolution of debt for an initially unlevered firm, issuing debt with a five-year maturity, using the same parameters as Figure 1. Leverage initially increases until it approaches approximately 40% of firm value. Once it exceeds that level, the firm issues new debt at a slower rate than its existing debt matures, and the total amount of debt declines. Overall, the firm’s debt level evolves gradually based on a weighted average of past earnings.
Figure 2: Simulation of Debt Evolution

$\mu = 2\%$, $\sigma = 40\%$, $\bar{\pi} = 30\%$, $c(1-\bar{\pi}) = r_f = 5\%$, $\xi = 20\%$

3.4. Full Commitment Benchmarks

To facilitate discussion, we now solve two other cases that serve as benchmark with commitment. In each case, the firm’s relevant state variables follow the same as in (17), except that the firm has the ability to commit to a certain future debt issuance policy.

No Future Debt Issues

We first consider the benchmark case in which the firm commits not to issue debt in the future, i.e. $g_t = 0$ always. We call this case the “No Future Debt” case, and indicate the corresponding solutions with the superscript “0” (as committing to $g = 0$)

Recall that our methodology developed in Section 2 first called for solving the equity value function as if there will be no future debt issues, even though the firm will choose to add debt equilibrium, because the lack of commitment dissipates the benefits of the debt tax shield. Therefore, from Proposition 2, the equity value $v^{\theta}(y) = v(y)$ with the same default boundary $y_b$. 
Indeed, the only change in this setting will be the debt pricing. Intuitively, new debt issues (despite being junior) harm existing creditors by accelerating default, and thus debt holders are willing pay more for the same promise today if the firm can commit not to issue more debt in the future. With such a commitment, the firm’s scaled cash flow $y$ evolves according to

$$\frac{dy}{y} = (\mu + \xi) dt + \sigma dZ , \quad (33)$$

and thus the HJB equation for debt price can be written as

$$rp^0(y) = c + \xi \left(1 - p^0(y)\right) + (\mu + \xi) y p^0'(y) + \frac{1}{2} \sigma^2 y^2 p^{0''}(y) , \quad (34)$$

with boundary conditions:

1. No recovery value: $p^0(y_b^0) = 0$, and

2. Risk-free pricing as the distance to default grows: $p^0(y) \to \frac{c + \xi}{r + \xi}$ as $y \to \infty$.

Using standard methods (see e.g. the proof of PROPOSITION 2) the solution for the debt price is

$$p^0(y) = \frac{c + \xi}{r + \xi} \left[1 - E\left[e^{-(r+\xi)\tau_b}\right]\right] = \frac{c + \xi}{r + \xi} \left[1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right], \quad (35)$$

where the constant $\gamma$ is again given by (23). We summarize these results as follows:

**PROPOSITION 5.** If the firm can commit not to issue future debt, i.e., $g = 0$, then the equity value is unchanged, as is the default boundary, relative to the no commitment case. Default is delayed, however, and thus the debt price improves by the value of the debt tax shield

$$p^0(y) - p(y) = \frac{\pi c}{r + \xi} \left[1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right]. \quad (36)$$

**Proof:** Equation (36) follows immediately from (26), (35) and $y_b^0 = y_b$.  

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From (36), we observe that when the firm can fully commit not to issue any debt in the future and hence is less likely to default, its debt will trade at a higher price than that of firms who cannot commit. More interestingly, the premium is equal to the value of the tax shield, consistent with the observation that, in the no commitment case, the firm issues new debt at a rate so that expected bankruptcy costs offset the expected tax benefit. Thus, commitment to \( g_t = 0 \) does not benefit equity holders, but does improve the value of the debt due to the reduction in bankruptcy costs, which is just the expected tax benefit.

**Fixed Face Value (Leland 1998)**

Another relevant benchmark for our model without commitment is Leland (1998), who assumes that firm commits to keep a fixed total face value \( F \). Specifically, in Leland (1998), the firm commits to replace the maturing debt (with intensity \( \xi \) ) by the same amount of newly issued debt with the same coupon, principal, and maturity. We denote this case using the superscript “\( \xi \)”, which requires \( g_t = \xi \) always.

The solution with constant face value is as follows. The scaled cash-flow \( y_t \) in this case follows

\[
\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t,
\]

and equity holders in equilibrium will default at a threshold \( y_b^\xi \) to be derived shortly. Then, using the same logic as we did to compute \( p^0 \), we have the analogous solution to (35):

\[
p^\xi(y) = \frac{c + \xi}{r + \xi} \left( 1 - \left( \frac{y}{y_b^\xi} \right)^{-\gamma_p^\xi} \right),
\]

where the constant \( \gamma_p^\xi \) is defined by effectively lowering the drift by \( \xi \) (the rate of new debt issues) in (23):

\[
\gamma_p^\xi = \frac{\mu - 0.5\sigma^2 + \sqrt{(\mu - 0.5\sigma^2)^2 + 2\sigma^2 (r + \xi)}}{\sigma^2} > 0.
\]
Next, the equity value $v^{\hat{\xi}}(y)$ must solve

$$\frac{n^{\hat{\xi}}(y)}{\text{required return}} = (1-\bar{\pi})(y-c) + \xi\left(p^{\hat{\xi}}(y) - 1\right) + \mu v^{\hat{\xi}'}(y) + \frac{1}{2}\sigma^2 y^2 v^{\hat{\xi}''}(y),$$

with boundary conditions $v^{\hat{\xi}}(y_b^{\hat{\xi}}) = v^{\hat{\xi}'}(y_b^{\hat{\xi}}) = 0$ and $v^{\hat{\xi}}(y) \to \frac{y(1-\bar{\pi}) + \bar{\pi}c}{r-\mu} + \frac{c + \xi}{r + \xi}$ as $y \to \infty$.

Note that the second term in (40) captures the rollover gains/losses when equity holders refinance the maturing debt, as emphasized by He and Xiong (2012): per dollar of face value, the firm must repay principal at rate $\xi$, while equity holders commit to replace the maturing debt by issuing $\xi$ new bonds at price $p^{\hat{\xi}}$.

In the Appendix, we follow the approach of Leland (1998) to derive the equity value function as

$$v^{\hat{\xi}}(y) = \frac{y(1-\bar{\pi}) + \bar{\pi}c}{r-\mu} + \frac{\bar{\pi}c}{r} - \left(\frac{y_b^{\hat{\xi}}(1-\bar{\pi}) + \bar{\pi}c}{r-\mu}\right)\left(\frac{y}{y_b^{\hat{\xi}}}\right)^{-\gamma^{\hat{\xi}} - 1} - \frac{c + \xi}{r + \xi} \left(1 - \left(\frac{y}{y_b^{\hat{\xi}}}\right)^{-\gamma^{\hat{\xi}} - 1}\right),$$

where the constant $\gamma^{\hat{\xi}}$ is defined as

$$\gamma^{\hat{\xi}} = \frac{\mu - 0.5\sigma^2 + \sqrt{\left(\mu - 0.5\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2} > 0,$$

and the endogenous default boundary $y_b^{\hat{\xi}}$ satisfying the smooth-pasting condition $v^{\hat{\xi}'}(y_b^{\hat{\xi}}) = 0$ is

$$y_b^{\hat{\xi}} = \frac{r-\mu}{(1+\gamma^{\hat{\xi}})(1-\bar{\pi})} \left[\frac{c + \xi}{r + \xi} \gamma^{\hat{\xi}} - \frac{\bar{\pi}c}{r + \xi} \gamma^{\hat{\xi}}\right].$$

In this case, the valuation multiple for the firm (TEV/EBIT) is given by

$$\frac{v^{\hat{\xi}}(y) + p^{\hat{\xi}}(y)}{y} = \frac{1-\bar{\pi}}{r-\mu} + \frac{\bar{\pi}c}{ry} - \left(\frac{1-\bar{\pi}}{r-\mu} + \frac{\bar{\pi}c}{ry_b^{\hat{\xi}}}\right)\left(\frac{y}{y_b^{\hat{\xi}}}\right)^{-\gamma^{\hat{\xi}} - 1}.$$
Figure 3: Equity Values for Alternative Debt Issuance Policies
Parameters are $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\bar{\pi} = 35\%$.

3.5. Model Comparisons and Implications

In this section we illustrate the implications of our model by comparing the no commitment solution to other benchmarks with commitment.

