Leverage and speculation
Preliminary and incomplete

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Abstract

We develop and analyze a continuous-time contracting model with speculation in which size plays a role. The agent controls the drift of a scaled arithmetic Brownian motion and may engage in secret speculation to enhance it; doing so exposes the process to large Poisson losses. The optimal contract must uses size as an additional instrument. There is downsizing with positive probability on the equilibrium path to preserve incentive compatibility. We characterize the value function and present important properties of the optimal contract. The reflecting barrier beyond which the agent is paid depends on the severity of the speculation problem. When it is severe, that reflecting barrier becomes completely absorbing: the domain of the value function is a singleton. This work is particularly pertinent to leverage regulation.

Keywords: asymmetric information; dynamic contracts; moral hazard; speculation.
JEL Classification: D82, D86, G28, L43.

1 Introduction

The Global Financial Crisis has rekindled a long–standing debate on both leverage and speculation in financial institutions. Leverage is typically seen as risky in its own right because it is associated with financial fragility. A firm that is too highly levered is less able to withstand adverse shocks such as credit losses; see for example Bonaccorsi di Patti et al. (2015). Because no firm operate in isolation, financially fragile firms make for a less resilient economy. Speculation is also, perhaps
rightly so, mistrusted on its own; some suggest it is unnecessarily wasteful and should be banned (see Posner and Weyl, 2014) [?].

An important dimension of the recent financial crisis is the conjunction of both of these characteristics: very large, highly levered, financial firms engaged in speculation and failed. This differs from failure due solely to speculation, for example that of Long–Term Capital Management in 1998, which did not precipitate such a major crisis. It also differs from the failures of other very large firms like Worldcom in 2002 (the largest then), Enron in 2001 or Chrysler and General Motors in 2009.

In this paper we present a model in which firm size governs the returns on speculation, and therefore influences the incentives to engage in it. Our’s is a dynamic–contracting model under moral hazard. The agent controls the profitability of a firm through a scalable arithmetic Brownian motion. That agent must be given incentives to not divert funds (equivalently, to exert effort); she can also engage in speculative activities that improve profitability but expose the firm to a Poisson process of very large losses. The firm is essential: it cannot be shut down. Hence, the threat of liquidation (and simultaneous termination of the agent) has no bite and a special resolution mechanism is required. This fits many large financial institutions such as Global Systemically Important Banks (there are 29 worldwide as of writing) and Global Systemically Important Insurers (there are 9).

We characterize incentive compatibility and show that firm size emerges as a necessary control to satisfy it. Thus, controlling the firm’s size is a contractual instrument that is required by incentive compatibility. The incentive compatible, optimal contract includes downsizing on the equilibrium path in order to preserve incentive compatibility, as well as a stopping time (termination and restart) that is driven by a minimum scale of operation. This downsizing option makes for particularly rich dynamics, and that process is characterized. If downsizing has to be so severe that the firm becomes inefficiently small, the principal prefers replacing the agent with a new one and restart the firm at the efficient scale. The contract also features a reflective boundary, beyond which cash is paid out because increasing the agent’s continuation value becomes too costly at that point. However, where that boundary exactly lies depends on the severity of the speculation problem. The value function reflects these characteristics.

If the speculation problem is mild, the dominant problem is that of cash diversion. However
because size matters for the agent’s incentives, it also matters for the termination condition that becomes the boundary condition of the HJB equation. The problem features a floating (lower boundary): rather than terminate, the principal downsizes the firm; we characterize the downsizing process. There is termination with positive probability. The value function is homogenous, with two benefits. First, that floating boundary becomes a sticky boundary and the downsizing process keeps track of agent’s sojourn at that boundary (at the principal’s will). Second, away from the boundary the differential equation that characterises the (scale adjusted) HJB equation is the same as that of DeMarzo and Sannikov (2006). In particular it features a reflecting barrier \( \tilde{W} \) beyond which the principal pays cash out to the agent instead of increasing the continuation utility.

