Job Insecurity

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Abstract
This paper examines the effect of job insecurity on productivity. We study a fixed wage relationship between a firm and a worker in which neither knows how well-suited the worker is to the job. The worker decides the level of effort, a choice that affects both learning and the firm’s bottom line. The employer, seeing the worker’s effort choice and outcome, decides whether or not to continue employing the worker. The employer cannot commit to retain the worker when she becomes pessimistic enough about the match quality. We show that, rather than aligning interests, this threat creates a perverse incentive not to attract attention: the worker strategically slows learning, harming productivity. As the firm anticipates this, job insecurity can be a self-fulfilling prophecy. We explicitly characterize the unique Markov perfect equilibrium in our continuous time dynamic game. Consistent with empirical evidence in organizational psychology, equilibrium exhibits a U-shaped relationship between job insecurity and productivity: a worker is least productive when his job is moderately secure.

Keywords: job security, dynamic agency, career concerns, low-powered incentives.

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1 Introduction

A wide body of literature, from economics and management to organizational psychology, studies the effects of job insecurity on employee productivity. The results are ambiguous. Repenning (2000) alludes to two competing schools of thought regarding the consequences of job insecurity. The ‘drive in fear’ school argues that fear of job loss can boost productivity by motivating employees, while the ‘drive out fear’ school argues that such fear distorts decision-making and thus harms productivity.

In Only the Paranoid Survive, Intel CEO Andy Groves extols the virtues of fear in organizational decision-making:

The most important role of managers is to create an environment in which people are passionately dedicated to winning in the marketplace. Fear plays a major role in creating and maintaining such passion…Simply put, fear can be the opposite of complacency…A good dose of fear may help sharpen [one’s] survival instincts.

On the other hand, among W. Edwards Deming’s fourteen key management principles is:

Drive out fear. No one can put in his best performance unless he feels secure…A common denominator of fear in any form, anywhere, is loss from impaired performance and padded figures.

Fear has real negative consequences, Deming argues, like the employee who “[would] like to understand better the reasons for some of the company’s procedures, but [doesn’t] dare to ask about them.” In short, fear induces employees to be overly cautious, impeding their productivity.

Empirical work, by both economists and organizational psychologists, has also found ambiguous productivity effects of job insecurity. Some organizational psychologists find positive associations, some negative, and some a more complex relationship. At least one such study has found the relationship to be non-constant, even for a given worker. Indeed, Selenko et al. (2013) find that, empirically, the impact of job insecurity on performance is U-shaped: a worker is least productive when his job is moderately secure.

Given the above qualitatively different management styles—driving in fear versus driving out fear—and the lacking empirical consensus, our goal is to understand the relationship between job insecurity and productivity in an agency framework. The celebrated

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1See Deming (1986).
2See Probst et al. (2007), Brockner, Tyler, and Cooper-Schneider (1992), and Sverke and Hellgren (2001).
3See Gilboa et al. (2008), Roskies and Louis-Guerin (1990), and Reisel et al. (2010).
4Here, see Staufenbiel and König (2010).
Shapiro and Stiglitz (1984) model of equilibrium unemployment—with workers paid a fixed wage strictly higher than their outside options—showcases how the threat of firing successfully mitigates an agency problem, but that model is ill-suited for the task at hand. Indeed, in their model, all workers exert high effort at all times, and thus are never fired for poor performance. That is, agents’ productivity and job security are subject to no variation whatsoever.

Toward this goal, we provide a simple framework to analyze the dynamics of a fixed wage principal-agent relationship where the agent faces a firing threat. A key feature of our model is that employees are sometimes fired for poor performance, as past performance heralds future performance. In our model, job insecurity and productivity are endogenously codetermined: fear of job loss feeds into optimal effort decisions, and productivity feeds into optimal firing choices. As a result, we show, employee productivity and job insecurity will exhibit a U-shaped relationship.

Model: We study an infinite horizon, continuous time game between a principal (the firm, “she”) and an agent (the worker, “he”). At any given moment, the principal decides whether or not to continue employing the agent at a fixed flow cost. While still employed, the agent collects a fixed wage, and chooses an effort level. The flow of output is stochastically determined by the agent’s effort and a relationship-specific ‘match quality’ between the worker and the firm. The players are symmetrically uninformed about the match quality: they both learn from observing the output and effort. We study the set of Markov perfect equilibria of this game, using the players’ beliefs about match quality as the state.

The agent is indifferent to firm output and his own action. He simply wants to stay employed as long as possible. The principal cares about the firm’s profit, and so weighs her expectations of output against the cost of employing the agent. The firm finds it profitable to employ a worker if and only if the two are well-matched. Finally, higher effort on the part of the worker, a complement of match quality, generates a higher payoff.

An important feature of our model is that the agent’s effort choices are perfectly observable. This yields a more tractable model: no player ever has private information (on or off path), so that their beliefs about match quality always agree.

Results: In such a simple model, it is perhaps surprising that the relationship will feature any rich dynamics in a Markov perfect equilibrium. As we will see, even with no cost of effort, no hidden action, and no private information, a nontrivial agency problem arises. Our first economic result, Proposition 1, shows that in equilibrium, the principal’s value

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5According to the U.S. Bureau of Labor Statistics (see http://www.bls.gov/opub/reports/minimum-wage/archive/characteristics-of-minimum-wage-workers-2014.pdf), 58.7% of wage and salary workers are paid a fixed hourly wage. Thus, we focus on environments without incentive pay, just as Shapiro and Stiglitz (1984) do.

6To highlight the incentive effects of job insecurity, effort is costless to the agent in our benchmark model. As we describe in Section 5, costly effort does not qualitatively change our results.
is as if the agent always exerted minimum effort. This tells us that the agency problem is severe, but it does not tell us how our players actually behave.

Our main result, Theorem 1, fully describes equilibrium and shows it is unique. Equilibrium play is characterized by three regimes. When the players are pessimistic about match quality, the principal immediately fires the agent. When the players are optimistic about match quality, the principal continues to employ the agent. At such beliefs, the agent enjoys a sense of job security. Here, he seeks to ‘avoid making waves’ for fear of risking his high status in the firm. For moderate beliefs, the principal sporadically fires the agent, who in turn provides a higher level of effort than when his job is secure. As can be seen in Figure 1, the equilibrium exhibits discontinuous behavior on part of both the players.

![Graph of Principal's firing rate and Agent's effort vs. Beliefs]

Figure 1: Players’ strategies

The structure of our equilibrium generates an interesting relationship between job security and productivity, formalized in Corollary 1. Consider how productivity varies as beliefs about match quality (and, with them, the agent’s job security) increase. First, in
the probationary region\textsuperscript{7} the agent barely earns his keep, so that productivity is constant. As beliefs pass to the (barely) optimistic region, the agent discontinuously switches to minimum effort: he is now less productive. As beliefs grow further through the optimistic region, the agent’s effort remains constant, but his expected match quality with the firm increases: he is in expectation more productive. Thus, our equilibrium provides plausible mechanics that generate the U-shaped pattern suggested in the empirical organizational psychology literature.

Our equilibrium speaks to the relationship between job insecurity and productivity in two ways. First, we match the organizational psychology literature’s empirical findings, as above. Second, we clarify the causal link debated by management experts. Our middle

![Figure 2: Observed relationship between job security and productivity](image)

region, with intermittent firing and higher effort, tells us why we should ‘drive in fear’: the threat of firing the worker faces at his current status motivates him to work harder. Our optimistic region highlights the reason to ‘drive out fear’. In this region, the agent puts in low effort, but not because his job is safe. Rather, he deliberately slows the flow of information for fear of entering the more pessimistic regime: future job insecurity causes a form of complacency on the part of the agent.\textsuperscript{8}

In addition to our economic results, we develop some technical results of general interest for the study of dynamic games. In single agent settings, the vast literature on optimal control and optimal stopping\textsuperscript{9} yields powerful methods for understanding well-behaved

\textsuperscript{7}Note, in the pessimistic region, there is no job to lose.
\textsuperscript{8}Indeed, if the principal’s strategy were to employ the agent forever—which it could not be in equilibrium—the agent would willingly exert high effort in our setting.
\textsuperscript{9}See Pham (2009) for example.
dynamic optimization problems in continuous time. Standard results from this literature, however, rely on the decision-maker facing an environment which varies smoothly with the underlying state. In strategic settings, one player’s best response may change discontinuously with the state, and so we may be forced to study the resulting discontinuous environment faced by the other player. Some past work avoids this issue in modeling strategic settings with only one ‘large’ strategic player.\textsuperscript{10} In our setting, with multiple large players, we show in the appendix that many of these powerful tools can still be made useful to characterize equilibrium. In a similar vein, we provide a candidate definition for sequential rationality in continuous time games.\textsuperscript{11} By characterizing its behavioral implications in our game, we show that it is a useful refinement, reducing a continuum family of qualitatively similar equilibria to a unique equilibrium (which is Pareto optimal within the family).

The remainder of the paper proceeds as follows. In Section 2, we describe our game and define our equilibrium concept. Section 3 presents and discusses our main economic results. In Section 4, we describe our broader contribution to the study of dynamic games in continuous time. Section 5 details some extensions of our model, while Section 6 surveys some of the related literature. Proofs are in the appendix.

2 Our Model

2.1 The Environment

We study an infinite horizon, continuous time game between a principal (firm) and an agent (worker). At each instant, the principal decides whether or not to irreversibly fire the agent, while the agent decides on his effort conditional on not being fired.

The worker-firm match quality $\theta \in \{0, 1\}$ is symmetrically unknown to the players, with prior $\mathbb{E}_0 \theta = p_0 \in (0, 1)$. Irrespective of effort, the stochastic output is profitable on average if and only if the agent is well-matched with the firm (something neither the principal nor the agent knows). The agent’s effort, along with his productive outcomes, are public. The principal only finds it worthwhile to employ the agent if he will be productive enough. The agent wants to stay employed and collect his fixed wage, and so cares about what his efforts reveal; to isolate the effects of ‘signal shaping’ motives, we assume that effort is costless.

At each instant, the agent makes an effort choice $A_t \in [\bar{a}, \bar{a}]$. Over the course of $[t, t + dt)$, the principal observes the allocation choice as well as the resulting (stochastic)

\textsuperscript{10}This assumption is a natural one when one agent has all of the market power. For examples of such settings, see Daley and Green (2012) and Board and Meyer-ter Vehn (2013).

\textsuperscript{11}See Section 4 for more details.
output. The players update their beliefs about the match quality, taking into account the effort choice. The principal decides whether to continue the relationship or fire the agent.

We stress that, no matter how the agent allocates his energy, our two players always (on path and off path) have the same information about the match quality, so that there can be no private beliefs.

### 2.2 Payoffs

We model the firm’s flow profit as a diffusion with drift depending on both the match quality and the agent’s effort. The firm also incurs a flow cost $c$ of hiring the agent. Let $X_t$ denote the cumulative output (net of the cost of hiring the agent). The law of motion of $X$ is then:

$$dX_t = (\theta A_t - c) \, dt + \sigma \, dB_t,$$

where $\{B_t\}$ is a standard Brownian motion. The firm’s accrued payoff over $[t, t+dt]$ is $dX_t$ while employing the worker, and the worker’s accrued payoff is equal to his collected wages $w \, dt$ (where $w > 0$ is a fixed wage). It may be natural to assume $w = c$, but we do not require this assumption. After the worker is fired, both players have continuation payoff normalized to zero. Therefore, if the worker is fired at (stochastic) time $\tau$, the principal’s and agent’s payoffs are

$$r \int_0^\tau e^{-rt} \, dX_t \quad \text{and} \quad r \int_0^\tau e^{-rt} w \, dt,$$

respectively.

We assume that $\bar{a} > a > c > 0$. So the firm would (myopically) strictly want to employ a well matched worker and fire a poorly matched worker. By counting time in different units, we may without loss of generality normalize $r = 1$. By counting profit and wages in different units, we may without loss of generality normalize $\sigma = w = 1$.

The constant volatility functional form is natural if the unpredictable component of output does not involve the agent. We use it here for expositional convenience, but only four features—which could easily be ensured by qualitative assumptions on primitives—actually matter for our results: (1) some learning always happens, (2) the most productive effort level alone yields the fastest learning, and (3) some other effort level alone yields the slowest learning, and (4) both of these two key effort levels are profitable for the firm if and only if the worker and firm are well matched.