Value Functions and Debt Issuance Policy

Figure 3 plots the equity value and debt issuance policies for the three models: the base model without commitment ($g = g^*$, solid thick line), full commitment to no future debt ($g = 0$, dashed thin line), and Leland (1998) commitment to a fixed face value ($g = \xi$, dash-dotted thin line).
As explained, the equity value in the no-commitment case coincides with the setting when there are no future debt issues. With a fixed face value policy, the equity value is lower when cash flows are low, as the firm is committed to continuing to issue debt even in the face of large rollover losses. This effect gives rise to a higher default boundary \( y_b^5 \) than the default boundary in the other two cases \( y_b^0 \), as indicated in the top plot of Figure 3. However, \( y_b^5 < y_b \) could potentially occur, especially when the tax benefit \( \pi \) is high.\(^{10}\) On the other hand, when cash flows are high, the equity value in the fixed face value case is higher than that in either case with \( g = 0 \) or \( g = g^* \). Relative to the \( g = 0 \) case where tax benefits are lost as debt matures, the firm in the fixed face value case maintains its debt and so enjoys greater tax benefits. On the other hand, the firm in the fixed face value case commits to a debt policy that is much less aggressive than the no-commitment case, hence incurring a much lower bankruptcy cost.

Figure 4 illustrates the debt price and valuation multiple for each policy. Not surprisingly, as shown in Proposition 5, the debt price with “no future debt issuance” \( g = 0 \) dominates that without commitment, simply because future debt issuance pushes the firm closer to the default boundary. This also explains why in the bottom panel, the TEV multiple without commitment is always lower than that under commitment of \( g = 0 \) (recall equity values are the same under these two cases).

From (36), we see that the debt price premium due to “no future debt issuance” grows with the firm’s distance to default. This implies that the debt of firms that cannot commit will exhibit large credit spreads even when the firm’s current total leverage is very low (but the future leverage might be high). In fact, even for almost zero current leverage, the credit spreads for firms without commitment are non-zero. In contrast, the credit spreads for almost zero leverage firms are zero for the other two benchmark cases.

\(^{10}\) Recall \( y_b \) is the default policy as if equity holder obtains no tax benefit, while for \( y_b^5 \), the firm with fixed debt face value indeed captures some tax benefit (hence, a high \( \pi \) pushes equity holders to default later).
Relative to our base case, the fixed face value (Leland 1998) case generates a lower debt price for low $y$ but higher debt price for high $y$. This is due to the endogenous issuance policy $g^*$ plotted in Figure 3. There, we observe that the debt issuance policy without commitment is increasing in $y$, and slower (faster) than the fixed face value policy when $y$ is low (high), and investors price the debt in anticipation of these future leverage polices.

The next proposition summarizes the comparison of debt values across three models, depending on the firm’s profitability state $y$. Figure 4 corresponds to the case of $y^*_b > y_b$, so for sufficiently low $y$, the debt price in the case of fixed face value $p^*$ drops below the other two cases.
**Proposition 6.** We always have \( p(y) < p^0(y) \). For \( y \to \infty \) we have

\[
p(y) < p^\xi(y) < p^0(y)
\]

For sufficiently low \( y \) so that \( y \to \min\left(y_b, y_b^\xi\right) \), we have

\[
p^\xi(y) < p(y) < p^0(y) \text{ if } y_b^\xi > y_b
\]

\[
p(y) < p^0(y) < p^\xi(y) \text{ if } y_b^\xi < y_b
\]

**Proof:** \( p(y) < p^0(y) \) is implied by (36). When \( y \to \infty \),

\[
p^\xi(y) = \frac{c + \xi}{r + \xi} \left(1 - \left(\frac{y}{y_b}\right)^{-\gamma^\xi_p}\right) \to \frac{c + \xi}{r + \xi}
\]

which exceeds

\[
p(y) = \frac{(1-\pi)c + \xi}{r + \xi} \left(1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right) \to \frac{(1-\pi)c + \xi}{r + \xi}
\]

To show that \( p^\xi(y) < p^0(y) \), we need to show that

\[
\left(\frac{y}{y_b}\right)^{-\gamma^\xi_p} < \left(\frac{y}{y_b}\right)^{-\gamma}
\]

holds when \( y \to \infty \), which is equivalent to \( \gamma^\xi_p < \gamma \); but the latter holds by comparing (39) and (23). The second part of result is obvious as the debt price drops to zero at the default boundary.

Finally, as indicated in (28), in our no commitment case the firm’s TEV multiple is strictly the scaled cash flow \( y \). Consequently, holding the level of cash flows \( Y \) fixed, total firm value decreases with the debt face value \( F \). In other words, in the no commitment equilibrium there is always a loss to total firm value from leverage – the tax benefits of debt more than offset the resulting bankruptcy costs due to future debt increases. In contrast, TEV multiples in both commitment cases have an interior maximum. This interior maximum is corresponding to the “optimal” leverage in the traditional trade-off theory.

**Leverage Dynamics**

In two benchmark cases with commitment, the scaled cash-flows follow a geometric Brownian motion with exogenous drifts, i.e.,

\[
dy_t/y_t = (\mu + \xi) dt + \sigma dZ_t
\]

for the case of no future debt issuance, and

\[
dy_t/y_t = \mu dt + \sigma dZ_t
\]

for the fixed face value case. In our model with no commitment, the equilibrium evolution of the firm’s scaled cash-flows is:
Figure 5: Cash-flows and leverage dynamics for three models. We fix the underlying cash-flow shocks \( \{dZ\} \). The left panel plots scaled cash-flow \( y_t \) while the right panel plots the market leverage \( p_t/(v_t + p_t) \). The firm in the fixed face value (Leland 1998) setting (black, dashed line) defaults first when its \( y_t \) hits \( y_y^F \), and the firm with no commitment (blue, solid line) defaults later when its \( y_t \) hits \( y_y \). The firm who commits no future debt issuance (red, dotted line) survives in this sample path. Parameters are \( r = 5\% \), \( c = 8\% \), \( \xi = 0.1 \), \( \mu = 2\% \), \( \sigma = 25\% \), \( \pi = 35\% \).

\[
dy_t = \left( \mu + \xi - g^*(y_t) \right) y_t dt + \sigma y_t dZ_t = \left[ \mu + \xi - \frac{(r + \xi)\pi c y_t^\gamma}{\gamma \left( (1 - \pi)c + \xi \right) y_y^\gamma} \right] y_t dt + \sigma y_t dZ_t. \tag{43}
\]

The equilibrium debt issuance policy \( g^*(y_t) \) in (27) is increasing in \( y_t \), implying that \( y_t \) grows slower when \( y_t \) is higher. In fact, the firm’s scaled cash-flows are mean-reverting towards the steady-state value (recall the definition of \( \hat{y}_g \) in (29)).

\[
\hat{y}_{g=\mu+\xi} \equiv y_y \left( \frac{\gamma (\mu + \xi)(c(1 - \pi) + \xi)}{(r + \xi)\pi c} \right)^{1/\gamma} \tag{44}
\]

We are interested in the firm leverage dynamics implied by three different models. For given underlying cash-flow shocks \( \{dZ_t\} \), the left panel of Figure 5 plots the dynamics of scaled cash-flow \( y_t \), which tracks one-to-one to the firm’s interest-coverage-ratio (or book leverage);

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11 Strictly speaking, to ensure \( y_t \) to mean revert over its equilibrium region \( y_t \in [y_y, \infty) \), one has to show that \( \hat{y}_{\mu+\xi} > y_y \) so that the drift of \( y_t \) is positive when \( y_t = y_y \), which is indeed the case for our baseline parameters. However, \( \hat{y}_{\mu+\xi} < y_y \) could occur for sufficiently large \( \sigma \) (so \( \gamma \to 0 \) in (44)).
the right panel plots the dynamics of the firm market leverage, defined as \( \frac{p_t}{p_t + v_t} \). The firm defaults whenever its market leverage hits 100%. Because the underlying shocks are the same, the differences across these three different models are purely due to their different debt issuance policies.

In this sample path, the firm starts with positive shocks early on, suffers a sequence of adverse events around year 5, and experiences another round of up-and-down shocks at year 8. In response to negative shocks in early years, the firm in our baseline no commitment case issues less bonds compared to the fixed face value case which commits to \( g = \xi \). As a result, \( y_t \) in the no commitment case (blue solid line) has a larger upward drift and thus defaults later around year 10, compared to the fixed face value case (black, dashed line) in which the firm default around year 8. In contrast, the firm that commits not to issue future debt \( (g = 0, \text{red dotted line}) \) has the highest upward drift, and as a result it survives throughout our sample path.