If speculation is the dominant problem, the reflecting barrier and the termination threshold become, loosely speaking, confounded. More precisely, disbursing cash at the efficient payment barrier \( \tilde{W} \) is not incentive compatible. Payments are possible only beyond the no-speculation threshold \( W_m > \tilde{W} \). At that point however it is “constrained-efficient” to pay cash out. But because paying out decreases the continuation utility, it triggers downsizing immediately. The domain of the principal’s (size-adjusted) value function becomes a singleton. This is costly in that at that point the agent is exposed to excessive volatility, for which he must be compensated. There is termination with probability 1. We also compute the cumulative payments to the agent (passed the reflecting barrier).

There is a connection between cash–flow diversion and speculation through the incentive contract, although these activities are independent. The more efficient diversion is, the more attractive the instantaneous return on speculation becomes, and so the more difficult it is to deter. Deterring speculation, thus, requires a higher continuation value; so more efficient cash flow diversion induces a higher social cost (because of the option to speculate).

The model maps naturally into standard regulatory instruments such as leverage ratios. However here leverage must be limited not to render the firm resilient to potential losses (such as credit losses) but to deter speculation. The reason is that limiting leverage increases the stake of the insiders (their continuation value) and so enforces incentive compatibility.\(^1\) This is especially relevant when the speculation problem dominates. Then the absorbing barrier is the unique leverage ratio:

\(^1\)This is not to say that leverage ratios are not useful to also buffer credit losses – but there are none in equilibrium in this model.
it is both a lower bound by incentive compatibility, and an upper bound by efficiency.

The papers closest to this work are He (2008), Biais, Mariotti, Rochet and Villeneuve (2010) (henceforth BMRV), Biais and Casamatta (1999), as well as DeMarzo, Lidvan and Tchistyi (2014) and Rochet and Roger (2015). He (2009) extends the work of DeMarzo and Sannikov (2006) (henceforth DS) to a model with a geometric Brownian motion. Incentive compatibility is characterised as in DS and, so, is independent of scale. When principal and agent have the same discount factor, the contract prescribes an absorbing state for the continuation value: incentives are free in that the agent’s state is large enough. BMRV study the problem of large Poisson risks (accidents or losses), the probability of which is controlled through the agent’s effort. Incentive compatibility dictates that the firm be downsized following an accident as a punishment to the agent. Here I also find that downsizing is a necessary instrument, however to deter speculation (not for spur effort). More precisely, the downsizing decision must be taken before the agent exerts her actions – so it is not a punishment. Biais and Casamatta (1999), like Rochet and Roger (2015) and [?] (independently) study the double moral hazard problem of diversion (or effort) and speculation. Biais and Casamatta (1999) use a static model and consider a restricted set of instruments (equity, debt and options) and show that the optimal contract always include debt and equity, and sometimes options. Debt turns the agent into the residual claimant and so is good to spur effort, as first established by Innes (1990). In Biais and Casamatta (1999) equity is necessary to deter risk shifting. Sometimes options are required to add to that (i.e. when the speculation problem is severe). Rochet and Roger (2015) and DeMarzo, Lidvan and Tchistyi (2014) develop continuous–time versions using arithmetic Brownian motions. Incentive compatibility requires the continuation value to remain sufficiently high, but because of the ABM it is size–independent. Rochet and Roger prescribe a deterministic termination when the continuation value is too low because liquidation of the project is ruled out. DeMarzo, Lidvan and Tchistyi (2014) suggest a stochastic termination rule to avoid inefficient termination (which is allowed). Here termination is an instrument too but only because the firm must operate at some efficient scale $\bar{X}$ (and cannot be stopped); in the limit ($\bar{X} \rightarrow 0$) there is no termination. Downsizing, instead of termination, is used to preserve incentive compatibility.