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12There are situations where one cannot maintain the stronger assumption that the learning-minimizing action is the least productive. In a previous version of this paper, for instance, the agent decided how to split his time between two activities, and learning was minimized by a form of multitasking.
2.3 Learning

We assume that the players (symmetrically) revise their beliefs about worker-firm match via Bayesian updating. We assume that players cannot signal what they do not know: nobody draws any direct inferences from the principal’s stopping choice or the agent’s activity choice. However, the agent’s choice \( A_t \) does indirectly affect learning, as it affects the stochastic law of \( dX_t \), and it affects the inferences (concerning \( \theta \)) the players draw from \( dX_t \).

Let \( P_t \in (0, 1) \) be the time \( t \) public expectation of \( \theta \). i.e. \( P_t \) is the common belief at time \( t \) that the worker and firm are well matched. Sometimes, it is more convenient to work with the state variable \( Z_t := \log \frac{P_t}{1-P_t} \). It is, of course, informationally equivalent: any \( z \) is associated with a probability \( p(z) := \frac{1}{1+e^{-z}} \). As noted in Daley and Green (2012) and in Bolton and Harris (1999) the process \( Z \) is a linear transformation of the output \( X \), and is therefore a Brownian diffusion process. Whenever the principal has not yet stopped, the belief variable \( Z \) follows the law of motion

\[
dZ_t = A_t \left[ dX_t - \left( \frac{1}{2} A_t - c \right) dt \right].
\]

Equivalently, \( P_t \) follows

\[
dP_t = A_t P_t (1 - P_t) \left[ dX_t - (P_t A_t - c) dt \right] \quad (2)
\]

\[
= A_t P_t (1 - P_t) \left[ A_t (\theta - P_t) dt + \sigma dB_t \right]. \quad (3)
\]

2.4 Player Strategies

Informally, our players both observe the history \( (A_i, X_i)_{i < t} \), and our principal and agent choose effort \( A_t \) and stopping rate \( S_t \) respectively (where \( S_t = \infty \) corresponds to firing the agent with certainty).

We restrict our attention to Markovian strategies, using beliefs about the match quality as our state variable. There are two reasons for our focus on Markov perfect equilibrium. First, we view it as a reasonable description of an employee’s career path within a firm. An employee has a “standing” in the firm (here, the belief about match quality), and his standing tomorrow depends only on his standing today and on his performance today. This seems sensible, especially in the context of a fixed wage relationship. Second, a restriction to Markovian strategies ensures that the stochastic process of players’ beliefs is uniquely determined by the realized path of output. Indeed it follows from Nakao (1972) that the law of motion of \( Z \) as governed by equation 1 has a unique strong solution, when players follow Markov strategies as defined below.

Definition 1. A Markov strategy profile is a pair \((a, s)\), where:
1. \(a : [0, 1] \rightarrow [a, \tilde{a}]\) is piecewise Lipschitz.\(^{13}\)

2. \(s : [0, 1] \rightarrow [0, \infty]\) is piecewise continuous; \(s\) is piecewise Lipschitz in any interval \(\subseteq [0, 1]\) over which \(s\) is bounded; and \(s^{-1}(\infty)\) is closed.

Let the space of above specified strategies for the agent be \(\mathcal{A}\) and for the principal be \(\mathcal{S}\).

Given \(P_0 \in [0, 1]\) and \((a, s) \in \mathcal{A} \times \mathcal{S}\), define the induced stochastic processes \(\{P_t, A_t, S_t\}\), by letting \(A_t = a(P_t)\) and \(S_t = s(P_t)\), and letting \(\{P_t\}\), follow equation 2.

In an abuse of notation, we identify any two strategies for a player which agree almost everywhere, and which (in the case of the principal) are finite-valued on the same set. For the remainder of the paper, every equation concerning players’ behavior (as a function of beliefs) is understood to hold only up to this equivalence.

Certain classes of strategies seem especially natural. For an effort level \(\hat{a} \in [a, \tilde{a}]\), the agent has a constant control strategy, with an effort level of \(\hat{a}\) at every belief; we simply call this strategy \(\hat{a} \in \mathcal{A}\). For a belief \(p \in [0, 1]\), the principal has a simple cutoff strategy \(s^p \in \mathcal{S}\) with \(s^p|_{[p, 1]} = 0\) and \(s^p|_{[0, p]} = \infty\). It is standard that, if the agent uses a constant control, then the principal optimally uses a simple cutoff.

By allowing \(S_t \in (0, \infty)\), we have permitted the principal to mix.\(^{14}\) That is, whenever \(S_t > 0\), the instantaneous probability of stopping is \(S_t \, dt\). As we will see, this mixing is a necessary component of equilibrium. In the present definition of an agent strategy, we allow the agent to observably choose interior efforts, but we have ruled out the possibility of explicit mixing. The latter is purely for notational convenience.\(^{15}\)

### 2.5 Defining Equilibrium

As the stochastic differential equation (SDE) 1 uniquely pins down the law of motion of beliefs given a strategy profile \((a, s) \in \mathcal{A} \times \mathcal{S}\) and initial condition \(P_0 = p \in [0, 1]\), there is no need to separately specify history-dependent beliefs in the specification of an equilibrium.

We can define the agent’s value as

\[
v(p|a, s) = \mathbb{E}_a \left[ \int_0^\infty e^{-r} e^{-\int_0^r S_t \, dt} \, dr \right],
\]

\(^{13}\)A function \(f : [x, \overline{x}] \rightarrow Y\) is piecewise Lipschitz if there exist \(x = x_0 < x_1 < \cdots < x_k = \overline{x}\) and Lipschitz functions \(f_j : [x_{j-1}, x_j] \rightarrow Y\) for each \(j \in \{1, \ldots, k\}\) such that \(f\) and \(f_j\) agree on \((x_{j-1}, x_j)\).

\(^{14}\)We have not allowed the probability of stopping at some belief \(p\) to be exactly \(q \in (0, 1)\). Notice: conditional on hitting \(z\), the state hits it again in the next \(\epsilon > 0\) time almost surely, so that this would be equivalent to setting \(s(p) = \infty\).

\(^{15}\)We can easily allow for behavioral strategies \(\kappa : [0, 1] \rightarrow \Delta[a, \tilde{a}]\) on the part of the agent—so that the agent tosses a coin to choose public \(A_i\) via measure \(\kappa(P_t)\)—but little is gained in terms of insight, and every resulting equilibrium yields players the same value as in our equilibrium.
where the term $e^{-\int_0^t S_\tau \, d\tau}$ is understood to be zero when $S_\tau = \infty$ for some $\tau \in [0, t)$.

Similarly, the principal’s value can be defined as

$$\pi(p|a, s) = E_a \left[ \int_0^\infty e^{-t} e^{-\int_0^t S_\tau \, d\tau} \, dX_t \right].$$

As is typical, we define equilibrium as a strategy profile satisfying mutual best response. Given our focus on strategies which are Markovian with respect to beliefs, and given that no proper subset of $(0, 1)$ is absorbing,\textsuperscript{16} we define best response from every possible posterior belief.

**Definition 2.** A Markov perfect equilibrium (or, in short, an equilibrium) is a pair of strategies $(a, s) \in A \times S$ such that for all $p \in (0, 1)$:

1. **Agent Optimality** The control solves the agent’s problem. i.e.,

   $$a \in \text{argmax}_{\hat{a} \in A} v(p|\hat{a}, s).$$

2. **Agent Sequential Rationality:** If $s = \infty$ in a neighborhood of $p$, then the distance between $a^*_\Delta(p|a, s)$ and $a(p)$ tends to zero as $\Delta$ does, where, for any $\Delta > 0$,

   $$a^*_\Delta(p|a, s) := \text{argmax}_{\hat{a} \in \tilde{A}} \left\{ (1 - e^{-\Delta}) + e^{-\Delta} E \left[ v(P_\Delta|a, s) \bigg| P_0 = p, A_\tau = \hat{a} \forall \tau \in [0, \Delta) \right] \right\}.$$

3. **Principal Optimality** The stopping rule solves the principal’s problem. i.e.,

   $$s \in \text{argmax}_{\hat{s} \in S} \pi(p|\hat{s}).$$

The requirement of sequential rationality in this framework is the only nonstandard one, its form being tailored for continuous time. We want to require that the agent is behaving optimally at every history. In a discrete time analogue of our game, a currently hired agent knows his action will be able to affect the public belief until the next period; our definition of sequential rationality recovers this intuition, having an agent best-respond as though he will at least retain his job for the next instant. As in discrete time dynamic games, this sequential rationality assumption constrains equilibrium play by limiting off path behavior to be credible.

\textsuperscript{16}This follows from the equation 1, which implies that the volatility of $Z$ is uniformly bounded below.
3 Main Results

The relationship in our model is a simple one. Neither player possesses any private information about match quality, nor about past behavior. The principal benefits (both myopically and from additional information) from having the agent exert high effort, while the agent suffers no direct cost from providing this benefit. Thus, one might expect a relationship free from conflict.

As the following Proposition clarifies, even with no direct conflict of interest within a given period, a conflict arises in equilibrium: the principal’s equilibrium profit is as though the agent always chose the principal’s least favorite action.

**Proposition 1.** In any equilibrium \((a, s)\), the principal’s value function is \(\pi(\cdot|a, s) = \sup_{\tilde{s} \in S} \pi(\cdot|a, \tilde{s})\).

Proposition 1 clarifies a sense in which the ongoing nature of a worker-firm relationship entails a severe agency problem. Recall that employing the worker is profitable if and only if he is well-matched to the job. Given this, the principal is certain to fire the agent if she becomes sufficiently pessimistic about their match quality. This contingency contaminates the relationship. However secure the worker’s job is today, he can never enjoy absolute job security in the future, as there is always a positive probability of “bad news” forcing the principal’s hand.

Suppose the principal strictly prefers to retain the worker at some belief. Currently enjoying some job security, and fearing for its future, the agent seeks to minimize the flow of information: this is achieved by choosing minimal effort. The ability to fire the agent benefits the principal, who may learn ex-post that the agent is not profitable to retain, but this threat comes at a cost, hindering productivity on the job.

By narrowing the possible equilibrium payoffs for the principal, Proposition 1 provides some useful information on the form of equilibrium behavior. There is some simple cutoff rule \(s^\beta\) which is a best response to the fixed control strategy \(a\), and one can further deduce that \(s^\beta\) is a principal best response to \(a\). However, \((a, s^\beta)\) is not an equilibrium. To see why, consider the incentives of an agent who finds himself facing \(s^\beta\) and yet employed at belief \(p < \bar{p}\). His job is so insecure that he has nothing to lose. Being sequentially rational, he will seek to maximize the probability that the posterior belief exceeds \(\bar{p}\) the following instant. His best possible chance of escaping the imminent firing threat is to choose an action that speeds up learning, i.e. induces the largest possible variation in posterior beliefs. This is achieved by choosing high effort at such beliefs. Consider the implications of this behavior if high effort is much more productive than low. Principal incentives unravel: given high effort in the firing region, the principal has a strict incentive to lower her cutoff.
The above reasoning shows that, if more effort is sufficiently valuable to the principal, then a simple cutoff strategy cannot be used in equilibrium. A similar argument can show that the principal cannot use a deterministic firing rule in equilibrium. However, given low effort, the principal is barely willing to retain the agent at $\bar{p}$. How then, can she be induced to retain the agent (even stochastically) when even more pessimistic? The agent must willingly exert higher effort over this mixing region.

Our main theorem, a complete characterization of the unique equilibrium, formalizes the above intuition.

![Figure 3: Players’ strategies](attachment:image.png)

**Theorem 1.** There is a unique equilibrium $(a^*, s^*)$. In equilibrium, there are two cutoffs $0 < p \leq \bar{p} < \frac{c}{a}$ such that:

- Above $\bar{p}$, the principal never fires the agent, who in turn exerts minimum effort.
- Between $p$ and $\bar{p}$, the principal mixes, the agent produces zero flow profit\(^{17}\) for

\(^{17}\)That is, his effort level at such belief $p$ satisfies $a^*(p)p - c = 0$. 

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the principal, and the agent’s value is affine. As \( p \) increases from \( \underline{p} \) to \( \bar{p} \), \( s^*(p) \) decreases continuously from \( \infty \) to some \( s > 0 \).

• Below \( \underline{p} \), the principal immediately fires the agent.

When higher effort is sufficiently valuable, the middle region is nondegenerate.\(^\text{18}\)

![Figure 4: Players’ value functions](image)

Theorem 1 confirms the observation made by Reisel et al. (2010) that, “...employees respond to job insecurity in ways that are counterproductive to organizational purpose.” Fearing for his future job security, an employee whose job is currently most secure exerts minimum effort, something that unambiguously undermines the organizational purpose of profit maximization.