### 3.6. Debt Maturity Structure

In our model, the firm commits to a constant debt maturity structure, i.e., all debt has an expected maturity of \( 1/\xi \). This assumption is common in much of the dynamic capital structure literature which treats the debt maturity structure as a parameter.\(^{12}\) It is beyond the scope of this paper to relax the full commitment assumption on the firm’s debt maturity structure policy; for some recent research, see He and Milbradt (2016).

We can consider, however, the consequences of the initial maturity choice. What if the firm can choose the debt maturity \( \xi \) to commit to for the future? In this section, we first show if the firm does not need to borrow a fixed amount upfront, then it is indifferent between all debt maturity choices. We then show that if the firm is restricted to borrow a fixed amount initially, then debt with shortest maturity becomes optimal. Finally, we analyze the role of leverage commitment in when firms are taking ultra-short-term debt which matures instantaneously.

\(^{12}\) To mention a few, Leland (1998), Leland and Toft (1996), He and Xiong (2012), and Diamond and He (2014).
Optimal Debt Maturity without Fixed Borrowing

We have seen that from (28) that the firm’s TEV multiple is decreasing in the scaled cash-flows \( y = Y / F \). Consequently, value-maximizing firms should set the optimal initial debt face value to be \( F_0^* = 0 \). Given this choice, (28) implies that the firm’s TEV multiple no longer depends on the debt maturity structure \( \xi \). This indifference result is deeper than it appears: Although the firm starts with no initial debt, recall that (32) says the firm will issue some positive amount of future debt, and the debt maturity \( \xi \) does affect those future debt contracts. Nevertheless, in our model this dynamic consideration has no bite on the optimal maturity choice by equity holders.

In fact, this irrelevance result is fairly general in our model. Imagine the following thought experiment, in which equity holders -- facing the state pair \((Y_t, F_t)\) and the existing debt maturity structure \( \xi \) -- are offered with a one-time chance of choosing \( \xi' \) for the firm’s future debt. That is, the firm’s existing debts continue to retire at the old speed \( \xi \), but the newly issued debts are with the new maturity and hence will retire at the new speed \( \xi' \). The following proposition shows that equity holders are indifferent among all choices of \( \xi' \).

**Proposition 7.** Focus on the equilibrium with smooth debt issuance policies. The equity value is independent of \( \xi' \) which is the maturity structure of the firm’s new debt.

**Proof:** For equilibria in which equity holders are taking smooth debt issuance polices, equity holders obtain zero profit by issuing future debt, and their value will be the same as if equity does not issue any future debt. As a result, the equity value depends on the maturity structure \( \xi \) of existing debt, but not on the maturity structure \( \xi' \) of future debt. ■

Optimal Debt Maturity with Fixed Borrowing

Now consider the situation in which the firm aims to raise an exogenously given target level of debt (in terms of market value), but can pick the optimal debt maturity structure. We show that short-term debt maximizes not only the firm value, but also the debt capacity, i.e., the maximum amount of debt that the firm can raise.

Our model highlights one advantage of short-term debt which allows firms to adjust their leverage burden in response to fundamental shocks in a more flexible way. This point has been
neglected in the Leland-style literature which often assumes the firm is committing to a future leverage policy with fixed debt face value. For instance, He and Xiong (2012) show that the longest possible debt maturity structure minimizes the rollover risk. As we will explain, the difference is driven by different assumptions on leverage policies.

Given initial cash-flow $Y_0$, the firm sets the initial debt face value $F_0$ to raise $D_0$ from debt holders. From (28) we know that the firm value is (recall $y_0 = Y_0 / F_0$)

$$Y_0 \frac{v(y_0) + p(y_0)}{y_0} = Y_0 \frac{1 - \frac{\pi}{r - \mu}}{r - \mu} \left[ 1 - \left( \frac{y}{y_0} \right)^{r-1} \right] < Y_0 \frac{1 - \frac{\pi}{r - \mu}}{r - \mu}$$

Hence, both the firm value $TEV_0$ and the debt value $D_0$ cannot exceed the upper bound $Y_0 \frac{1 - \frac{\pi}{r - \mu}}{r - \mu}$.

**PROPOSITION 8.** We have the following two results.

i) For any target debt value $D_0 < Y_0 \frac{1 - \frac{\pi}{x}}{r - \mu}$, the optimal debt maturity structure that maximizes the levered firm value (and hence the equity value) is $\xi^* = \infty$.

ii) The debt capacity $\sup D_0$, which is the highest debt value that the firm is able to raise, equals $Y_0 \frac{1 - \frac{\pi}{r - \mu}}{r - \mu}$ by setting $F_0 \uparrow Y_0 / y_b$ and $\xi^* \to \infty$.

**PROOF:** The first claim follows by show that $\xi = \infty$ always achieves the upper bound of $Y_0 \frac{1 - \frac{\pi}{r - \mu}}{r - \mu}$. Because

$$\gamma = \frac{\mu + \xi - 0.5\sigma^2 + \sqrt{(0.5\sigma^2 - \mu - \xi)^2 + 2\sigma^2 (r + \xi)}}{\sigma^2} \to \infty$$

so $(y / y_b)^{r-1}$ vanishes in (45) as $\xi \to \infty$, which proves the claim. For the second claim, from (26) we have

$$D_0 = F_0 p \left( \frac{Y_0}{F_0} \right) = F_0 \frac{c(1 - \frac{\pi}{x} + \xi)}{r + \xi} \left[ 1 - \left( \frac{y_0}{y_b} \right)^{r-1} \right]$$
By setting $\xi \to \infty$ the term in the parentheses vanishes, while $F_0 \uparrow \frac{Y_0}{y_b}$ delivers the upper bound debt value $D_0 \uparrow Y_0 \frac{1-\overline{\pi}}{r-\mu}$.

Figure 6 illustrates Proposition 8 by plotting the firm value $TEV_0$ (left panel) and debt value $D_0$ (right panel), both as a function of face value $F_0$. We consider three debt maturities: long-term debt with a 100-year maturity ($\xi = 0.01$, dash-dotted), medium-term debt with a 10-year maturity ($\xi = 0.1$, dashed), and short-term debt with 3-day maturity ($\xi = 100$, solid). The left panel shows that the firm value is maximized by using 3-day maturity debt. This is simply because $\xi \to \infty$ implies that $\gamma \to \infty$, and hence for any $y > y_b$ the firm value achieves its upper bound $Y_0 \frac{1-\overline{\pi}}{r-\mu}$ in (45). Of course, a too high $F_0$ pushes the scaled cash-flow $y_0$ below $y_b$, triggering default---and the firm value drops to zero. Although firms with the shortest-term debt---with the highest default threshold $y_b$---face the tightest constraint in setting a high $F_0$,\textsuperscript{13} the

\textsuperscript{13} Under a mild sufficient condition $r > c(1-\overline{\pi})$, one can show that the firm defaults earlier with shorter-term debt, i.e., default threshold $y_b$ in (24) is increasing in debt maturity $\xi$. This is a general result in Leland-style models.

---

\textbf{Figure 6.} Firm value and debt value as a function of initial debt face value $F_0$, for three levels of debt retiring rate $\xi$'s. Blue solid line is for shortest debt maturity $1/\xi = 0.01$, red dash-dotted line is for medium debt maturity $1/\xi = 10$, and black dashed line is for longest debt maturity $1/\xi = 100$. Both firm and debt values hit zero when $F_0 = Y_0/y_b$. Parameters are $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\overline{\pi} = 35\%$. 
right panel of Figure 6 shows that they achieve the highest market value of debt, thanks to the “Laffer” curve effect: Because long-term debt makes it difficult for the firm to reduce leverage in response to shocks, at some point an increase in the face value is more than offset by the increase in credit risk. As shown, the upper bound of \( D_0, \frac{1 - \pi}{r - \mu} \), is achieved by setting \( F_0 \) at (slightly below) the default boundary for firms using 3-day debts.

Our model differs from Leland type models in the firm’s future leverage policy. In both settings, the debt maturity structure \( \xi \) captures the speed of debt retiring. In Leland, committing to refinance those retiring bonds (to maintain the aggregate debt face at a constant) leads equity to bear rollover gains/losses \( \xi \left( p^\xi (y) - 1 \right) \) -- see Equation (40) -- and hence the rollover risk. The higher the \( \xi \), the stronger the rollover risk.\(^{14}\) To mitigate this rollover risk, a value-maximizing firm will set \( \xi = 0 \), corresponding to a consol bond that is free of rollover concerns.