This paper also connects to the literature on leverage and risk–taking, especially in financial institutions. In a series of papers, Anat Admati (and at times her coauthors) makes the argument for less leverage on grounds of less fragility for individual firms, less subsidies from society and
greater systemic resilience. I add a simple but salient point: with smaller (less levered) institutions, there is also less speculation. The optimal contract can be implemented with a minimal equity requirement imposed on the firm together with a leverage ratio (these jointly completely control size). The purpose of which is to ensure the agent has enough at stake not to engage in excessive risk-taking. Van Hoose (2007) provides a survey that suggests a persistent lack of consensus as to the role and benefit of capital requirements. [?] establish that asset risk decreases when the capitalization of a bank increases. Milne (2002) observes that a bank’s portfolio choice depends on its capitalization. This model accords well with both, and minimum capital requirements induce the institution to choose the less risky path. The reason is that breaching the capital requirement triggers downsizing. Morrison and White (2005) propose a model of adverse selection and moral hazard in which capital requirements are also used to solve the moral hazard problem and to screen out bad banks (or bankers). Last, this work connects to a more recent literature on interventions and bailouts. Zentefis (2014) shows the nature of the rescue matters: if the institution is burdened by excessively large repayments ex post (as a debtor, for example) it has incentives to default. In this model there is no default but early intervention takes the form of downsizing. Mariathasan et al. (2014) show empirically that the provision of implicit guarantee enhances risk-taking. Instead here the resolution mechanism is explicit and the only guarantee is that of intervention (to preserve incentive compatibility).

2 Model

We consider a principal-agent problem set in continuous time framework. The principal (shareholders or a regulator) must rely on the expertise of an agent (manager or firm) to operate their business. All parties are risk-neutral; the principal discounts future payments at rate $r > 0$ and the agent is (weakly) more impatient, as her discount rate $\rho \ge r$. In order to formally describe the principal–agent interaction, let us introduce the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. At each date $t \ge 0$, the agent chooses an action $a_t \in \{0, 1\}$. If the agent chooses the action 0 we say that she has speculated, whereas if her choice is 1 we say she has been prudent. Let $\mu, \Delta \mu$ and $\lambda$ be strictly
positive and define the functions

\[ \mu(a) := \begin{cases} 
\mu + \Delta \mu, & \text{if } a = 0; \\
\mu, & \text{if } a = 1,
\end{cases} \quad \text{and} \quad \lambda(a) := \begin{cases} 
\lambda, & \text{if } a = 0; \\
0, & \text{if } a = 1.
\end{cases} \] (2.1)

For a given strategy \( a = \{a_t, 0 \leq t\} \) we define \( N(a) := \{N_t(a_t), 0 \leq t\} \) as the Poisson process whose intensity is given by \( \lambda(a) = \{\lambda(a_t), 0 \leq t\} \). We shall also make use of the standard Brownian motion \( Z = \{Z_t, 0 < t\} \) and, from this point on, we assume that \( \mathbb{F} = (\mathcal{F}_t^a, t \geq 0) = \sigma(Z, N(a)) \) is the natural filtration. In the sequel we denote by \( \mathcal{A} \) the set of all admissible strategies for the agent. We are now in a position to introduce the firm’s operating cash flows, which, for \( a \in \mathcal{A} \), follow the process

\[ dS_t^a = X_t(\mu(a_t)dt + \sigma dZ_t - LdN_t(a_t)), \quad S_0 = 0, \] (2.2)

where, for \( t \geq 0 \), \( X_t \) is the firm’s size (for a bank one could think of its balance sheet). Indeed, speculation, or asset shifting, increases the drift of the cash–flows process but introduces exposure large losses. For the time being we assume that the latter are sufficient to wipe out the firm – more on this later.

The \( \mathbb{F} \)–predictable process \( X = \{X_t, 0 \leq t\} \), which is chosen by the principal, is non–increasing. In other words, we contemplate the possibility of downsizing but not of growth. Downsizing is undesirable for the principal, as it reduces the cashflows, but its presence will be necessary for incentive purposes. We normalize the initial size \( X_0 \) to one and assume there is some \( 1 > X > 0 \) that corresponds to the minimal project scale that the principal is willing to keep in operation. As soon as \( X_t = X \) the agent is fired and the firm is restructured (or liquidated). The restructuring process brings the firm’s scale back to one and results in costs \( \kappa > 0 \), which also include the cost of searching for a new agent. Liquidation earns the principal \( \Pi \). We first focus on liquidation as it is more tractable, and analyze restructuring as an extension.\(^2\) We suppose the agent’s outside option is 0.