\(^{18}\)More precisely, \( \underline{p} < \bar{p} \) if and only if \( \bar{a} > a + \frac{a-c}{\alpha} \), where \( \delta = \frac{1}{2} \left( \sqrt{1 + \frac{1}{s^2}} - 1 \right) \).
Notice that, as shown in Figure 3, both players’ behavior is discontinuous in the unique equilibrium. Discontinuity of the firm’s retention decisions highlights how employees with very similar performance histories can experience markedly different treatment from their employers.

The region \((p, \bar{p})\) can be thought of as a probationary period. Here, the employee works harder in the face of an imminent threat of losing his job. Should enough good news accrue, the employee is ‘promoted’ to a region above \(\bar{p}\), where he faces no such imminent threat. The discontinuity of the worker’s effort decision confirms an observation made by Engellandt and Riphahn (2005), Ichino and Riphahn (2005) and Jacob (2010): employees’ productive effort systematically falls as a probationary period ends.

In passing, we note that our model is also consistent with the empirical observation in the labor economics literature\(^{19}\) that an employee’s hazard rate of separation from a firm decreases in the duration of employment. This is true because the firm, making its firing choices based on evidence of worker productivity, induces a selection bias among retained workers.

### 3.1 Job Security and Productivity

In our model, a worker’s continuation value \(v\) is his expected discounted future employment, i.e. his job security. If he exerts effort \(a\) while believed to be well-matched to the firm with probability \(p\), then his current expected flow profitability, i.e. his productivity, is \(ap - c\). Therefore, in equilibrium, belief about match quality is a common determinant of both job security and productivity. Given this, a relationship between the latter two follows directly from the theorem.

**Corollary 1.** Consider an equilibrium \((a^\ast, s^\ast)\), and let \(v^\ast\) be the associated agent value function. For any level \(v \in (0, 1)\) of job insecurity, there is a unique belief \(p(v) \in (0, 1)\) such that \(v^\ast(p(v)) = v\). Let \(F(v) := p(v)a^\ast(p(v)) - c\), the agent’s current productivity at this level of job security. Then there exists \(\hat{v} \in [0, 1)\) such that:

- On \([0, \hat{v})\), \(F\) is zero.
- On \((\hat{v}, 1)\), \(F\) is strictly convex and strictly increases from negative to positive.

When higher effort is sufficiently valuable, \(\hat{v} > 0\).

When the worker’s job is very insecure (below \(\hat{v}\)), he is productive enough to barely be worth employing for the moment. A more secure worker exerts minimum effort, but what are the productivity implications? A very secure worker is productive on average, because he is likely to be well-matched to the firm. A moderately secure worker (with

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\(^{19}\)See Miller (1984), for example.
job security barely above \( \hat{v} \) loses money for the firm today: he is rationally retained only because of the possibility of becoming more productive in the future.

Thus, equilibrium play in our model generates the U-shaped pattern suggested in the empirical organizational psychology literature: a worker is least productive when his job is moderately secure.

4 Dynamic Games in Continuous Time

Beyond clarifying the relationship between job security and productivity, the technical side of our paper may be of more general interest to those who study dynamic games in continuous time. First, we extend the definition of sequential rationality to one which mirrors discrete time intuitions and yet has refining power in continuous time. Second, we extend the direct applicability of powerful methods from continuous time dynamic optimization to settings in which multiple (long-lived, strategic, ‘large’) players interact.

4.1 Defining Sequential Rationality

Our environment gives cause to consider players’ optimal behavior at contingencies they never expect to reach in equilibrium, i.e. off the path of play. Indeed, consider a belief at which the principal plans to immediately fire the agent. To assess how reasonable this principal decision is, one must consider the agent’s hypothetical behavior in the event that the principal retains him.

But how should we assess the agent’s rational behavior at such a belief? In a discrete
time setting, the agent would take his current employment as given, and would seek to maximize his prospects from the next period onward. In continuous time, “the next period” will arrive immediately. Thus, the agent is indifferent: his choice is inconsequential to his future. This is unsatisfactory if one wants to interpret the continuous time model (as we do) as simply a mathematically convenient approximation of a discrete time model.

To remedy this dissonance, we offer a definition of sequential rationality appropriate for continuous time games. Analogous to a discrete time setting, we assume that a player responds optimally, taking for granted that his current decision will be relevant for the next instant. More formally, at an off path state, we could consider the agent’s optimal effort $\Delta a$ if he were sure to retain his job for the next $\Delta > 0$ units of time. Our refinement then requires his behavior be a limit of $a\Delta$ as $\Delta$ vanishes.

While the ‘period length’ implicit in our refinement is small, the behavioral implications are substantial. In our model, absent a sequential rationality assumption, there would be a continuum family of equilibria with distinct on path behavior. Indeed, letting $(p, \bar{p})$ be the cutoffs described in Theorem 1, there would be an equilibrium with cutoffs $(p_L, \bar{p})$ for every $p_L \in [p, \bar{p}]$. Our refinement yields a unique selection (of a Pareto optimal member) from this family.

This sequential rationality condition, suitably generalized, enables a recursive approach to a broad class of continuous time multistage games of observable action. In ongoing work, we are studying its implications in a more general model of strategic relationships in continuous time.

### 4.2 Dynamic Optimization Methods with Multiple Players

Stochastic control in continuous time yields powerful methods that have been applied to dynamic games. Two prevalent approaches to solving decision problems in continuous time involve the Hamilton-Jacobi-Bellman (HJB) equation associated with a forward-looking player’s incentives.

The first approach is the so-called verification approach, which focuses on sufficient conditions for an optimum. The analyst heuristically derives a candidate policy for the decision maker, and with it a candidate value function. Then, there are results—verification theorems—as the literature calls them—which tell us: if the candidates satisfy the HJB and the value function is sufficiently smooth, then the candidates are in fact optimal.

The direct applicability of such results to strategic settings has limits, for at least two reasons. First, one player’s behavior—and therefore the environment faced by the other

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20 For example, Daley and Green (2012), Board and Meyer-ter Vehn (2013), Bohren (2016), Kolb (2015), and Dilmé (2012).

21 For example, see Pham (2009) for optimal control verification theorems and Øksendal (2003) for an optimal stopping verification theorem.
optimizing player—may be discontinuous in equilibrium. Indeed, in our game, Theorem 1 shows that this is necessarily the case: both players employ discontinuous strategies, so that our agent and principal face discontinuous discounting and volatility, respectively. Second, the ability to certify a candidate value function as optimal is less useful in multi-agent settings, unlike a single-agent decision problem which has a unique optimal value function. Even if the verification approach is used to confirm that some strategy profile is an equilibrium, such results are silent on whether there are other equilibria yielding a different optimal value function for a player. In our setting, therefore, we must focus mainly on necessary conditions for optimality.

The second approach is to show that the decision maker’s value function necessarily satisfies smoothness properties and derive necessary conditions of optimal play which follow from the HJB. In The Art of Smooth Pasting, Dixit (1993) heuristically derives optimal policies in many dynamic optimization problems using this approach. When is this approach valid? As described in Strulovici and Szydlowski (2012), value functions will exhibit some smoothness and HJB equations will be satisfied as long as the underlying optimization problem is sufficiently smooth.

However, as mentioned above, one player’s optimal play may be discontinuous, violating the required regularity conditions for the other player. Given this complication, one must extend existing tools. On intervals (of the space of beliefs) where players’ behavior is Lipschitz continuous, nothing new is required to show that value functions are $C^2$ and HJB equations satisfied. At beliefs where players change their behavior discontinuously, we derive smooth pasting conditions. The argument applies the viscosity approach (see Crandall, Ishii, and Lions (1992) for instance) to stochastic optimal control. With piecewise $C^2$ value functions and smooth pasting in hand, we are able to derive useful necessary conditions toward characterizing equilibrium in our setting. An important wrinkle is the possibility that the agent may effectively face discontinuous discounting (because the principal may use a discontinuous stopping rule). By modifying existing viscosity-based arguments, we show that discontinuous drift, volatility, and discounting do not destroy smooth pasting. Thus we formalize the intuition of Dixit (1993), and extend the applicability of such arguments in dynamic games.

5 Extensions

Our model is a stylized one, but it is also flexible. In this section, we consider some straightforward extensions of our model and briefly describe how the worker-firm relationship is affected.
5.1 Costly Effort

If the agent suffers a private cost of exerting effort, the observable implications of our model are very similar. Equilibrium still consists of a three-region structure, with firing in the pessimistic region, mutual mixing in the middle region, and low effort with no firing in the optimistic region. In that model, sequential rationality requires the agent to instead exert low effort in the pessimistic region. As a result, there is a multiplicity of qualitatively similar equilibria. The basic insights are unchanged.

5.2 Varying Wages

Suppose the agent receives a wage that varies with his standing in the firm. If his wage \( w(p) \) is weakly concave in belief \( p \) \(^{22}\) (for example, it could be linear as in Holmström (1999)), then our main results again go through.

This clarifies that ours is a model of job insecurity paired with low-powered incentives. In particular, the logic of Proposition 1 relies on the agent facing job insecurity following poor performance and no steep reward following good performance.

5.3 Promotion

One way of providing incentives in an ongoing worker-firm relationship is through promotions. Our model can accommodate this.

Suppose there is an exogenous cutoff \( \hat{p} \) above which the worker has exogenous continuation value \( \hat{w} \) upon reaching \( \hat{p} \). For instance, the worker could be tenured with wage \( \hat{w} \). If \( \hat{w} \leq w \), then the structure of our unique equilibrium persists: even the lure of tenure cannot incentivize a currently untenured agent to exert high effort.

More interesting is the case where \( \hat{w} > w \): a worker is unambiguously better off once promoted. As the agent nears the promotion threshold, the possibility of promotion serves as a high-powered incentive: he exerts high effort, aiming to accelerate this reward. One can show that equilibrium will exhibit a four-regime structure as in Figure 6, with this very productive regime (paired with the firm retaining the worker) at the most optimistic beliefs.

5.4 Career Concerns

What if, rather than learning about a worker-firm match quality, our players learn about the agent’s ability. In principle, this might change the analysis, as information about a

\(^{22}\)Also assume that the associated cost of employing the agent varies little enough that the principal still myopically prefers a better matched agent. In the differentiable case, this just means that \( c' < a \).
worker’s ability affects his career even after losing his current job. Would such concern for job security in future employment change incentives on the job?

To address this, consider a simple economy populated by an equal population of homogeneous firms and heterogeneous workers. A matched worker and firm have wages, employment costs, and output exactly as in our model. Once a firm fires a worker, each engages in undirected search on the two-sided market. Each meets a random participant of the other side of the market with constant hazard rate $\mu$. When a firm meets a worker, it sees the worker’s history of employment and output, and a new relationship begins. Consider what happens in a steady state equilibrium, where each relationship is identical conditional on the worker’s perceived ability, and play within a relationship is Markovian in worker reputation.

If the firm fires the worker, then its continuation profit is $\gamma \hat{\pi}$, where $\hat{\pi} = \mathbb{E}_p \pi(p)$ is the profit associated with hiring the average searching worker, and $\gamma = \frac{\mu}{1+\mu}$ is the

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23The results are unchanged for fixed wages or, as in the Varying Wages extension, linear or concave wages.
discounting associated with searching for another worker. Thus the firm’s incentives are as in our model, but with a fixed outside option of $\gamma \hat{\pi}$; equivalently, it is as if the flow cost of retaining the worker is $\tilde{c} = c + \gamma \hat{\pi}$. A worker fired at reputation $p$ must search before beginning a new relationship, and will therefore have continuation value $\gamma v(p)$. Therefore, he faces an ‘effective firing rate’ of $S_t(1 - \gamma)$ when the true firing rate is $S_t$.

Given the above, any given worker-firm relationship is nearly identical to in our setting. The main theorem is correct as stated, with the new firing rate being higher (by a factor of $\frac{1}{1-\gamma} = 1 + \mu$) in the mixing region.

6 Related Literature

First and foremost, we join the literature studying how job insecurity relates to employee performance. As mentioned in the introduction, the literature in organizational psychology yields ambiguous results. Of note there, Selenko et al. (2013) and Borg and Braun (1992) both show that, empirically, the relationship between job insecurity and productivity is U-shaped. Our results, in particular Corollary 1, confirm this cross-sectional relationship.

The economic literature on this topic has focused on the differing incentives faced by probationary employees and permanent ones. In early theoretical work, Shapiro and Stiglitz (1984) show that an equilibrium firing threat can alleviate a moral hazard problem. This perspective is empirically confirmed in many studies of exogenous variation in a worker’s degree of job security. For instance, Ichino and Riphahn (2005) and Jacob (2010) show empirically that employment protection provisions are associated with increased employee absenteeism. Engellandt and Riphahn (2005) find that temporary employees work harder than permanent employees, and are significantly more likely to work unpaid overtime. In experimental settings, both Falk, Huffman, and MacLeod (2008) and Corgnet, Hernán-González, and Rassenti (2015) find that the flexibility to dismiss a worker yields better worker performance. The above findings all speak to the incentive effects of job insecurity in the presence of a moral hazard problem. In settings without moral hazard, such as ours, the threat of firing can in fact harm employee incentives.