When the firm can change its future debt burden freely, there emerges an important benefit of short-term debt. Given that short-term debt retires quickly, it allows the firm to swiftly adjust its leverage in response to profitability shocks. This result follows from the firm’s mean-reverting leverage dynamics illustrated in Section 3.3, as (27) implies that the firm issues more (less) debt following positive (negative) performance.

It is worth noting that, typically, the flexibility brought on by short-term debt comes with a potential cost of lack of commitment;\(^{15}\) in our particular setting the former benefit dominates the latter cost. We also caution that PROPOSITION 8 heavily relies on a strong, and perhaps counterfactual, assumption that underlies all of these models: the firm is facing a frictionless equity market through which equity could inject liquidity at any time in a costless way. Following a sequence of negative shocks, to repay the mounting debts that are maturing instantaneously, firms issue equity as needed. We expect that modelling an equity market with

\(^{14}\) The rollover risk exacerbates inefficient default in the Leland framework (He and Xiong, 2012; Diamond and He, 2014). To see this, during bad times the falling debt price \( p^\xi \) imposes heavy rollover losses on equity holders, leading to a lower option value of keeping the firm alive and hence early default.

\(^{15}\) For instance, long-term debt, because of its slow retiring speed, could potentially serve as a commitment device of leverage policy. He and Milbradt (2016) study a firm who cannot commit to debt maturity structure, and gives an example where the cost of lacking commitment dominates the benefit of flexibility.
realistic frictions will change many of these qualitative results, an interesting question left for future research.

**Ultra-Short-term Debt and Commitment**

The previous section shows that in our model it is optimal to set the debt maturity to be ultra-short-term, so that the debt matures instantaneously every \( dt \) (much like demand deposits).

In the literature, some papers (e.g., Tserlukievich, 2008) suggest that when the firm can adjust its leverage freely in response to the cash-flows shocks, then it is optimal to set \( F_t \approx Y_t / c \) always so that the firm avoids default while capturing the entire debt tax shield. In particular, it seems that ultra-short-term debt, which at any moment matures entirely and allows the firm to reset its leverage to its favorable level in response to cash-flow shocks, can achieve this goal. Essentially, this captures the flexibility advantage offered by short-term debt.

Although the flexibility benefit does apply in our model, the above argument implicitly assumes that equity holders can commit to the first best leverage policy. In our model, the inability to commit to certain future leverage policy matters in a significant way – equity holders continue to raise debt until the likelihood of default impacts its price. In the limit, even with instantaneously maturing debt, there is always a risk of bankruptcy in our model, and the implied bankruptcy cost just offsets the tax benefit.

**4. Extensions**

We consider two extensions of our baseline model in this section. The first extension considers the realistic tax policy under which only profitable firms receive a tax benefit. The second extension allows the equity holders to choose investment so that the firm’s cash-flows evolve endogenously.

**4.1. Increasing Tax Rate**

We have so far assumed that the firm receives a tax benefit even if the firm incurs losses. Suppose instead, as in practice, that there is no tax benefit of leverage when the firm has negative earnings. Specifically, let \( \pi(y - c) = \bar{\pi} \times \max(y - c, 0) \), so that:
\[
\pi'(y-c) = \begin{cases} 
\bar{\pi} > 0 & \text{if } y > c \\
0 & \text{if } y < c.
\end{cases}
\]

Clearly, this loss of tax benefits has no impact on the results of the prior section, if the firm chooses to default before earnings become negative. That is, if
\[
y_b = \gamma \frac{r - \mu}{1 + \gamma} c + \frac{r - \mu}{1 + \gamma} \left[ c + \frac{\xi}{1 - \bar{\pi}} \right] \geq c \Leftrightarrow \xi \geq \frac{c(1 - \bar{\pi})}{\eta - 1},
\]
then the equilibrium is as described in **Proposition 3**.

When (46) fails, we can rewrite the equation for equity value (20) as
\[
(r + \xi) v(y) = \begin{cases} 
(y-c)(1-\bar{\pi})-\xi+(\mu+\xi) yv'(y)+\frac{1}{2} \sigma^2 y^2 v''(y) & \text{if } y > c \\
(y-c)-\xi+(\mu+\xi) yv'(y)+\frac{1}{2} \sigma^2 y^2 v''(y) & \text{if } y < c
\end{cases}.
\]

It is easiest to analyze the solution for each region separately. When \(y > c\), the limiting value (21) still applies. Given the to-be-determined value \(v(c)\) at the boundary of the tax regime, the equity value when profits are positive is given by (recall \(\bar{v}(y)\) is given in (21)):
\[
v^+(y) = \bar{v}(y) + (v(c) - \bar{v}(c)) \left( \frac{y}{c} \right)^{\gamma}.
\]

On the other hand, when \(y < c\), we have the general form of the solution:
\[
v^-(y) = \bar{v}_0(y) + a_1 \left( \frac{y}{y_b} \right)^{\gamma} + a_2 \left( \frac{y}{c} \right)^{\eta},
\]
where \(\bar{v}_0(y)\) is given by (21) with a zero tax rate (i.e., setting \(\bar{\pi} = 0\) in (21)), and \(\eta\) is given by: \(^{16}\)
\[
\eta = \left( \frac{0.5 \sigma^2 - \mu - \xi}{\sigma^2} \right) + \frac{\sqrt{(0.5 \sigma^2 - \mu - \xi)^2 + 2 \sigma^2 (r + \xi)}}{\sigma^2} > 1.
\]

We then solve for four unknowns \((a_1, a_2, y_b, v(c))\) using the following boundary conditions:

\(^{16}\) Here, one can show that \(r > \mu\) implies that \(\eta > 1\). Recall we impose \(r > \mu\) throughout to ensure that the firm’s discounted present value of cash flows is finite.
1. Optimal default: \( v^-(y_b) = v'^-(y_b) = 0 \), and

2. Value matching and smooth pasting: \( v^-(c) = v'^-(c) \), and \( v''(c) = v''(c) \).

The following lemma gives the details of constructed equity value function.

**Proposition 9.** Suppose there is no tax credit in the event of operating losses, and 
\( \xi < \frac{c(1-\pi)}{\eta-1} \). Then the equity value function is twice differentiable, and given by (48) and (49) with the following parameters:

\[
a_1 = \frac{1-\eta}{\gamma + \eta} \left( \frac{y_b}{r-\mu} \right) + \frac{\eta}{\gamma + \eta} \left( \frac{c+\xi}{r+\xi} \right) > 0, \text{ and } \]

\[
a_2 = -\frac{\pi(1+\gamma)c}{(\gamma+\eta)(r-\mu)\eta} < 0, \quad (51)
\]

where the default boundary \( y_b \) is the unique solution in \((0, c)\) of

\[
\frac{\pi c}{\eta} \left( \frac{y_b}{c} \right)^\eta - y_b + \frac{\gamma}{1+\gamma} \frac{c+\xi}{r+\xi} (r-\mu) = 0. \quad (52)
\]

**Proof.** Value matching implies

\[
v^-(c) = v_0'(c) + a_1 \left( \frac{c}{y_b} \right)^{-\gamma} + a_2 = v'^-(c) = v(c),
\]

while smooth pasting implies

\[
v''(c) = v_0''(c) - \gamma a_1 \left( \frac{c}{y_b} \right)^{-\gamma} \left( a_1 \left( \frac{c}{y_b} \right)^{-\gamma} + a_2 \right) + \gamma v'(c) = v''(c) = v'(c).
\]

Combining these,

\[
c v_0'(c) - \gamma a_1 \left( \frac{c}{y_b} \right)^{-\gamma} + \eta a_2 = c v'(c) + \gamma v(c) - \gamma v_0'(c) + a_1 \left( \frac{c}{y_b} \right)^{-\gamma} + a_2
\]

which we can solve for \( a_2 \):
\[ a_2 = \frac{c(\bar{v}'(c) - \bar{v}_0'(c)) + \gamma(\bar{v}(c) - \bar{v}_0(c))}{\gamma + \eta} = \frac{\pi c}{\eta} \left( \frac{\gamma}{r + \xi} - \frac{\gamma + 1}{r - \mu} \right). \]

The default conditions imply

\[ v^{-}(y_b) = 0 = \bar{v}_0(y_b) + a_1 + d_2 \left( \frac{y_b}{c} \right)^\eta \quad \text{and} \quad v^{-'}(y) = 0 = \frac{\gamma}{y_b} a_1 + \frac{\eta}{y_b} a_2 \left( \frac{y_b}{c} \right)^\eta. \]