Only the agent observes the operating cashflows \( dS_t^a \), which she reports to the principal. This generates to two sources of frictions. First, in the spirit of DS, she might report \( d\hat{S}_t < dS_t^a \) and appropriate the difference. A dollar diverted brings the agent \( \eta \leq 1 \) dollars, i.e. misreporting results

\(^2\)“Restructuring” means the firm cannot be liquidated – for example, a very large bank. Instead the agent is fired and replaced, and a new contract is initialized.
in the instantaneous profits $\eta|dS_t^n - d\hat{S}_t|$. Second, the agent can secretly engage in excessively risky ("speculative") activities that generate the additional cash-flow $X_t \Delta \mu dt$ per unit of time but expose the firm to the catastrophic losses $L$. For example, the firm sells (but does not buy) CDS (like AIG or Morgan Stanley during the Global Financial Crisis) or issues options. Or it speculates on electricity contracts (like Enron in the late 1990’s).

3 The Contract

The principal seeks to maximizes the ex-ante value of the firm in the presence of moral hazard. For the time being we assume that it is optimal for the principal to deter speculation. The contract between the principal and the agent is designed and agreed upon at date $t = 0$ and we assume that all parties can commit long-term. A contract $\Xi = (X, I, \tau)$ stipulates, contingent on the history of reported cashflows, a downsizing process $X$, a non-decreasing process $I$ of cumulative payments to the agent and a (random) termination time $\tau$. The fact that $I$ is non-decreasing reflects the agents limited liability. For a given contract $\Xi$, the agent chooses her strategy by solving

$$\sup_{a \in A} \mathbb{E} \left[ \int_0^\tau e^{-\gamma s} dI_s + \eta|dS_t^n - d\hat{S}_t| \right]$$

The contract is incentive compatible if it is designed in such a way that the agent never finds it optimal to divert cash, nor to engage in speculative activities. The principal takes this problem as a constraint and designs a contract $\Xi$ to maximize his net payoff.

Following Spear and Srivastava (1987), who introduced the recursive approach to contracting, any contract can be characterized by the stochastic process $W$ describing the continuation payoff of the agent when the contract $\Xi$ is executed. If the agent chooses strategy $a$ then

$$W_t^a(\Xi) = \mathbb{E}_t \left[ \int_t^\tau e^{-\gamma (s-t)} d\tilde{C}_s^a + e^{-\gamma (\tau-t)} W_\tau^a \right]$$

where $d\tilde{C}_s^a := dI_s + \eta|dS_t^n - d\hat{S}_t|$ is the consumption process of the agent. In order to characterize the dynamics of $W^a(\Xi)$ we make use of the Martingale Representation Theorem for jump-diffusion processes (henceforth MRT)\(^3\), which in a nutshell states that any process $Y$ that is a martingale with respect to the filtration $F$ generated by the Wiener process $Z$ and the Poisson process $N$ with

\(^3\)The theorem’s precise statement may be found in the Appendix
intensity $\lambda$ can be written as

$$Y_t = y_0 - \int_0^t h_s \{dN_s - \lambda_s ds\} + \int_0^t g_s dZ_s,$$

where $h = (h_t, t \geq 0)$ and $g = (g_t, t \geq 0)$ are unique, $\mathcal{F}$-adapted processes. Let us define the auxiliary process (the agent’s total utility)

$$\hat{\psi}_t^a(\Xi) := \mathbb{E}\left[\int_0^T e^{-\rho_s} d\hat{C}_s^a \left| \mathcal{F}_t^a \right.\right],$$