In addition to affecting incentives, firing can help with assignment: it is ex-post efficient for an employer to fire a worker who is not well-suited to the job. As Lazear (1990) argues, reducing employment protection can in fact lead to lower unemployment by making firms more efficient. In the context of our model, the enhanced efficiency from firing unproductive workers comes at a cost: by distorting incentives, it results in lower employee effort.

Our paper is also related to the literature on dynamic contracting with unknown agent quality, as in Prat and Jovanovic (2014) and DeMarzo and Sannikov (2016)). The focus in these papers is on optimal contracts under moral hazard. We focus on equilibria in a fixed-
wage environment in which agent choice is perfectly observed. Notice, observability of
the agent’s action leaves no room for the agent to bias the principal’s beliefs. So, while
the agent may shape the principal’s signal (by speeding up or slowing down the flow of
information), he cannot jam the principal’s signal.

Finally, we contribute to the literature applying continuous time optimal control to
economics. Some work, for instance Strulovici and Szydlowski (2012), adapts powerful
methods of optimal control and optimal stopping to economic applications. As discussed
in Section 4, some of our technical groundwork in the appendix may broaden the appli-
cability of this toolkit to strategic settings.
References


Brockner, Joel, Tom R Tyler, and Rochelle Cooper-Schneider. 1992. “The influence of prior commitment to an institution on reactions to perceived unfairness: The higher they are, the harder they fall.” *Administrative Science Quarterly* 241–261.


Falk, Armin, David Huffman, and W Bentley MacLeod. 2008. “Institutions and contract enforcement.”


## 7 Appendix

### 7.1 Technical Preliminaries

Players’ value functions in an equilibrium might be very wild. In particular, as one player’s behavior might change with beliefs in an erratic way in equilibrium—and may be *required to* by his/her own best response property—neither value function is known to be the optimal value function to a well-behaved decision problem. Even so, one can obtain various regularity properties, which will be helpful for bringing the toolset of stochastic calculus to bear in understanding our setting.
Lemma 1. Given any \((a, s) \in \mathcal{A} \times \mathcal{S}\), the value functions \(\pi(\cdot|a, s), v(\cdot|a, s)\) and the optimal value functions \(\sup_{a \in \mathcal{A}} \pi(\cdot|a, s), \sup_{a \in \mathcal{A}} v(\cdot|a, s)\) are continuous on \((0, 1)\).

Proof. Given \(0 < q < p < q + \epsilon < 1\), let \(\{P^p_t\}\) denote the belief process starting at \(P_0 = p\), and define the \(P^p\)-measurable stopping time \(\tau_{q,\epsilon}^p := \inf\{t \geq 0 : P_t^p \notin (q, q + \epsilon)\}\)

To begin with, we prove the following claim: For any \(s \in \mathcal{S}\) and \(q \in (0, 1)\), there are real constants \(K, L > 0\) and \(\epsilon > 0\) such that

\[
\left|v(p|a, s) - v(q|a, s)\right|, \left|\pi(p|a, s) - \pi(q|a, s)\right| \leq K\mathbb{E}_a(\tau_{q,\epsilon}^p) + L\frac{p-q}{\epsilon}
\]

for all \(a \in \mathcal{A}\).

Toward the claim, fix \((a, s) \in \mathcal{A} \times \mathcal{S}\) and \(q \in (0, 1)\), and let \(g \in \{v(\cdot|a, s), \pi(\cdot|a, s)\}\). Then there is a bounded, piecewise Lipschitz function \(f : [0, 1] \to \mathbb{R}\) such that\(^{24}\)

\[
g(p) = \mathbb{E}_a\left[\int_0^\infty e^{-t}e^{-\int_0^t s(P^p_t)dt}f(P_t)dt \mid P_0 = p\right].
\]

For now, focus on the case of \(s(q) < \infty\). Let \(\epsilon \in (0, 1 - q)\) be small enough to ensure \(s(\cdot|q+\epsilon)\) is bounded above by a constant \(\tilde{s} \in \mathbb{R}^+\). Let \(\beta^{p}_{t} := e^{-t}e^{-\int_0^t s(P^p_t)dt}\) and \(\tau^p := \tau_{q,\epsilon}^p\). Then, for any \(p \in (q, q + \epsilon)\):

\[
|g(p) - g(q)| = \left|\mathbb{E}_a\left[\int_0^{p} \beta^{p}_{t} f(P^p_t) dt\right] - g(q)\right|
= \left|\mathbb{E}_a\left[\int_0^{p} \beta^{p}_{t} f(P^p_t) dt + \beta^{p}_{t} g(P^p_t) \mid P_0 = p\right] - g(q)\right|
= \left|\mathbb{E}_a\left\{\int_0^{\tau^p} \beta^{p}_{t} f(P^p_t) dt - (1 - \beta^{p}_{\tau^p})g(q) + \beta^{p}_{\tau^p} \left[g(P^p_{\tau^p}) - g(q)\right]\right\}\right|
= \left|\mathbb{E}_a\left\{\int_0^{\tau^p} \beta^{p}_{t} f(P^p_t) dt - (1 + s(P^p_t))g(q) + \beta^{p}_{\tau^p} \left[g(P^p_{\tau^p}) - g(q)\right]\right\}\right|
\leq \mathbb{E}_a\int_0^{\tau^p} \beta^{p}_{t} \left|f(P^p_t) - [1 + s(P^p_t)]g(q)\right| dt + \mathbb{E}_a\beta^{p}_{\tau^p} \left|g(P^p_{\tau^p}) - g(q)\right|
\leq \mathbb{E}_a\int_0^{\tau^p} \left|f(P^p_t) - [1 + s(P^p_t)]g(q)\right| dt + |g(q + \epsilon) - g(q)|\mathbb{P}_a[P^p_t = q + \epsilon].
\]

So let \(K := \|f\|_\infty + (1 + \tilde{s})|g(q)|\) and \(L := |g(q + \epsilon) - g(q)|\). Then, \(|g(p) - g(q)| \leq K\mathbb{E}_a(\tau^p) + L\frac{p-q}{\epsilon}\), as \(P^p_t\) is a martingale.

\(^{24}\)Indeed \(f(p) = 1\) if \(g = \pi(\cdot|a, s)\), and \(f(p) = a(p)p - c\) if \(g = \pi(\cdot|a, s)\).
Now consider the case of \( s(q) = \infty \). Let \( \hat{\tau}_p := \inf\{t \geq 0 : P_t^p = q\} \). Then

\[
|g(p) - g(q)| = \left| \mathbb{E}_a \left[ \int_0^{\hat{\tau}_p} \beta_t^p f(P_t^p) \, dt \right] - g(q) \right|
\]

\[
= \left| \mathbb{E}_a \left[ \int_0^{\hat{\tau}_p} \beta_t^p f(P_t^p) \, dt + \beta_{\hat{\tau}_p}^p g(q) \right] - g(q) \right|
\]

\[
= \left| \mathbb{E}_a \left[ \int_0^{\hat{\tau}_p} \beta_t^p f(P_t^p) \, dt \right] \right|
\]

\[
\leq \mathbb{E}_a \int_0^{\hat{\tau}_p} |f(P_t^p)| \, dt,
\]

so that letting \( K := \|f\|_\infty \) (with any \( L, \epsilon \)) completes the proof of the claim.

Now, standard dynamic programming arguments, together with the martingale property of beliefs, tell us that \( h(p) := \sup_{a \in \mathcal{A}} \mathbb{E}_a(\tau_{q, \epsilon}^p) \) is concave in \( p \), so that \( \sup_{a \in \mathcal{A}} \mathbb{E}_a(\tau_{q, \epsilon}^p) = \mathbb{E}_a(\tau_{q, \epsilon}^p) \).

Therefore, we can find constants \( K, L, \epsilon > 0 \) such that, for every \( p \in (q, q + \epsilon) \), the (control-independent) number,

\[
M^p = K \mathbb{E}_a(\tau_{q, \epsilon}^p) + L^{\frac{p-q}{\epsilon}},
\]

is a uniform upper bound for \( \left\{ \left| v(p|a, s) - v(q|a, s) \right|, \left| \pi(p|a, s) - \pi(q|a, s) \right| : a \in \mathcal{A} \right\} \).

Finally, \( M^p \to 0 \) as \( p \to q \) because \( \mathbb{E}_a(\tau_{q, \epsilon}^p) \) does.

The above argument implies that \( v(\cdot|a, s) \), \( \pi(\cdot|a, s) \) are right-continuous at \( q \) for fixed control, and that \( \sup_{a \in \mathcal{A}} v(\cdot|a, s) \) is right-continuous there. An identical argument shows that these functions are left-continuous there as well.

Finally, fix any \( a \in \mathcal{A} \). The optimal value function \( \pi = \sup_{a \in \mathcal{S}} \pi(\cdot|a, s) \) is a supremum of continuous functions, and so is lower-semicontinuous. Therefore, its zero set is closed, so that \( s := \inf_{a \in \mathcal{S}} \pi(\cdot|a, s) \) is an element of \( \mathcal{S} \). It is easy to see that \( s \) is a principal best response to \( a \), so that \( \pi = \pi(\cdot|a, s) \) is continuous.

All that remains is to check continuity at an endpoint \( p \in \{0, 1\} \). Notice that\(^{25}\) for any \( \epsilon, \phi \in (0, 1) \), there exists \( \epsilon' \in (0, \epsilon) \) such that

\[
\inf_{a \in \mathcal{A}} \mathbb{E}_a \left[ \int_0^{\infty} e^{-t} \mathbf{1}_{|p_t - p| < \epsilon'} \, dt \right] \geq \phi
\]

whenever \( P_0 \) is within \( \epsilon' \) of \( p \). Therefore:

\(^{25}\)This can be seen by boundedness of \([q, a]\) and the law of motion 1.
• If $s(p) = \infty$, continuity\textsuperscript{26} of $s$ in a neighborhood of $p$ implies that the value functions tend to zero near $p$.

• If $s(p) < \infty$, then play is Lipschitz in a neighborhood of $p$. It follows that the limiting agent payoff is 1 and the limiting principal payoff is $a(p)p - c$.

In either case, the limiting payoff agrees with the payoff at $p$ itself by direct computation. □

**Lemma 2.** Fix $0 \leq p_0 < p_1 \leq 1$. Suppose $s \in \mathcal{S}$ is such that $s|_{[p_0,p_1]}$ is finite-valued and Lipschitz, and $a \in \mathcal{A}$ such that $a|_{[p_0,p_1]}$ is Lipschitz. Let $v^* := \sup_{a \in \mathcal{A}} v(\cdot|a, s)$ and $v := v(\cdot|a, s)$. Then $v^*, v$ are $C^2$ on $[p_0, p_1]$\textsuperscript{27} and satisfy:

\[
0 = \frac{1}{2} [a(p)p(1 - p)]^2 v''(p) - [1 + s(p)]v(p) + 1
\]

\[
0 = \sup_{a \in [p_0,p_1]} \left[ \frac{1}{2} [\hat{a}p(1 - p)]^2 v''(p) - [1 + s(p)]v^*(p) + 1 \right].
\]

Moreover, if a $C^2$ function $f : [p_0, p_1] \to \mathbb{R}$ satisfies

\[
0 = \frac{1}{2} [a(p)p(1 - p)]^2 f''(p) - [1 + s(p)]f(p) + 1
\]

together with the boundary conditions $(f(p_0), f(p_1)) = (v(p_0), v(p_1))$, then $f = v|_{[p_0, p_1]}$; similarly for $v^*$.

**Proof.** This follows immediately from from (Krylov, 2008, Ch. 1, Thm. 5). □

In our setup the principal solves an optimal stopping problem. In the terminology of the previous section, the principal chooses the stopping rate $s(p)$. Standard game-theoretic reasoning tells us that if $s(p) \in (0, \infty)$, then the principal must be indifferent between stopping and continuing. Therefore, fixing the agent’s strategy, we can only focus on principal’s stopping strategies that are pure strategies. That is, $s(x) \in [0, \infty)$. In other words, the principal decides a stopping time $\tau$.

The potential difficulty is that the principal’s stopping problem may not be well-behaved, as the agent’s behavior—and thus the flow payoffs and volatility that the principal faces—can change discontinuously with the state. In spite of this problem, the next result allows one to verify that a given principal strategy is in fact a best response. The theorem is close to Theorem 10.4.1 from Øksendal (2003).