Combining these we have

\[ a_1 = y_b \bar{v}_0'(y_b) - \eta \bar{v}_0(y_b) = \frac{1 - \eta}{\gamma + \eta} \left( \frac{y_b}{r - \mu} \right) + \frac{\eta}{\gamma + \eta} \left( \frac{c + \xi}{r + \xi} \right), \quad (53) \]

and also

\[ (\gamma + \eta) a_2 \left( \frac{y_b}{c} \right)^\eta + \frac{1 + \gamma}{r - \mu} y_b - \gamma \frac{c + \xi}{r + \xi} = 0. \quad (54) \]

Next, because \(-\gamma + \eta = 1 - \frac{2(\mu + \xi)}{\sigma^2}\), and \(\gamma \eta = \frac{2(r + \xi)}{\sigma^2}\), one can verify that

\[ \frac{\gamma}{r - \mu} \frac{r - \mu}{r + \xi} = \eta - \frac{1}{\eta}, \quad \text{and} \quad \frac{\gamma + 1}{r - \mu} - \frac{\gamma}{r + \xi} = \frac{1 + \gamma}{\eta(r - \mu)}. \quad (55) \]

These results imply that

\[ a_1 = \frac{1 - \eta}{\gamma + \eta} \left( \frac{y_b}{r - \mu} \right) + \frac{\eta}{\gamma + \eta} \left( \frac{1 - \eta}{r + \xi} \right) = \frac{\eta}{\gamma + \eta} \left( \frac{1 - \eta y_b}{r - \mu} + \frac{c + \xi}{r + \xi} \right) = \frac{\eta}{\gamma + \eta} \left( \frac{c + \xi - (1 + \gamma^\nu)y_b}{r - \mu} \right) > 0, \]

and

\[ a_2 = -\frac{\pi c}{(\gamma + \eta)(r - \mu)\eta} < 0. \]

We use this expression for \(a_2\) and (54) to derive (52). Also, the condition that

\[ y_b = \frac{\gamma}{1 + \gamma} \frac{r - \mu}{r + \xi} \left[ \frac{c + \xi}{1 - \pi} \right] < c \]

becomes
\[
\frac{\eta - 1}{\eta} \left[ c + \frac{\xi}{1-\pi} \right] < c \iff \frac{\xi(\eta - 1)}{1-\pi} < c.
\] (56)

We now show that there exists a positive solution for (52) which takes value in the interval \((0, c)\) if (56) holds. Denote \(y_b/c = x\) then the equation we are solving is

\[
Q(x) \equiv \frac{(1+\gamma)c}{r-\mu} \left[ -\frac{\pi x^n}{\eta} + x \right] - \gamma \frac{c + \xi}{r + \xi} = 0.
\]

Clearly, \(Q(0) < 0\). Evaluating \(Q(x)\) at \(x = 1\) (i.e., \(y_b = c\)), we have the following result, where we have used (55) in the second equality and (56) in the third equality:

\[
Q(1) = \frac{(1+\gamma)c}{r-\mu} \left[ -\frac{\pi + 1}{\eta} \right] - \gamma \frac{c + \xi}{r + \xi} = \frac{(1+\gamma)}{r-\mu} \left[ -\frac{\pi c}{\eta} + c - \frac{\eta - 1}{\eta} (c + \xi) \right]
= \frac{1+\gamma}{(1-\pi)(r-\mu)} \left[ (1-\pi)c - (\eta - 1)\xi \right] > 0.
\]

Finally, we show the equity value is twice differentiable and convex. Note first that

\[
\frac{\sigma^2 y^2}{2} v''''(y_b) = c + \xi - y_b > 0.
\]

Next, using (47), note that

\[
\frac{1}{2} \sigma^2 y^2 v''''(y) = -(y-c)(1-\pi) + \xi - (\mu + \xi) y v'''(y) + (r + \xi) v''(y),
\]

\[
\frac{1}{2} \sigma^2 y^2 v''''(y) = -(y-c) + \xi - (\mu + \xi) y v'(y) + (r + \xi) v(y).
\]

When \(y = c\), the smooth pasting and value matching condition implies that \(v''''(c) = v''''(c)\).

With the constructed equity value function \(v(y)\) in hand, one can further construct other equilibrium objects following the methodology of Proposition 3. However, depending on whether the constructed equity value is convex or not (which ultimately depends on parameter values), the resulting equilibrium may be in different nature.
Equilibrium with Smooth Issuance Policy

Importantly, the procedure outlined above works if the constructed equity value function \( v(y) \) is convex, as required by \textbf{Proposition 1}. In this case, the assumed interior smooth debt issuance policy \( g(y) \) is indeed optimal. From inspecting (48), because \( \bar{v}(y) \) is linear, the constructed equity value \( v^+(y) \) in the region of positive earnings is convex if and only if its value evaluated at zero profit point \( y = c \) dominates the corresponding value without default option, i.e.,

\[ v(c) \geq \bar{v}(c). \]  

(57)

One can further show that \( v(y) \) is indeed globally convex if and only if the condition (57) holds:

\textbf{Proposition 10.} Suppose there is no tax credit in the event of operating losses.

(i) When \( \xi \geq \frac{c(1 - \bar{\pi})}{\eta - 1} \), the equilibrium is the same as the one constructed in \textbf{Proposition 3}.

(ii) When \( \xi < \frac{c(1 - \bar{\pi})}{\eta - 1} \), then under the condition (57), the equity value function \( v(y) \) characterized in \textbf{Proposition 9} is convex, and the equilibrium debt issuance policy is given by

\[ g^*(y) = \begin{cases} \frac{\pi c}{y^2 v^{++}(y)} = \frac{\pi c}{\gamma(\gamma + 1)(v(c) - \bar{v}(c))} \left(\frac{y}{c}\right)^\gamma & \text{if } y > c \\ 0 & \text{if } y \leq c \end{cases} \]  

(58)

with corresponding (continuous and differentiable) debt price

\[ p(y) = \begin{cases} yv'^+(y) - v^+(y) & \text{if } y > c \\ yv^-(y) - v^-(y) & \text{if } y \leq c \end{cases} \]

\textbf{Proof.} The first case i) is obvious. For the second case ii), note that at the cutoff point \( y = c \), \( v^{++}(c) > 0 \) is equivalent to \( v(c) > \bar{v}(c) \). To see this, \( v^{++}(c) \propto \gamma (1 + \gamma)(v(c) - \bar{v}(c))y^{-\gamma-2} > 0 \) as \( \gamma > 0 \) using (48). Now suppose counterfactually that we have \( v''(y) < 0 \) for some \( y \in [y_b, c] \); then it must be that there exists two roots for \( v''(y) = 0 \) for \( y \in [y_b, c] \). However,
\[
y^2v^{\prime\prime}(y) = \gamma(1 + \gamma) a_1 \left( \frac{y}{y_b} \right)^{-\gamma} + \eta(\eta - 1) a_2 \left( \frac{y}{c} \right)^\eta
\]

only admits at most one positive solution, contradiction.

Finally, the differentiability of \( p \) following from the fact that \( v^{\prime\prime}(c) = v^{\prime\prime}(c) \). Note that \( g^*(y) = 0 \) when \( y < c \) because \( \pi'(y - c) = 0 \). □

The equilibrium is similar in nature to the baseline model with constant tax rate, although there is one interesting feature worth highlighting. As (58) suggests, the firm’s issuance policy is “myopic,” in that the firm stops issuing any new debt whenever entering the loss region \( y \leq c \), i.e., whenever the instantaneous marginal debt tax subsidy disappears. This result might seem surprising since any newly issued debt (which is long-term in our model) would earn strictly positive tax benefits once the firm’s fundamentals improve and push it back to the profit region \( y > c \). However, the firm—given the optimal continuation policy in equilibrium—has zero marginal gain in the future even in the profit region, and hence the instantaneous tax subsidy is sufficient to capture the incentive to issue new debt.

The last point echoes the key theoretical observation that allows us to construct the equilibrium with a smooth debt issuance policy. That is, the equity value without commitment is identical to the hypothetical equity value if the firm is committed not to issue any future debt— as explained, all tax benefits are exhausted by the dead-weight bankruptcy cost. As shown shortly, when the condition (57) does not hold so that constructed equity value function fails to be globally convex, we can construct an equilibrium with discrete debt issuance policies, in which equity holders enjoy some strictly positive tax benefit.