**Lemma 3.** Let $a \in \mathcal{A}$. Suppose $\psi : [0, 1] \to \mathbb{R}_+$ and $\hat{p} \in [0, 1]$ satisfy the following properties:

\textsuperscript{26}This is the only place in the paper where piecewise continuity of $s$ is required. In particular, it could be replaced with the weaker condition that $\lim_{p \to \hat{p}} s(\hat{p}) = \infty$ if $p \in \{0, 1\}$ and $s(p) = \infty$, with no change to our results.

\textsuperscript{27}Where derivatives at $p_0, p_1$ are taken to be limits of derivatives.
• $\psi$ is $C^1$ and piecewise $C^2$.
• $\psi|_{[0,\hat{p}]} = 0$.
• For a.e. $p \in [0,1]$, 
  \[ \psi(p) - \frac{1}{2}[a(p)p(1-p)]^2\psi''(p) \geq a(p)p - c, \]
  with equality if $p > \hat{p}$.

Then $\psi = \pi(\cdot|s_\hat{p}a) = \max_{s \in S} \pi(\cdot|s, a)$. That is, $s_\hat{p}$ is an optimal strategy, yielding principal value function $\psi$.

Proof. Fix $p \in (\hat{p},1)$, and consider the belief process $\{P_t\}_t$ defined by $a \in \mathcal{A}$ and $P_0 = p$. Let $\tau^* := \inf\{t : P_t \notin (\hat{p},1)\}$, and let $\tau$ be an arbitrary $\{P_t\}_t$-measurable (stochastic) stopping rule.

By Ito’s formula,
\[
e^{-\tau^*} \psi(P_{\tau^*}) = \psi(p) + \int_0^{\tau^*} e^{-u} \left[ -\psi(P_u) + \frac{1}{2}[a(P_u)P_u(1-P_u)]^2\psi''(P_u) \right] du
+ \int_0^{\tau^*} e^{-u}\psi'(P_u)[a(P_u)P_u(1-P_u)] dB_u
\]
\[
\Rightarrow \psi(p) = e^{-\tau^*} \psi(P_{\tau^*}) + \int_0^{\tau^*} e^{-u} \left[ \psi(P_u) - \frac{1}{2}[a(P_u)P_u(1-P_u)]^2\psi''(P_u) \right] du
- \int_0^{\tau^*} e^{-u}\psi'(P_u)[a(P_u)P_u(1-P_u)] dB_u
\]

As the integrand of the stochastic integral is bounded, we can take expectations, yielding:
\[
\psi(p) = \mathbb{E}e^{-\tau^*} \psi(P_{\tau^*}) + \mathbb{E} \int_0^{\tau^*} e^{-u} \left[ \psi(P_u) - \frac{1}{2}[a(P_u)P_u(1-P_u)]^2\psi''(P_u) \right] du
\geq \mathbb{E}e^{-\tau^*} \psi(P_{\tau^*}) + \mathbb{E} \int_0^{\tau^*} e^{-u}[a(P_u)P_u - c] du
\]

By dominated convergence,
\[
\psi(p) \geq \mathbb{E}e^{-\tau} \psi(P_\tau) + \mathbb{E} \int_0^\tau e^{-u}[a(P_u)P_u - c] du
\]
As $\tau$ was arbitrary, we conclude that $\psi(p) \geq \sup_{s \in S} \pi(p|a, s)$. 

27
In the above argument, if \( \tau = \tau' \), then the inequalities hold with equality, so that

\[
\psi(p) = \mathbb{E}e^{-\tau}\psi(P_{\tau'}) + \mathbb{E} \int_0^{\tau} e^{-u}[a(P_u)P_u - c] \, du = 0 + \pi(x|a, s_{\bar{\rho}})
\]

Together, we have

\[
\psi(p) \leq \pi(x|a, s_{\bar{\rho}}) \leq \sup_{s \in S} \pi(x|a, s) \leq \psi(p).
\]

Recall that \( \rho \in (\hat{\rho}, 1) \) was arbitrary, so that (by Lemma 1) \( \pi(\cdot|a, s_{\bar{\rho}}) = \sup_{s \in S} \pi(\cdot|a, s) = \psi \) on \([\hat{\rho}, 1] \). On \([0, \hat{\rho}] \), we see \( \pi(\cdot|a, s_{\bar{\rho}}) = 0 = \psi \).

Now, take any \( \rho \in [0, \hat{\rho}] \), and consider the belief process \( \{P_t\} \) defined by \( a \in \mathcal{A} \) and \( P_0 = p \). Let \( \tau := \inf\{t : P_t \notin (0, \hat{\rho})\} \), and let \( \tau \) be an arbitrary \( \{P_t\}_t \)-measurable (stochastic) stopping rule.

\[
\mathbb{E} \int_0^\tau e^{-t}[a(P_t)P_t - c] \, dt = \mathbb{E} \int_0^{\tau \wedge \tau^*} e^{-t}[a(P_t)P_t - c] \, dt + \mathbb{E} \int_{\tau \wedge \tau^*} e^{-t}[a(P_t)P_t - c] \, dt
\]
\[
\leq \mathbb{E} \int_{\tau \wedge \tau^*} e^{-t}[a(P_t)P_t - c] \, dt = \mathbb{E}1_{t \leq \tau} \int_{\tau}^\tau e^{-t}[a(P_t)P_t - c] \, dt
\]
\[
\leq \mathbb{E}1_{t \leq \tau} e^{-\tau} \sup_{s \in S} \pi(P_\tau|a, s) \, dt = \mathbb{E}1_{t \leq \tau} e^{-\tau} \sup_{s \in S} \pi(\hat{\rho}|a, s) \, dt
\]
\[
= 0.
\]

As \( \tau \) was arbitrary, we have \( \sup_{s \in S} \pi(\cdot|a, s) \leq 0 \) on \([0, \hat{\rho}) \) as well.

\[
\square
\]

Since we know that \( v \) is \( C^2 \) over any interval where \( s \) is finite-valued and Lipschitz, the only difficulty is at the (isolated) spots where \( s \) discontinuous. In this section, we show that smooth pasting obtains even at these problem points.

**Lemma 4.** Let \( p \in (0, 1) \) be such that \( s(p) \) is finite, \( \bar{s} \) be an upper bound for \( s \) in a neighborhood of \( p \), and \( \underline{\sigma} \in (0, q_p(1 - p)) \) a lower bound for volatility in the same neighborhood. Let \( v = v(\cdot|a, s) \) for some fixed \( a \in \mathcal{A} \), or let \( v = \sup_{a \in \mathcal{A}} v(\cdot|a, s) \).

1. If \( \psi : [0, 1] \to \mathbb{R} \) is \( C^2 \) and convex, \( \psi(p) = v(p) \), and \( \psi(q) > \psi(q) \) for \( q \neq p \) in some neighborhood of \( p \), then \( (1 + \bar{s})v(p) \geq 1 + \frac{1}{2}\underline{\sigma}^2 \psi''(p) \).

2. If \( \psi : [0, 1] \to \mathbb{R} \) is \( C^2 \) and concave, \( \psi(p) = v(p) \), and \( \psi(q) < \psi(q) \) for \( q \neq p \) in some neighborhood of \( p \), then \( v(p) \leq 1 + \frac{1}{2}\underline{\sigma}^2 \psi''(p) \).
The proof of these results is based on the so-called viscosity approach to differential equations. While pre-existing results do not carry over directly, our proof closely mimics arguments from Pham (2009).

Proof. That $s(p)$ is finite and $\psi$ continuous imply $\psi(p) = v(p) > 0$, so that $\psi > 0$ near $p$.

Fix some $\delta > 0$ small enough that $s \leq \bar{s}$, $\sigma \geq \bar{\sigma}$, and $0 < \psi \not\equiv v$ on $(p - \delta, p + \delta) \setminus \{p\}$.

Fix some sequence $\{p_m\}_m \subseteq (p - \delta, p + \delta) \setminus \{p\}$ converging to $p$. Let $\eta_m := v(p_m) - \psi(p_m)$, which converges to zero by continuity of $\nu$.

Let $\{P^{\alpha_m, p_m}\}_m$ denote the belief process associated with control $\alpha \in \mathcal{A}$ given $P^{\alpha, p_m}_0 = p_m$.

Define the stopping time $\tau_m^\alpha := \inf\{u \geq 0 : |P^{\alpha, p_m}_u - p_m| \geq \frac{\delta}{2}\} \land \sqrt{|\eta_m|}.

Given any $\zeta > 0$, define a sequence of controls $\{a_m\}_m \subseteq \mathcal{A}$ as follows: if $\nu = \nu(\cdot|\alpha, s)$, let $a_m := \alpha$; otherwise, if $\nu = \sup_{\alpha \in \mathcal{A}} \nu(\cdot|\alpha, s)$, let $a_m$ satisfy

$$
\mathbb{E} \left[ \int_0^{\tau_m^a} e^{-\int_0^u s(P^{\alpha_m, p_m}_u)\text{d}u} \text{d}u + e^{-\int_0^{\tau_m^a} s(P^{\alpha_m, p_m}_u)\text{d}u} v(P^{\alpha_m, p_m}_u) \right] \geq v(p_m) - \zeta \sqrt{|\eta_m|},
$$

which is achievable by some $a_m$ by the principle of optimality. As a shorthand, let $P^m := P^{a_m, p_m}$ and $\tau_m := \tau_m^a$.

For $r = 1, 1 + \bar{s}$, Ito’s formula delivers:

$$
e^{-\tau_m^a} \psi(P^m_{\tau_m^a}) = \psi(p_m) - r \int_0^{\tau_m^a} e^{-ru} \psi(P^m_u) \text{d}u + \int_0^{\tau_m^a} e^{-ru} \frac{1}{2} [a(P^m_u) P^m_u (1 - P^m_u)]^2 \psi''(P^m_u) \text{d}u + \int_0^{\tau_m^a} e^{-ru} a(P^m_u)^2 P^m_u (1 - P^m_u) (\theta - P^m_u) \psi'(P^m_u) \text{d}u + \int_0^{\tau_m^a} e^{-ru} a(P^m_u) P^m_u (1 - P^m_u) \psi'(P^m_u) \text{dB}_u
$$

Taking expectations (note, $\psi$ is bounded), together with $\eta_m \psi'' \geq 0$, gives us:

$$
\eta_m \left\{ \mathbb{E} e^{-\tau_m^a} \psi(P^m_{\tau_m^a}) - \psi(p_m) \right\}
= \eta_m \left\{ -r \mathbb{E} \int_0^{\tau_m^a} e^{-ru} \psi(P^m_u) \text{d}u + \mathbb{E} \int_0^{\tau_m^a} e^{-ru} \frac{1}{2} [a(P^m_u) P^m_u (1 - P^m_u)]^2 \psi''(P^m_u) \text{d}u \right\}
\geq \eta_m \left\{ -r \mathbb{E} \int_0^{\tau_m^a} e^{-ru} \psi(P^m_u) \text{d}u + \mathbb{E} \int_0^{\tau_m^a} e^{-ru} \frac{1}{2} \sigma^2 \psi''(P^m_u) \text{d}u \right\}
= \eta_m \mathbb{E} \int_0^{\tau_m^a} e^{-ru} \left[ \frac{1}{2} \sigma^2 \psi''(P^m_u) - r \psi(P^m_u) \right] \text{d}u.
$$

Now, we reason separately for the two cases.
First, suppose \( \psi \) is convex and (around \( p \)) below \( v \). Then,

\[
\eta_m = v(p_m) - \psi(p_m) \\
\geq \mathbb{E} \left[ \int_0^{\tau_m} e^{-u} e^{\int_0^u \eta(P_m^m) \, d\tilde{u}} \, du + e^{-\tau_m} e^{\int_0^{\tau_m} \eta(P_m^m) \, d\tilde{u}} v(P_m^m) \right] - \psi(p_m) \\
\geq \mathbb{E} \left\{ \int_0^{\tau_m} e^{-(1+\bar{\delta})u} \, du + \left[ e^{-(1+\bar{\delta})\tau_m} \psi(P_m^m) - \psi(p_m) \right] \right\} \\
\geq \mathbb{E} \left[ \int_0^{\tau_m} e^{-(1+\bar{\delta})u} \left[ 1 + \frac{1}{2} \sigma^2 \psi''(P_m^m) - (1 + \bar{\delta})\psi(P_m^m) \right] \, du \right]
\]

Taking limits as \( m \to \infty \), (and applying the mean value theorem) tells us \( 0 \geq 1 + \frac{1}{2} \sigma^2 \psi''(p) - (1 + \bar{\delta}) v(p) \), as required.

Next, suppose \( \psi \) is concave and (around \( p \)) above \( v \).

By choice of \( a_m \),

\[
\eta_m - \zeta \sqrt{|\eta_m|} = v(p_m) - \psi(p_m) - \zeta \sqrt{|\eta_m|} \\
\leq \mathbb{E} \left[ \int_0^{\tau_m} e^{-u} e^{\int_0^u \eta(P_m^m) \, d\tilde{u}} \, du + e^{-\tau_m} e^{\int_0^{\tau_m} \eta(P_m^m) \, d\tilde{u}} v(P_m^m) \right] - \psi(p_m) \\
\leq \mathbb{E} \left[ \int_0^{\tau_m} e^{-u} \, du + e^{-\tau_m} v(P_m^m) \right] - \psi(p_m) \\
\leq \mathbb{E} \left[ \int_0^{\tau_m} e^{-u} \left[ 1 + \frac{1}{2} \sigma^2 \psi''(P_m^m) - \psi(P_m^m) \right] \, du \right]
\]

\[
\implies -\sqrt{|\eta_m|} - \zeta \leq \mathbb{E} \left[ \frac{\tau_m}{\sqrt{|\eta_m|}} \int_0^{\tau_m} e^{-u} \left[ 1 + \frac{1}{2} \sigma^2 \psi''(P_m^m) - \psi(P_m^m) \right] \, du \right]
\]

Taking limits now yields \( -\zeta \leq 1 + \frac{1}{2} \sigma^2 \psi''(p) - v(p) \). Since this is true for arbitrary \( \zeta > 0 \), we conclude that \( v(p) \leq 1 + \frac{1}{2} \sigma^2 \psi''(p) \), as required.

\[\square\]

**Lemma 5.** Let \( p \in (0, 1) \) be such that \( s(p) \) is finite. Let \( \nu = \nu(|a, s) \) for some fixed \( a \in \mathcal{A} \), or let \( \nu = \sup_{a \in \mathcal{A}} \nu(|a, s) \). Then \( \nu \) is differentiable at \( p \).

**Proof.** By Lemma 1, \( \nu \) is continuous at \( p \). Then, by Lemma 2, both one-sided derivatives exist: call them \( \nu'_-(p) \) and \( \nu'_+(p) \) from the left and right, respectively. It remains to show that \( \nu'_-(p) = \nu'_+(p) \).
Let \( \bar{s} \in \mathbb{R}_+ \) be some local upper bound for \( s \) around \( p \), and let \( \sigma \in (0, p(1 - p)q) \) a local lower bound for \( q \mapsto q(1 - q)a(q) \) around \( p \). Take any \( \lambda > 0 \).

If \( \exists \kappa \in (v'(p), v'_+(p)) \), then letting \( \psi(q) := v(p) + \kappa(q - p) + \lambda(q - p)^2 \), Lemma 5(1) tells us that \( v(p) \geq \frac{1 + \sigma^2}{4 + \bar{s}} \). If \( \exists \kappa \in (v'_+(p), v'_-(p)) \), then letting \( \psi(q) := v(p) + \kappa(q - p) - \lambda(q - p)^2 \), Lemma 5(2) tells us that \( v(p) \leq 1 - \frac{\sigma^2}{\bar{s}} \).

In either case, the ability to make \( \lambda \) arbitrarily large would violate \( 0 \leq v \leq 1 \). So it must be that the one-sided derivatives match. \( \square \)

### 7.2 Necessary Conditions for Equilibrium

**Definition 3.** A weak equilibrium is a pair of strategies \((a, s) \in \mathcal{A} \times \mathcal{S}\) such that for all \( p \in (0, 1) \), the strategies \( a \) and \( s \) satisfy agent optimality and principal optimality, as stated in Definition 2.

In this subsection, we collect some features that must be true of an arbitrary equilibrium. Together, these results will leave us with a unique candidate.

The following lemma is our first step towards characterizing the entire set of equilibria. We show that, as the agent’s wishes to be employed for as long as possible, he seeks to minimize the flow of information whenever his job is secure for the moment. Moreover, as the agent’s behavior is consistent at such beliefs, the principal’s behavior takes a similar form to that of a standard one-player bandit problem: the set of beliefs at which the principal optimally stops irrespective of agent behavior. This implies \( v < 1 \) globally, so that the above equation implies \( v''(p) < 0 \). Therefore, choosing \( a \) to maximize \( a^2 v''(p) \) is the same as minimizing \( a^2 \), which is uniquely attained by \( a = \bar{a} \). So over any subinterval of \( I \) where \( v \) is \( C^2 \), we have

\[
v(p) = 1 + \frac{1}{2} p^2 (1 - p)^2 \max_{a \in [\bar{a}, \bar{a}]} \{ a^2 v''(p) \}.
\]

For \( \bar{p} \approx 0 \), the principal optimally stops irrespective of agent behavior. This implies \( v < 1 \) globally, so that the above equation implies \( v''(p) < 0 \). Therefore, choosing \( a \) to minimize \( a^2 v''(p) \) is the same as minimizing \( a^2 \), which is uniquely attained by \( a = \bar{a} \). So over any subinterval of \( I \) where \( v \) is \( C^2 \), we have \( a = \bar{a} \) (almost everywhere). Then, Lemma 2 tells us that \( I \) is the finite union of such subintervals, so that \( a = \bar{a} \) (a.e.) on \( I \).

Now, the principal’s flow payoff is \( f(p) := ap - c \) for \( p \in (p_0, p_1) \). Fix \( \tilde{p} \in (p_0, p_1) \). As \( \pi \) is zero at any endpoint \( < 1 \) (by maximality of the interval and
Lemma 1), and belief 1 is a.s. never reached, it must be that

$$\pi(\tilde{p}) = \int_{p_0}^{p_1} f \, d\mu$$

for some positive, absolutely continuous measure $\mu$ on $(z_0, z_1)$. As $\pi(\tilde{p}) > 0$, it must be that $f$ is positive on a set of positive measure. Since $f$ is increasing, it is positive in a neighborhood of $p_1$. Finally, applying Rüschendorf, Urusov et al. (2008, Proposition 2.9), it cannot be optimal for the principal to stop at $p_1$ if $p_1 < 1$. That is, the principal would like to continue, despite a possibly negative flow payoff, to the right of $p_1$. This would contradict the maximality of $p_1$. Therefore, $p_1 = 1$, proving the claim.

Now, using the claim, we prove the lemma. First, notice that there is some $p \in (0, 1)$ such that $\pi(p) > 0$. Indeed, this property is guaranteed for $p > 0$, since such states are guaranteed strictly positive flow profit. Next, notice that any $p$ with $\pi(p) > 0$ belongs to some open interval with the same property by Lemma 1. The claim then tells us that $\pi|_{[p_1, 1)} > 0$ and $a|_{[p_1, 1)} = \underline{a}$. Finally, let

$$p_H := \inf\{p \in [0, 1] : \pi(p) > 0\}.$$ 

We now know that $\pi|_{[p_H, 1)} > 0$, $a|_{[p_H, 1)} = \underline{a}$, and $\pi|_{(0, p_H)} = 0$. Lastly, by Lemma 1, $\pi(0) = \pi(p_H) = 0$ too.

Lemma 7. In any weak equilibrium $(a, s)$, the principal’s value function is $\pi(\cdot|a, s) = \sup_{\tilde{s} \in S} \pi(\cdot|a, \tilde{s})$.

Proof. Let $\pi := \pi(\cdot|a, s)$ and $\pi_* := \sup_{\tilde{s} \in S} \pi(\cdot|a, \tilde{s})$. For all $\tilde{s} \in S$,

$$\pi = \pi(\cdot|a, s) \geq \pi(\cdot|a, \tilde{s}) \geq \pi(\cdot|a, \tilde{s}),$$

so that (taking the supremum) $\pi \geq \pi_*$.

Now, let $p_H \in (0, 1)$ be the cutoff guaranteed by Lemma 6. As $(a, s)$ is a weak equilibrium, $\pi$ is the principal’s optimal value function given agent strategy $a$. Lemma 6 tells us that $\pi^{-1}(0) = [0, p_H]$, so that the simple cutoff strategy $s^{pu}$ is also a principal best response to $a$. That is, $\pi(\cdot|a, s^{pu}) = \pi$. But $\pi(\cdot|a, s^{pu}) = \pi(\cdot|a, s^{pu})$, as they feature exactly the same behavior on path. Therefore,

$$\pi = \pi(\cdot|a, s^{pu}) = \pi(\cdot|a, s^{pu}) \leq \pi_*.$$ 

Proposition 1 follows immediately from Lemma 7.
Notation 1. Let \( \delta := \frac{1}{2} \left( \sqrt{1 + \frac{1}{8a}} - 1 \right) \), \( \bar{p} := \frac{\delta}{a(1+\delta)-c} \), and \( p := \min \{ \bar{p}, \frac{c}{\bar{p}} \} \).

Lemma 8. Take any weak equilibrium \((a, s)\).

The simple cutoff strategy \(s^0 \in S\) is a best response to \(a \in \mathcal{A}\), and the value function \(\pi = \pi(\cdot|a, s)\) satisfies

\[
\pi(p) = \left[ ap - c + (c - a\bar{p}) \left( \frac{p}{\bar{p}} \right)^{\delta} \left( \frac{1}{1 - \bar{p}} \right)^{1+\delta} \right] 1_{p \geq \bar{p}}.
\]

Proof. Let \(\psi\) be the proposed function on the right-hand side.

On \((\bar{p}, 1]\), \(\psi\) is \(C^2\) and satisfies \(\psi(p) = ap - c + \frac{1}{2}[ap(1-p)]^2\psi''(p)\). On \([0, \bar{p})\), \(\psi\) is zero and flow payoffs are \(\leq 0\). Lastly, \(\psi\) is differentiable at \(\bar{p}\). Lemma 3 then tells us that \(\psi = \sup_{s \in S} \pi(\cdot|a, \bar{s})\), and \(s^0\) is optimal for the principal against \(a\). That \(\pi = \psi\) then follows from Lemma 7.

Lemma 9. In any weak equilibrium \((a, s)\), on any open interval \(I \subseteq [0, \bar{p}]\) where the agent has a strictly positive value, the agent uses \(a(p) = \frac{c}{p}\).

Proof. Consider any \(p \in I\) at which \(a\) is continuous. Note that \(s(p) < \infty\) because \(v(p) > 0\). So consider an \( \epsilon > 0 \) and \( \bar{s} \in \mathbb{R}_+ \) such that \((p - \epsilon, p + \epsilon) \subseteq I\) and \(s_{|I-p-\epsilon,p+\epsilon} \leq \bar{s}\), and consider the belief process \(\{P_t\}_t\) associated with agent control \(a\) and \(P_0 = p\). Let \(\tau_\epsilon := \inf \{t : |P_t - p| \geq \epsilon\}\). We can decompose \(\pi(p)\) as follows:

\[
0 = \pi(p) = \mathbb{E} \left[ \int_0^{\tau_\epsilon} e^{-t} e^{-\int_0^t s(P_u) \, du} [a(P_t)P_t - c] \, dt + e^{-\tau_\epsilon} e^{-\int_0^{\tau_\epsilon} s(P_u) \, du} \pi(P_{\tau_\epsilon}) \right]
\]

\[
= \mathbb{E} \int_0^{\tau_\epsilon} e^{-t} e^{-\int_0^t s(P_u) \, du} [a(P_t)P_t - c] \, dt
\]

\[
\implies 0 = \mathbb{E} \frac{1}{\tau_\epsilon} \int_0^{\tau_\epsilon} e^{-t} e^{-\int_0^t s(P_u) \, du} [a(P_t)P_t - c] \, dt, \text{ where } e^{-\int_0^{\tau_\epsilon} s(P_u) \, du} \in [e^{-\frac{\epsilon}{\epsilon}}, 1].
\]

Taking limits as \(\epsilon \to 0\) and appealing to the mean value theorem, \(a(p)\) \(\rho - c = 0\). This condition holds at every \(p \in I\) at which \(a\) is continuous, and so it holds almost everywhere, as required.

Lemma 10. In any weak equilibrium \((a, s)\), on any open interval \(I \subseteq [0, \bar{p}]\) where the agent has a strictly positive value, \(v(p)\) is affine.

Proof. Since \(v|_I > 0\), we know \(s|_I\) is finite-valued.
Consider an open subinterval \( J \subseteq I \setminus \left\{ \frac{a}{2} \right\} \) over which \( s \) is Lipschitz. By Lemma 2, for \( p \in J \),

\[
a(p)v''(p) = \frac{[1 + s(p)]v(p) - 1}{\frac{1}{2}p(1 - p)} = \sup_{\hat{a} \in [\underline{a}, \bar{a}]} \hat{a}v''(p).
\]

As \( a(p) \neq \underline{a}, \bar{a} \), it must be that \( v''(p) = 0 \).