**Equilibria with Discrete Issuance Policy**

**Proposition 1** focuses on the smooth issuance policy in which the firm gradually issues debt until the tax shield is exhausted, which is an equilibrium as long as \( v(c) > \overline{v}(c) \) holds. Because \( v(c) \) includes the value of equity’s default option (while \( \overline{v}(c) \) does not), this condition will hold when volatility is sufficiently high. But if volatility decreases to the point that \( v(c) = \overline{v}(c) \), the equity value \( v(y) = \overline{v}(y) \) becomes linear in (48) for \( y > c \), and the debt issuance speed in (58) becomes infinite.
This suggests that when condition (57) fails, the resulting equilibrium might feature a discrete jump in the debt issuance/repurchase policy, which is in a different nature from the one we have focused so far with smooth gradual adjustment. The above observation suggests an equilibrium of the following form: Whenever \( y > c \), the firm immediately increases leverage by the discrete amount \( dF_i \) to the point \( y_i = \frac{Y_i}{F_i + dF_i} = c \), and does not actively adjust leverage at all (i.e., \( dF_i = 0 \)) when \( y < c \).

The desirability of this policy is that equity holders actively keep a sufficiently high leverage for the firm so that its taxable income is always negative, hence paying no taxes at all. Equity holders are absorbing negative cash flows \( (y_i - c - \xi)dt < 0 \) all the way, but receive debt issuance proceeds \( p(c) \cdot dF_i = [c v'(c) - v(c)] \cdot dF_i \), occasionally whenever \( y_i \) hits \( c \) from below. This policy is particularly attractive for the low volatility case, as in this situation equity holders are less concerned about bankruptcy associated with high (book) leverage.

It turns out that our model in general admits multiple equilibria, as this class of equilibria may also exist even when the condition (57) holds. We now formally characterize this class of equilibria with the above discrete debt issuance policy:

\[
dF_i = \begin{cases} \frac{Y_i}{c} - F_i & \text{if } y > c, \\ 0 & \text{if } y \leq c. \end{cases}
\]  

(59)

The corresponding equity value function is given by

\[
v(y) = \begin{cases} v(c) + v'(c) \left( y - c \right) & \text{if } y \geq c, \\ \nu_0(y) + a_1 \left( \frac{y}{y_b} \right)^\gamma + a_2 \left( \frac{y}{c} \right)^\eta & \text{if } y \leq c, \end{cases}
\]

(60)

which implies the debt price \( p(y) = yv'(y) - v(y) \). Here, \( a_1 \) is given by (50), \( a_2 \) is given by

\[
a_2 = -\frac{a_1 \gamma (\gamma + 1)}{\eta (\eta - 1)} \left( \frac{c}{y_b} \right)^\gamma.
\]

(61)

The default boundary \( y_b \), if exists, solves the following nonlinear equation

\[ \frac{\nu_0(y) + a_1 \left( \frac{y}{y_b} \right)^\gamma + a_2 \left( \frac{y}{c} \right)^\eta}{\nu_0(y) + a_1 \left( \frac{y}{y_b} \right)^\gamma + a_2 \left( \frac{y}{c} \right)^\eta} = 0. \]
\[
\frac{\gamma (r - \mu)}{\eta - 1} \left( \frac{c + \xi}{c + \hat{\xi}} \right) \left( \frac{y_b}{c} \right)^{\gamma + \eta} - \frac{\gamma c}{\eta} \left( \frac{y_b}{c} \right)^{\gamma + \eta + 1} - y_b + \frac{\gamma (r - \mu)}{1 + \gamma} \frac{c + \hat{\xi}}{c + \hat{\xi}} = 0. \tag{62}
\]

Under some condition that can be easily verified ex post, the above procedure constructs an equilibrium with discrete debt issuance policies, as shown by the next proposition.

**Proposition 11.** Suppose there is no tax credit in the event of operating losses. For some real-valued solution \( y_b \) to (62), the above debt issuance policy (59) and value functions \( v(y) \) and \( p(y) \) in (60) constitute an equilibrium, if and only if the implied slope of equity value function evaluated at \( y = c \) satisfies \( v'(c) \in \left[ \frac{1 - \pi}{r - \mu}, \frac{1}{r - \mu} \right] \).

**Proof:** For \( y > c \), under the given policy debt will be immediately increased to the point \( y = Y / F = c \), that is, \( F = Y / c \). Therefore,

\[
V(Y, F) = V \left( Y, \frac{Y}{c} \right) + \left( \frac{Y}{c} - F \right) p(c),
\]

or equivalently (note that \( p(c) = cv'(c) - v(c) \))

\[
v(y) = (v(c) + p(c)) \frac{y}{c} - p(c) = v(c) + v'(c)(y - c).
\]

For \( y \in [y_b, c] \), we have the following HJB equations for the debt price and equity value:

\[
(r + \xi) p(y) = c + \xi + (\mu + \xi) yp'(y) + \frac{\sigma^2}{2} y^2 p''(y) \tag{64}
\]

\[
(r + \xi) v(y) = y - c - \xi + (\mu + \xi) yv'(y) + \frac{\sigma^2}{2} y^2 v''(y) \tag{65}
\]

with boundary conditions:

1. Value matching and smooth pasting at default: \( p(y_b) = v(y_b) = v'(y_b) = 0 \),

2. Smooth pasting at \( c \): \( p'(c) = 0, v'(c) = \left( v(c) + p(c) \right)/c \).
Note that if we set \( p(y) = yv'(y) - v(y) \), then \( p(y_v) = 0 \), \( v'(c) = \left( v(c) + p(c) \right)/c \), and differentiating (65), multiplying by \( y \), and subtracting (64), we obtain

\[
\frac{r + \xi}{p(y)} (yv'(y) - v(y)) = c + \xi + \left( \mu + \xi \right) \frac{y^2v''(y)}{yp'(y)} + \frac{\sigma^2}{2} \left( y^2v''(y) + y^3v'''(y) \right).
\]

Therefore, the only additional restriction imposed by the debt price is the smooth pasting condition \( p'(c) = 0 \), which is equivalent to the super-contact condition \( v''(c) = 0 \). Thus we have reduced the problem to solving (65) with boundary conditions \( v(y_b) = v'(y_b) = v''(c) = 0 \). More specifically, \( v(y_b) = v'(y_b) = 0 \) (see the derivation in (53) and (54) in the proof of Proposition 9) give us (50) and

\[
(y + \eta) a_2 \left( \frac{y_b}{c} \right)^\eta + \frac{(1 + \gamma)}{r - \mu} y_b - \gamma \frac{c + \xi}{r + \xi} = 0;
\]

and the condition \( v''(c) = 0 \) gives

\[
a_1 \gamma (\gamma + 1) \left( \frac{c}{y_b} \right)^\gamma + a_2 \eta (\eta - 1) = 0.
\]

Hence (61) follows from (67), and (62) follows from plugging in \( a_1, a_2 \) expressions in (66).

Now we prove that \( v(y) \) is convex for \( y \in \left[ y_b, c \right] \), which implies that it is optimal not to issue any debt in that region. Suppose counterfactually that we have \( v''''(y) < 0 \) for some \( \hat{y} \in \left[ y_b, c \right] \); then since \( v''''(y_v) > 0 \) it must be that there exists two roots for \( v''''(y) = 0 \) for \( y \in \left[ y_b, c \right] \) (one is \( c \) and another inside \( \left[ y_b, \hat{y} \right] \)). However,

\[
y^2v''''(y) = \gamma (1 + \gamma) a_1 \left( \frac{y}{y_b} \right)^\gamma + \eta (\eta - 1) a_2 \left( \frac{y}{c} \right)^\eta
\]