The above tells us that, if \( s \) is piecewise Lipschitz over a subinterval \( J \) of \( I \), then \( v \) is piecewise affine over said subinterval. Moreover, by Lemma 5, \( v|_J \) is then affine.

Therefore, \( v \) is affine in a neighborhood of any \( p \in I \). As a result, it is affine over \( I \).

**Lemma 11.** In any weak equilibrium \((a, s)\), there is a cutoff \( p_L \) such that the agent’s zero value set is \([0, p_L]\).

**Proof.** Consider any \( \hat{p} \in (0, 1) \) with \( v(p) > 0 \). Let us show that \( v|_{[\hat{p}, 1]} > 0 \). In the proof below, we will use Lemma 1 many times.

First, note that \( \hat{p} \) belongs to some maximal open interval \((p_0, p_1)\) with the property that \( v|_{(p_0, p_1)} > 0 \). We know that the principal best response requires stopping for sufficiently low beliefs, so that \( p_0 > 0 \). By the intermediate value theorem, \( v(p_0) = 0 \). By Lemma 10, \( v \) takes the form \( v(p) = \gamma(p - p_0) \) for \( p \in (p_0, p_1) \), where \( \gamma \in \mathbb{R} \) is some constant. As \( v(\hat{p}) > 0 \), it must be that \( \gamma > 0 \), implying \( v(p_1) > 0 \). But then (continuity of \( v \) and) maximality of \( p_1 \) implies that \( p_1 = 1 \). So \( v|_{[\hat{p}, 1]} > 0 \).

Letting \( p_L := \inf\{p \in [0, 1]: v(p) > 0\} \) concludes the proof. \( \square \)

**Lemma 12.** Take any weak equilibrium \((a, s)\), and let \( p_L \) be as delivered by Lemma 11. Then \((a, s)\) is an equilibrium if and only if \( a|_{(0, p_L)} = \hat{a} \).

**Proof.** For the proof of this lemma, it is convenient to work with log-likelihood ratios rather than beliefs. For \( z \in \mathbb{R} \), let \( p(z) := \frac{1}{1 + e^{-z}} \), so that \( z = \log \frac{p(z)}{1 - p(z)} \). Let \( \hat{v}(z) := v(p(z)|a, s) \) for any \( z \in \mathbb{R} \).

Let \( z_L := \log \left( \frac{p_L}{1 - p_L} \right) \), and take any \( z_0 < z_L \). We want to show that sequential rationality implies \( a(p(z_0)) = \hat{a} \). To this end, fix arbitrary \( \hat{a} \in [a, \bar{a}] \).

For any \( a_0 \in [\underline{a}, \bar{a}] \), let \( \{Z^{a_0}_t\} \) be the \( X \)-measurable stochastic process for log-likelihood ratios associated with initial condition \( Z^{a_0}_0 = z_0 \) and constant control \( a_0 \in \mathcal{A} \). That is, \( Z^{a_0} \) satisfies the SDE (see equation 1)

\[
dZ^{a_0}_t = a_0 \left[ a_0 \left( \theta - \frac{1}{2} \right) dt + dB_t \right].
\]

Crucially, the “\( dB_t \)” term is the same for every different \( a_0 \), so that these stochastic processes are highly correlated.

To begin with, we prove the following claim: There are constants \( \gamma, \hat{\Delta} > 0 \) such that, for every \( \Delta \in (0, \hat{\Delta}] \) and \( a_0 \in [\underline{a}, \bar{a}] \), we have

\[
Z^{\hat{\Delta}}_{\Delta} \geq z_L + \gamma \text{ if } Z^{a_0}_{\Delta} \geq z_L.
\]
To see this, observe that:

\[
\frac{dZ_t^\hat{a}}{dt} - \frac{dZ_t^{a_0}}{dt} = \hat{a} \left[ \hat{a} \left( \theta - \frac{1}{2} \right) dt + dB_t \right] - a_0 \left[ a_0 \left( \theta - \frac{1}{2} \right) dt + dB_t \right] = (\hat{a}^2 - a_0^2) \left( \theta - \frac{1}{2} \right) dt + (\hat{a} - a_0) dB_t
\]

\[
= \hat{a}(\hat{a} - a_0) \left( \theta - \frac{1}{2} \right) dt + \frac{\hat{a} - a_0}{a_0} \frac{dZ_t^{a_0}}{dt} \geq -\frac{\hat{a}}{2}(\hat{a} - a) dt + \frac{\hat{a} - a_0}{a_0} dZ_t^{a_0}
\]

\[
\implies Z_{\Delta}^\hat{a} \geq Z_{\Delta}^{a_0} - \frac{\hat{a}}{2}(\hat{a} - a)\Delta + \frac{\hat{a} - a_0}{a_0} (Z_{\Delta}^{a_0} - z_0).
\]

In particular, if \( Z_{\Delta}^{a_0} \geq z_L > z_0 \), then

\[
Z_{\Delta}^\hat{a} \geq Z_{\Delta}^{a_0} - \frac{\hat{a}}{2}(\hat{a} - a)\Delta + \frac{\hat{a} - a_0}{a_0} (Z_{\Delta}^{a_0} - z_0)
\]

\[
\geq z_L - \frac{\hat{a}}{2}(\hat{a} - a)\Delta + \frac{\hat{a} - a_0}{a_0} (z_L - z_0).
\]

Taking any \( \gamma \in \left( 0, \frac{\hat{a} - a}{\hat{a}} (z_L - z_0) \right) \) with small enough \( \Delta > 0 \) will witness the claim.

We now use the claim to prove the lemma.

Notice next that there is a constant \( M \geq 0 \) such that \( \hat{v}(z) \leq M(z - z_L) \), for all \( z \in \mathbb{R} \).

Indeed, \( \nu(\cdot, s) \) is affine on \([p_L, \bar{p}]\) by Lemma 10, concave on \([\bar{p}, 1]\) by the proof of Lemma 6, and differentiable at \( \bar{p} \) if \( \bar{p} > p_L \) by Lemma 5; therefore it is concave on \([\bar{p}, 1]\). Next, the right-hand derivative \( m \) of said function is finite—by Lemma 10 if \( \bar{p} > p_L \), by the proof of Lemma 6 and direct computation if \( \bar{p} = p_L \). Moreover, these same two lemmata tell us that the function is increasing on \([p_L, 1]\), so that (being constant on \([0, p_L]\)) it is Lipschitz of constant \( m \). Then, as a composition of Lipschitz functions, \( \hat{v} \) is itself Lipschitz of some constant \( M \). The bound then follows directly.

Now, for any \( a_0 \in [a, \hat{a}] \) and \( \Delta \in (0, \hat{\Delta}) \):

\[
\frac{\mathbb{E}(\hat{v}(Z_{\Delta}^{a_0}))}{\mathbb{E}(\hat{v}(Z_{\Delta}^{a_0}))} \leq \frac{\mathbb{E}(\hat{v}(Z_{\Delta}^{a_0}))}{\mathbb{E}(\hat{v}(Z_{\Delta}^{a_0}))} = \frac{\mathbb{P}(Z_{\Delta}^{a_0} \geq z_L \mid Z_{\Delta}^{a_0} \geq z_L + \gamma)}{\mathbb{P}(Z_{\Delta}^{a_0} \geq z_L + \gamma \mid Z_{\Delta}^{a_0} \geq z_L)} (\text{since } \hat{v} \text{ is zero on } [0, z_L])
\]

\[
\leq \frac{\mathbb{P}(Z_{\Delta}^{a_0} \geq z_L \mid Z_{\Delta}^{a_0} \geq z_L)}{\mathbb{P}(Z_{\Delta}^{a_0} \geq z_L \mid Z_{\Delta}^{a_0} \geq z_L + \gamma)} (\text{by the claim})
\]

\[
\leq \frac{\mathbb{P}(Z_{\Delta}^{a_0} \geq z_L \mid Z_{\Delta}^{a_0} \geq z_L)}{\mathbb{P}(Z_{\Delta}^{a_0} \geq z_L \mid Z_{\Delta}^{a_0} \geq z_L + \gamma)} (\text{by monotonicity of } \nu)
\]

\[
\leq \frac{M}{\hat{v}(z_L + \gamma)} \mathbb{E}(\hat{v}(Z_{\Delta}^{a_0} \geq z_L \mid Z_{\Delta}^{a_0} \geq z_L))
\]

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Conditional on $\theta \in [0, 1]$, the random variable $Z^0_\Lambda - z_L$ is normally distributed with mean $-(z_L - z_0) + a_0^2(\theta - \frac{1}{2})\Delta$ and standard deviation $a_0 \sqrt{\Delta}$. Therefore, applying the known formula for the expectation of truncated normal:

$$
\mathbb{E}_\theta \left[ Z^0_\Lambda - z_L \mid Z^0_\Lambda - z_L \geq 0 \right] = \mathbb{E}_\theta \left[ Z^0_\Lambda - z_L \right] + \text{StdDev}_\theta \left[ Z^0_\Lambda - z_L \right] \cdot \frac{\phi \left( \frac{z_L - z_0 - a_0^2(\theta - \frac{1}{2})\Delta}{a_0 \sqrt{\Delta}} \right)}{1 - \Phi \left( \frac{z_L - z_0 - a_0^2(\theta - \frac{1}{2})\Delta}{a_0 \sqrt{\Delta}} \right)}
$$

$$
= -(z_L - z_0) + a_0^2(\theta - \frac{1}{2})\Delta + a_0 \sqrt{\Delta} \cdot \frac{\phi \left( \frac{z_L - z_0 - a_0^2(\theta - \frac{1}{2})\Delta}{a_0 \sqrt{\Delta}} \right)}{1 - \Phi \left( \frac{z_L - z_0 - a_0^2(\theta - \frac{1}{2})\Delta}{a_0 \sqrt{\Delta}} \right)}
$$

$$
= [(z_L - z_0) - a_0^2(\theta - \frac{1}{2})\Delta] \left[-1 + \epsilon^{a_0,\theta} \frac{\phi \left( \frac{1}{\epsilon^{a_0,\theta}} \right)}{1 - \Phi \left( \frac{1}{\epsilon^{a_0,\theta}} \right)} \right],
$$

where $\epsilon^{a_0,\theta} := \frac{a_0 \sqrt{\Delta}}{z_L - z_0 - a_0^2(\theta - \frac{1}{2})\Delta}$. Notice that $\epsilon^{a_0,\theta} > 0$ for sufficiently small $\Delta$, and that $\lim_{\Delta \to 0} \sup_{a_0 \in [\underline{a}, \bar{a}]} \epsilon^{a_0,\theta} = 0$. Now, observe that:

$$
\lim_{\epsilon \to 0} \epsilon \frac{\phi \left( \frac{1}{\epsilon} \right)}{1 - \Phi \left( \frac{1}{\epsilon} \right)} = \lim_{\epsilon \to 0} \frac{\epsilon e^{-\frac{1}{2\epsilon^2}}}{\int_{\epsilon}^{\infty} e^{-\frac{y^2}{2}} dy} = \lim_{\epsilon \to 0} \frac{\epsilon e^{-\frac{1}{2\epsilon^2}} \left[1 + \epsilon (-\frac{1}{2})(-2)\epsilon^{-3} \right]}{-e^{-\frac{1}{2\epsilon^2}} (-\epsilon^{-2})} = \lim_{\epsilon \to 0} \frac{1 + \epsilon^{-2}}{\epsilon^{-2}} = \lim_{\epsilon \to 0} (1 + \epsilon^2) = 1
$$

$$
\Rightarrow \lim_{\Delta \to 0} \sup_{a_0 \in [\underline{a}, \bar{a}]} \mathbb{E}_\theta \left[ Z^0_\Lambda - z_L \mid Z^0_\Lambda \geq z_L \right] = (z_L - z_0) \left[\left[-1 + \lim_{\Delta \to 0} \sup_{a_0 \in [\underline{a}, \bar{a}]} \epsilon^{a_0,\theta} \frac{\phi \left( \frac{1}{\epsilon^{a_0,\theta}} \right)}{1 - \Phi \left( \frac{1}{\epsilon^{a_0,\theta}} \right)} \right] = 0
$$

Finally, this tells us that:

$$
\lim_{\Delta \to 0} \sup_{a_0 \in [\underline{a}, \bar{a}]} \frac{\mathbb{E}(Z^0_\Lambda)}{\mathbb{E}(Z^0_\Lambda)} \leq \frac{M}{\mathbb{E}(Z^0_\Lambda)} \lim_{\Delta \to 0} \sup_{a_0 \in [\underline{a}, \bar{a}]} \mathbb{E}_\theta \left[ Z^0_\Lambda - z_L \mid Z^0_\Lambda \geq z_L \right]
$$

$$
\leq \frac{M}{\mathbb{E}(Z^0_\Lambda)} \lim_{\Delta \to 0} \sup_{\theta \in [0, 1]} \mathbb{E}_\theta \left[ Z^0_\Lambda - z_L \mid Z^0_\Lambda \geq z_L \right] = 0.
$$

Therefore, for sufficiently small $\Delta > 0$, we have $\sup_{a_0 \in [\underline{a}, \bar{a}]} \mathbb{E}(Z^0_\Lambda) < \mathbb{E}(Z^0_\Lambda)$. That is, $a^*_\Lambda(p|a, s) = \arg\max_{a_0 \in [\underline{a}, \bar{a}]} \mathbb{E}(Z^0_\Lambda) \subseteq (\hat{a}, \bar{a})$. But then $\hat{a} \in [\underline{a}, \bar{a})$ was arbitrary.

We are now equipped to prove both directions of the lemma, given $p \in (0, 1)$. First, given any $a_0 \in [\underline{a}, \bar{a})$, the set $a^*_\Lambda(p|a, s)$ is contained in $[\frac{\hat{a} + a_0}{2}, \bar{a}]$ for sufficiently small $\Delta$, so that $a(p) \neq a_0$ if $(a, s)$ is to be an equilibrium. Second, it is straightforward that the maximum defining $a^*_\Lambda(p|a, s)$ is attained by some $a_0 \in [\underline{a}, \bar{a})$. But then $a_\Lambda \to \bar{a}$ for sufficiently small $\Delta$ for any $\hat{a} \in [\underline{a}, \bar{a})$. Therefore $a_\Lambda \to \bar{a}$ as $\Delta \to 0$, proving sequential rationality of $\bar{a}$.

\[\square\]
**Lemma 13.** Take any equilibrium \((a, s)\), and let \(p_L\) be as delivered by Lemma 11. Then \(p_L = \bar{p}\).

**Proof.** First, suppose that \(p_L < \bar{p}\). For all \(p \in (p_L, \bar{a})\), we know from Lemma 9 that \(\bar{a} \geq a(p) = \frac{c}{\bar{p}}\), so that \(p \geq \frac{c}{\bar{a}}\). Considering \(p\) very close to \(p_L\) then tells us that \(p_L \geq \frac{c}{\bar{a}}\).

So either \(p_L \geq \bar{p}\) or \(p_L \geq \frac{c}{\bar{a}}\). That is, \(p_L \geq \bar{p}\).

Next, by Lemma 8, the principal has strictly positive value above \(\bar{p}\), and so does not stop there. Therefore, the agent has strictly positive value above \(\bar{p}\) too, so that \(p_L \leq \bar{p}\).

All that remains is to show that \(p_L \leq \frac{c}{\bar{a}}\). By principal best response, there cannot be an open subinterval of \([0, p_L]\) on which the principal’s flow profit \(p \mapsto a(p)p - c\) is strictly positive. For \(p\) in such an interval, the associated flow profit is simply \(\bar{a}p - c\). Focusing on such intervals contained in a small neighborhood of \(p_L\), it must be that \(\bar{a}p_L - c \leq 0\), as desired. \(\square\)

### 7.3 Sufficient Conditions for Equilibrium

In this subsection, we lay the groundwork to show that our candidate equilibrium is in fact an equilibrium.

**Lemma 14.** There is a unique continuous function \(v : [0, 1] \rightarrow \mathbb{R}\) such that:

- \(v\) is \(C^2\) on \([0, 1] \setminus \{p, \bar{p}\}\) and differentiable on \((p, 1]\).
- \(v|_{[0, p]} = 0\) and \(v(1) = 1\).
- \(v\) is affine on \([p, \bar{p}]\) and satisfies \(v(p) = 1 + \frac{1}{2}(\bar{a}p(1 - p))^2v''(p)\) for \(p \in (\bar{p}, 1)\).

Moreover, this function is concave and strictly increasing over \((\bar{a}, 1)\), and so \((0, 1)\)-valued there.

**Proof.** Given any function \(v\) of the desired form, there are constants \(\beta, K\) such that

\[
v(p) = \begin{cases} 
0 & : p \in [0, p_L) \\
\beta(p - p_L) & : p \in (p_L, \bar{p}) \\
1 - K(1 - p)^{1+\delta}p^{-\delta} & : p \in [\bar{p}, 1]. 
\end{cases}
\]

If \(p = \bar{p}\) (so that \(\beta\) is irrelevant), we need to see that there is a unique constant \(K\) such that the value matching equation \(1 - K(1 - \bar{p})^{1+\delta}\bar{p}^{-\delta} = 0\) holds. This is clearly so, with \(K = (1 - \bar{p})^{-(1+\delta)}\bar{p}^{\delta}\).

\[\text{Notice that the 2nd order linear DE over } (\bar{p}, 1) \text{ has solutions } p \mapsto 1 - K'p^{1+\delta}(1 - p)^{-\delta} - K(1 - p)^{1+\delta}p^{-\delta}\text{ for constants } K', K'. \text{ But then continuity and } v(1) = 1 \text{ imply } K' = 0.\]

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If \( p < \bar{p} \), we need to see that there is a unique pair of constants \( \beta, K \) such that the value matching and smooth pasting equations

\[
\beta(\bar{p} - p) = 1 - K(1 - \bar{p})^{1+\delta} \bar{p}^{-\delta} \\
\beta = K(1 - \bar{p})^{\delta} \bar{p}^{-(1+\delta)}[p + \delta]
\]

hold. The solution set is the intersection of a positively sloped line and a negatively sloped line in \((\beta, K)\) space, delivering a unique solution.

Finally, direct computation shows that \( \beta = \bar{p}^{1+\delta} - p^{-\delta} \), \( K = (1 - \bar{p})^{1+\delta} \), and \( v|_{[0,1]} \) is increasing and concave.

**Notation 2.** Define \( a^* \in \mathcal{A} \) and \( s^* \in \mathcal{S} \) as follows.

\[
a^* = \begin{cases} 
a & \text{if } p \in [\bar{p}, 1] \\
c/p & \text{if } p \in (p, \bar{p}) \\
\bar{a} & \text{if } p \in [0, p] \end{cases}
\]

\[
s^* = \begin{cases} 
0 & \text{if } p \in [\bar{p}, 1] \\
1 - v(p) & \text{if } p \in (p, \bar{p}) \\
\infty & \text{if } p \in [0, p] \end{cases}
\]

where \( v \) is the function specified in Lemma 14.

**Lemma 15.** Let \( A : [0, 1] \Rightarrow [a, \bar{a}] \) be given by either \( A(p) = [a, \bar{a}] \) \( \forall p \) or \( A(p) = \{a^*(p)\} \) \( \forall p \). Then there is a unique continuous function \( f : [0, 1] \rightarrow [0, 1] \) such that:

- \( f \) is \( C^2 \) on \([0, 1] \setminus \{p, \bar{p}\}\) and differentiable on \((p, \bar{p})\).
- \( f|_{[0,\bar{p}]} = 0 \) and \( f(1) = 1 \).
- \( f \) satisfies the HJB

\[
[1 + s^*(p)]f(p) = 1 + \frac{1}{2} \sup_{a \in A(p)} \{(ap(1 - p))^2 f''(p)\}
\]

for \( p \in (p, \bar{p}) \cup (\bar{p}, 1) \).

**Proof.** Let \( v \) be the function specified in Lemma 14. By that Lemma, and by definition of \( s^* \), the function \( v \) satisfies the conditions of the present lemma. Let \( f \) be any such function, and let \( h := f - v \). We must show that \( h = 0 \).

First, notice that \( h \) is continuous on \([0, 1]\), differentiable on \((\bar{p}, 1)\), zero on \([0, p] \cup \{1\}\), and \( C^2 \) on \([0, 1] \setminus \{p, \bar{p}\}\).
For \( p \in (\bar{p}, 1) \),
\[
1 \geq f(p) = 1 + \frac{1}{2} \sup_{a \in A(p)} \{ [ap(1-p)]^2 f''(p) \}
\]
\[
\implies 0 \geq \sup_{a \in A(p)} \{ [ap(1-p)]^2 f''(p) \}
\]
\[
\implies 0 \geq f''(p)
\]
\[
\implies f(p) = 1 + \frac{1}{2} [ap(1-p)]^2 f''(p)
\]
\[
\implies h(p) = \frac{1}{2} [ap(1-p)]^2 h''(p).
\]

For \( p \in (\underline{p}, \bar{p}) \),
\[
[1 + s^*(p)] f(p) = 1 + \frac{1}{2} \sup_{a \in A(p)} \{ [ap(1-p)]^2 [f''(p) - 0] \}
\]
\[
= [1 + s^*(p)] h(p) + \frac{1}{2} \sup_{a \in A(p)} \{ [ap(1-p)]^2 h''(p) \}
\]
\[
\implies [1 + s^*(p)] h(p) = \frac{1}{2} \sup_{a \in A(p)} \{ [ap(1-p)]^2 h''(p) \}.
\]

Take any subinterval \( I \) of \((p, 1)\) on which \( h \) is strictly positive [negative]. By the above equations, it is strictly convex [concave] over each of \( I \cap (p, \bar{p}) \) and \( I \cap (\underline{p}, 1) \). But then differentiability of \( h \) on \( I \) tells us that it is strictly convex [concave] over \( I \).

Assume, for contradiction, that \( h \) is nonzero. Lemma 1 then lets us say more.

There is then some maximal interval \((\underline{p}, p_0) \subseteq (0, 1)\) over which it is nonzero. As \( h|_{[0,p_0]} \cup [1] = 0 \), it must be that \( h(p_0) = h(p_1) = 0 \), and \( h|_{(p_0, p_1)} \) is either strictly positive or strictly negative. By the above argument, \( h|_{(p_0, p_1)} \) is either positive and convex or negative and concave. This contradicts \( h(p_0) = h(p_1) = 0 \) being zero at its endpoints. \( \square \)

### 7.4 Proof of Theorem 1

Consider any equilibrium. Lemmata 6 and 8 tell us that, above \( \bar{p} \), the agent exerts low effort and the principal never fires. Lemmata 11, 12, and 13 tell us that, below \( \underline{p} \), the agent exerts high effort and the principal immediately fires. Lemmata 9, 10, and 13 tell us that, over \((p, \bar{p})\), the agent chooses effort to give zero flow payoff to the principal, and the agent’s value is affine in beliefs. Lemma 14 then tells us that the principal’s firing behavior agrees with \( s^* \) over this interval. Summing up, if there is any equilibrium, then it is \((a^*, s^*)\).

We now verify that our unique candidate is in fact an equilibrium. Lemma 3 tells us that \( s^p \) is a principal best response to \( a^* \) for every \( p \in [\underline{p}, \bar{p}] \). It follows that the mixture \( s^p \) is as well.

Lastly, consider the two agent value functions \( v(\cdot|a^*, s^*) \) and \( \sup_{a \in A} v(\cdot|a, s^*) \). Both are functions \([0, 1] \rightarrow [0, 1] \) which take value 0 over \([0, \underline{p}] \) and 1 at 1. By Lemma 1,
both are continuous. By Lemma 2, both are $C^2$ on $(p, \bar{p}) \cup (\bar{p}, 1)$. By Lemma 5, both are differentiable at $\bar{p}$. Therefore, by Lemma 15, they coincide: $a^*$ is an agent best response to $s^*$ on path. Finally, Lemma 12 tells us that $a^*$ is sequentially rational against $s^*$.

\[
\Box
\]

### 7.5 Proof of Corollary 1

Let $\hat{v} := v(\bar{p})$. The belief function $v \mapsto p(v)$ is simply the inverse of the strictly increasing function $v^*|_{(p, 1)}$.

On $p \in (p, \bar{p})$, $a^*(p)p - c = 0$ by definition of $a^*$. Therefore, $F$ is zero over $(0, \hat{v})$.

For $p \in [\bar{p}, 1]$, $v^*(p) = 1 - K(1 - p)^{1+\delta}p^{-\delta}$ for $K > 0$, so that $v^*$ is concave and increasing there. Therefore, its inverse is convex and increasing. Finally, $p \mapsto ap - c$ is affine, increasing, negative at $p = a$, and positive at $p = 1$. Therefore, $F$ takes the desired form over $(\hat{v}, 1)$.

\[
\Box
\]