only admits at most one positive solution, contradiction.
We briefly verify the optimality of the policy in (59) given the value function, if
\[ v'(c) \in \left[ \frac{1-\pi}{r-\mu}, \frac{1}{r-\mu} \right] \] holds. Focus on the region \((Y, F)\) with \(y = Y / F > c\), and consider any policy \(dF \leq Y / c - F\). Since \(V(Y, F)\) in (63) is linear in \(Y\), we have \(V_{yF} = 0\), and the instantaneous gain is (the first equation uses (63))
\[
\begin{align*}
&\left[ Y dt - \bar{\pi}(Y - cF) dt - (c + \xi) F dt + dF \cdot p(c) + \xi F Y V_F(Y, F) dt - \left( V(Y, F) - V(Y, F - dF) \right) \right] + \mu V_{yF}(Y, F) dt - rV(Y, F) dt \\
&= Y dt - \bar{\pi}(Y - cF) dt - (c + \xi) F dt + \xi F Y V_F(Y, F) dt + \mu V_{yF}(Y, F) dt - rV(Y, F) dt \\
&= \left[ \left(1 - \bar{\pi}\right) \left(y - c\right) - \xi + (\mu + \xi) y v'(y) - (r + \xi) v(y) \right] F dt \\
&= \left[ \left(1 - \bar{\pi}\right) y - (r - \mu) v(y) - (1 - \bar{\pi}) c - \xi + (\mu + \xi) p(c) \right] F dt
\end{align*}
\]
Recall \(v(y)\) is linear with slope \(v'(c)\) in this region, and \(v'(c) > \frac{1-\pi}{r-\mu}\). As a result, to ensure the above instantaneous gain to be non-positive, we only need to check this condition for \(y = c\), i.e.
\[(1-\bar{\pi}) c - (r-\mu) v(c) - (1-\bar{\pi}) c - \xi + (\mu + \xi) p(c) = -(r-\mu) v(c) - \xi + (\mu + \xi) p(c) \leq 0\]
But this is just (65) with \(v''(c) = 0\) and \(p(c) = cv'(c) - v(c)\). For \(dF > Y / c - F\), the firm gets to the region of decreasing debt price, which is strictly less optimal as in Proposition 1.

Figure 7 illustrates the multiple equilibria in our model by plotting the slope of equity value \(v'(y)\) in upper panels, and debt price \(p(y)\) in lower panels. The left panels depict the case with a relatively high \(\sigma = 22.8\%\), so that condition (57) holds and the smooth equilibrium (solid line) exists. In contrast, the right panels have a relatively low \(\sigma = 18\%\) so that condition (57) fails. Graphically, there is a slight bump in \(v'(y)\) and \(p(y)\) right before \(y = c = 8\%\), and as a result the smooth issuance policy (solid line) does not constitute an equilibrium.

For the class of equilibria with discrete issuance, the model has two equilibria for the high \(\sigma\) case (left panel, dashed and dotted lines). They correspond to two solutions to the nonlinear equation (62), and each \(y_b\) yields an equity value slope \(v'(c)\) that lies in the interval
These two Pareto-ranked equilibria reflect an interesting coordination between equity and debt holders: equity holders will default earlier (a larger $y_b$) if they expect future debt prices to be low; but future debt prices will be indeed low if equity holders default earlier. As a result, self-fulfilling prophecy emerges, akin to Calvo (1998) and Cole and Kehoe (2000) in the sovereign debt literature.

Figure 7: **Multiple Equilibria.** Parameters are $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\pi = 35\%$.

For the low $\sigma$ case so that (57) fails, only the solution with a smaller $y_b$ (a higher debt price, dotted line) survives as the equilibrium. The earlier-default solution with a larger $y_b$ (dashed line) is not an equilibrium: as the top-right panel shows, $v'(c)$ lies below $\frac{1 - \pi}{r - \mu}$, violating
the condition in Proposition 11. Further, at the knife-edge case so that (57) holds exactly, this solution (dashed line) coincides with the solution with smooth issuance policy (solid lines).

We can further show that the equity value in equilibrium with discrete issuance policies strictly dominates the hypothetical equity value with the firm committing not to issue any future debt, which is also the equity value in the smooth equilibrium. Thus, in equilibria with discrete debt issuance, the tax savings from the equilibrium debt issuance policy more than offsets the increased default costs. Here, the restriction on tax subsidies for negative earnings actually allows the firm to gain the benefit from a positive (net) tax savings initially!

We can understand this result through the lens of commitment. The existence of no tax subsidy region $y < c$ provides a “partial commitment device” for the firm to credibly refrain itself from issuing any more debt, and as a result equity holders are able to extract a strictly positive gain. In contrast, when the firm gets tax subsidies always, the lack of this partial commitment device annihilates this class of equilibria with discrete debt issuance. It remains as open question whether there exist other Markov equilibria with discrete debt issuance policies and positive tax savings for the baseline constant tax rate case.

### 4.2. Endogenous Investment

**General analysis**

Now we allow equity holders to choose the firm’s endogenous investment policy $i$, which affects the drift of cash-flow process by $\mu(Y, i)$ at a cost of $K(Y, i)$. Both functions are smooth with $\mu_i(Y, i) > 0$, $K_i(Y, i) > 0$ and $K_{ii}(Y, i) > 0$.

For illustration purpose, we carry out the general analysis with investment under the general cash-flow diffusion process without jumps; the analysis for cash-flow process as in (1) with jumps is similar. We also take the simplifying assumption that

---

17 The underlying economics is that, when $v'(c) < (1 - \pi) / (r - \mu)$, then in the state $Y > cF_c$ equity holders can gain by not issuing the discrete amount of debt which brings the scaled cash-flows to $Y = c$. See the proof of Proposition 11 for details.

18 This explains the observation made in the beginning of this subsection, that the speed of debt issuance in (58) becomes infinite when (57) holds exactly.

19 In equilibria with discrete issuance policies, in general $v'(c) < (1 - \pi) / (r - \mu)$ holds strictly. In contrast, under the smooth equilibrium, (25) implies that $v'(y) \to \bar{v}'(y) = (1 - \pi) / (r - \mu)$ for sufficiently large $y$. 

The HJB equation for equity holders, who are choosing the firm’s debt issuance $G$ and investment $i$, can be written as

$$rV(Y, F) = (1 - \bar{\pi})Y - \left((1 - \bar{\pi})c + \xi\right)F + \text{required return after-tax cash-flows net coupon-principal payment}$$

$$\max_{G,i} \left[p(Y,F)G + (G - \xi F)\mu Y(Y,F) + \mu(Y,i)\mu Y(Y,F) + \frac{\sigma(Y)^2}{2}V_{yy}(Y,F) - K(Y,i)\right]$$

As before we focus on the equilibrium where $G$ takes interior solutions, which implies that $p = -V_F$, and we can solve the equity value function as if there is no issuance so $G = 0$. Suppress the arguments for $V(Y,F)$, equation (68) with $G = 0$ becomes

$$rV = \max_{i^*} \left[(1 - \bar{\pi})Y - \left((1 - \bar{\pi})c + \xi\right)F - \xi FV_F - K(Y,i) + \mu(Y,i)\mu Y(Y,F) + \frac{1}{2}\sigma(Y)^2V_{yy}\right]$$

The first-order condition for the optimal investment policy $i^*$ is

$$K_i(Y,i^*) = \mu_i(Y,i^*)V_Y$$

The underlying assumption that (70) characterizes the optimal investment policy $i^*$ is that $\mu(Y,i)V_Y - K(Y,i)$ is concave in $i$.

Plugging the optimal investment policy $i^*$ into (69), and taking derivative with respect to $F$ further, we have

$$rV_F = -\left((1 - \bar{\pi})c + \xi\right) - \xi FV_{FF} + \mu(Y,i^*)\mu Y_{YY} + \frac{1}{2}\sigma(Y)^2V_{yy}$$

As before, we have used the Envelope theorem that we can ignore the dependence of the optimal policy $i^*$ on $F$.\textsuperscript{20} Using $p = -V_F$, we have

$$-rp(Y,F) = -\left((1 - \bar{\pi})c + \xi\right) - \mu(Y,i^*)p_Y(Y,F) + \xi p(Y,F) + \xi Fp_F(Y,F) - \frac{\sigma(Y)^2}{2}p_{yy}(Y,F)$$

The valuation equation for debt price $p(Y,F)$ in (5), with the equilibrium evolution of state variables $(Y,F)$, is

---

\textsuperscript{20} The envelope theorem readily applies if $i^*$ takes interior solutions always. However, even if the optimal investment policy takes a binding solution, our logic goes through as long as the constraint is independent of $F$. 

45
\[ \frac{r p(Y, F)}{\text{required return}} = c + \xi (1 - p(Y, F)) + (G - \xi F) p_F(Y, F) + \mu(Y, i^*) p_F(Y, F) + \frac{\sigma(y^*)}{\kappa} p_{FF}(Y, F) \]  

(72)

Combining (71) and (72) gives rise to the exact same equilibrium debt issuance policy as in (12):

\[ G(Y, F) = \frac{\bar{\pi} c}{-p_F(Y, F)} = \frac{\bar{\pi} c}{V_{FF}(Y, F)} > 0. \]

Again, the equilibrium requires the condition in Proposition 1, i.e. \(-p_F(Y, F) = V_{FF}(Y, F) > 0\).

**Log-normal cash-flow process and quadratic investment cost**

Consider the setting with a log-normal cash-flow process studied in Section 3, with 
\[
\mu(Y, i) = (\mu + i) Y \quad \text{and} \quad K(Y, i) = 0.5 \kappa^2 Y.
\]

Here, the investment \(i\) increases the cash-flow growth rate linearly, and the cost is proportional to cash-flow size \(Y\) but quadratic in investment \(i\).

Without debt, our model is similar to Haryashi (1982), and the first-best investment policy \(i_{FB}\) is given by:

\[
i_{FB} = r - \mu - \sqrt{(\mu - r)^2 - 2(1 - \pi)/\kappa}.
\]

Now we derive the solution to our model with leverage without commitment. Denote the optimal investment rate by \(i^*\) so that the evolution of scaled cash-flow \(y_t = Y_t/F_t\) is

\[
\frac{dy_t}{y_t} = (\mu + i^*_t + \xi - g_t) dt + \sigma dZ_t.
\]

Equity holders default when \(y_t\) hits the endogenous default boundary \(y_b\). The scaled equity value \(v(y)\) without debt issuance satisfies

\[
(r + \xi) v(y) = \max_y (1 - \bar{\pi})(y - c) - \xi - \frac{\kappa^2}{2} y + (\mu + i + \xi) y v'(y) + \frac{1}{2} \sigma^2 y^2 v''(y).
\]

(73)

Given the optimal investment \(i^*(y) = v'(y)/\kappa\), the above equation becomes

\[
(r + \xi) v(y) = (1 - \bar{\pi})(y - c) - \xi + \frac{y[v'(y)]^2}{2\kappa} + (\mu + \xi) y v'(y) + \frac{1}{2} \sigma^2 y^2 v''(y),
\]

(74)
with two boundary conditions: \( v(y) = \kappa i_B y - \frac{(1 - \pi) c + \xi}{r + \xi} \) for \( y \to \infty \), and \( v(y_b) = 0 \). The default boundary \( y_b \) is determined by the smooth-pasting condition \( v'(y_b) = 0 \). The debt price is given by \( p(y) = yv'(y) - v(y) \), and the debt issuance policy is \( g(y) = \frac{\pi c}{yp'(y)} > 0 \).

Finally, we need to verify the key condition \( v''(y) > 0 \) (or equivalently \( p'(y) > 0 \)) in Proposition 1. Although we no longer have closed-form solution in the model with investment, in the next proposition we show that \( v''(y) > 0 \) holds by analyzing the ODE (74) satisfied by the equity value.

**Proposition 8.** In the log-normal cash-flow model with investment, equity value is strictly convex, i.e., \( v''(y) > 0 \), so that the debt price is decreasing in debt face value.

**Proof:** Define a constant \( B \equiv \kappa i_B \) and \( w(y) = v(y) - By + \frac{(1 - \pi) c + \xi}{r + \xi} \); then \( w(\cdot) \) is concave if and only if \( v(\cdot) \) is concave. Using (74) and \( v'(y) = w'(y) + B \), we have:

\[
(r + \xi) w(y) = (r + \xi) v(y) - (r + \xi) By + (1 - \pi) c + \xi \\
= (1 - \pi) y + \frac{y (w'(y) + B)^2}{2\kappa} + (\mu + \xi) y (w'(y) + B) + \frac{\sigma^2 y^2}{2} w''(y) - (r + \xi) By \\
= y \left[ 1 - \pi + \frac{B^2}{2\kappa} + (\mu + \xi) B - (r + \xi) B \right] + \frac{2Bw'(y) + (w'(y))^2}{2\kappa} + (\mu + \xi) yw'(y) + \frac{\sigma^2 y^2}{2} w''(y) \\
= yw'(y) \left[ \frac{B}{\kappa} + \mu + \xi \right] + y \left( \frac{w'(y))^2}{2\kappa} + \frac{\sigma^2 y^2}{2} w''(y) \right.
\]

Hence \( w(y) \) satisfies the following ODE:

\[
(r + \xi) w(y) = yw'(y) \left[ \frac{B}{\kappa} + \mu + \xi \right] + y \left( \frac{w'(y))^2}{2\kappa} + \frac{\sigma^2 y^2}{2} w''(y) \right.
\]

There are two steps to show that \( w''(y) > 0 \) for all \( y > y_b \).
**Step 1.** \(w(y) > 0\) for all \(y \geq y_b\). We know that at default \(w'(y_b) = v'(y_b) - B = -B < 0\), and \(w(\infty) = 0\). This implies that if \(w(y) \leq 0\) ever occurs, then the global minimum must be nonpositive and interior. Pick that global minimum point \(y_1\); we must have \(w'(y_1) = 0\) and \(w''(y_1) > 0\). Suppose that \(w(y_1) < 0\); evaluating (75) at \(y_1\), we find that the LHS is strictly negative while the RHS is positive, contradiction. Suppose that \(w(y_1) = 0\); then there must exist some local maximum point \(y_2 > y_1\), so that \(w(y_2) > 0\), \(w'(y_2) = 0\) and \(w''(y_2) < 0\). But the same argument of evaluating (75) at \(y_2\) leads to a contradiction.

**Step 2.** Because \(w(y)\) approaches 0 from above when \(y \to \infty\), we know that for \(y\) to be sufficiently large \(w(y)\) is convex. Suppose counterfactually that \(w(y)\) is not convex globally; we can take the largest inflection point \(y_2\) with \(w''(y_3) = 0\). We must have \(w'(y_3) < 0\) and \(w'''(y_3) > 0\) (it is because for \(y > y_1\) the function \(w(y)\) is convex and decreasing to zero from above). At this point, differentiate (75) and ignore the term with \(w''(y_3) = 0\), and we have

\[
\left(r - \mu - \frac{B}{\kappa}\right)w'(y_3) - \frac{\left(w'(y_3)\right)^2}{2\kappa} = \frac{\sigma^2 y_3^2}{2} w'''(y) .
\] (76)

Recall \(B = \kappa(r - \mu) - \sqrt{\kappa^2 (r - \mu)^2 - 2\kappa}\) which implies

\[
\left(r - \mu - \frac{B}{\kappa}\right)w'(y_3) = \sqrt{\kappa^2 (r - \mu)^2 - 2\kappa \cdot w'(y_3)} < 0
\]

As a result, the LHS of (76) is negative while the RHS of (76) is positive, contradiction. This implies that \(w(y)\) is convex globally. ■

Combining all the results above, we have shown that \(v''(y) > 0\) for \(y > y_b\). ■
Figure 6 plots equity value, investment rate, debt issuance, and firm TEV multiple in one numerical example. As expected, the well-known debt overhang effect (Myers 1977, Hennessey, 2004) causes investment to increase with the firm’s distance to default, while other objects exhibit similar qualitative features as the baseline model without investment. We also present results for two levels of debt maturities, one for long-term debt $\xi = 0.1$ and the other for short-term debt $\xi = 0.2$. As shown in Section 3.5, short-term debt is more efficient in delivering a higher firm value as well as a higher investment (when the firm sits on the right hand side of Laffer curve for debt price).
5. References

To be added.


6. Appendix.


Following Leland (1998), we can first solve for the scaled levered firm value $TEV$ as

$$
\frac{TEV}{F} = \left(1 - \pi \right) y + \frac{\pi c}{r} - \left( \frac{\pi c}{r} + \frac{1 - \pi}{r - \mu} \right) \left( \frac{y}{y_b^\xi} \right)^{-\gamma^\xi} \tag{77}
$$

where the constant $\gamma^\xi = \frac{\mu - 0.5\sigma^2 + \sqrt{(0.5\sigma^2 - \mu)^2 + 2\sigma^2 r}}{\sigma^2}$.

Then, the equity value $v^\xi(y) = \frac{TEV}{F} - p^\xi(y)$

equals:

$$
v^\xi(y) = \left(1 - \pi \right) y + \frac{\pi c}{r} - \left( \frac{\pi c}{r} + \frac{1 - \pi}{r - \mu} \right) \left( \frac{y}{y_b^\xi} \right)^{-\gamma^\xi} - c + \xi \left( \frac{y}{y_b^\xi} \right)^{-\gamma^\xi}\left(1 - \frac{y}{y_b^\xi} \right)^{-\gamma^\xi} \tag{78}
$$

7. Numerical Solution for Discrete Time Model

To be added.