which clearly is a $\mathcal{F}$-martingale. From this point on we refrain from writing $\Xi$ as an argument, unless there are grounds for confusion. Applying the MRT, there exist $P^a = (P_t^a, t \geq 0)$ and $\beta^a = (\beta_t^a, t \geq 0)$ such that

$$\hat{\psi}_t^a = \hat{\psi}_0^a - \int_0^t e^{-\rho_s} P_s^a \{dN_s - \lambda(s)ds\} + \int_0^t e^{-\rho_s} \beta_s^a dZ_s,$$

where $e^{-\rho_s} > 0$ is a scaling factor. Moreover, Expression (3.3) can be rewritten as

$$\hat{\psi}_t^a = \int_0^t e^{-\rho_s} d\hat{C}_s^a + e^{-\rho t} W_t^a,$$

(3.5)

From Equations (3.4) and (3.5) we have that

$$\hat{\psi}_t^a = \hat{\psi}_0^a - \int_0^t e^{-\rho_s} P_s^a \{dN_s - \lambda(a_s)ds\} + \int_0^t e^{-\rho_s} \beta_s^a dZ_s = e^{-\rho t} W_t^a + \int_0^t e^{-\rho_s} d\hat{C}_s^a,$$

(3.6)

In differential form Expression (3.6) becomes, after eliminating the term $e^{-\rho t}$ and rearranging

$$dW_t^a = \rho W_t^a dt + \beta_t^a dz_t - d\hat{C}_t^a = P_t^a \{dN_t - \lambda(a_t)dt\}.$$

Here $\beta_t^a$ represents the sensitivity of the agent’s continuation payoffs to the volatility of cash flows, and $P_t^a$ to large losses. As in DS and others since, incentive compatibility can be characterized by simple conditions on these sensitivity parameters.

### 3.1 Incentive compatibility

In this section we derive conditions on the processes $\beta^a$ and $P^a$, which can be controlled through the choice of $\Xi$, that guarantee incentive compatibility. Since we must account for both diversion and speculation, it is here that our results start to markedly deviate from DS, Sannikov (2008) and He (2009), for instance.
Proposition 1 For any strategy \( a \) there is no cash diversion as long as

\[
\beta_t^a \geq \eta \sigma =: \beta,
\]

and there is no speculation, i.e. \( a_t \equiv 1 \), if and only if

\[
P_t^a = P_t \geq \eta \frac{\Delta \mu}{\lambda} X_t
\]

Combining these two expressions, a contract deters both speculation and cash diversion only if

\[
P_t \geq \frac{\beta_t}{\sigma} \frac{\Delta \mu}{\lambda} X_t,
\]

and by limited liability

\[
W_t \geq \frac{\beta_t}{\sigma} \frac{\Delta \mu}{\lambda} X_t =: W_m(X_t).
\]

Observe that size enters the second constraint but not the first one. While size has no bearing on the cash diversion (equivalently, effort provision) problem, it is very much relevant for the speculation problem.

The first part of the proposition is akin to the results of DS and He (2009). By exposing the agent to the random part of the cash–flow, the principal can deter diversion because the agent steals from her own pocket. To gain some intuition, the term

\[
\left[ \frac{\beta_t^a}{\sigma} - \eta \right] \left[ dS_t^a - \hat{dS}_t^a \right]
\]

showcases the compromise the agent makes when deciding whether or not to divert funds: she enjoys the instantaneous benefit \( \eta[dS_t^a - \hat{dS}_t^a] \) but forgoes \( [\beta_t^a / \sigma][dS_t^a - \hat{dS}_t^a] \). This is true for any strategy \( a \), as we show formally in the Appendix.

Turning to speculation, consider two strategies as follows: \( a_t \equiv 1 \) (always prudent) and \( \tilde{a} = \{ a_s = 0, s < t; a_s = 1, s \geq t \} \) (speculate until date \( t \), then be prudent), for some \( t > 0 \). Since \( s, t \) are arbitrary, what follows holds true for any strategy \( a \). To deter speculation, for any processes \( C_t \) and \( \beta_t^a \), the principal needs to make the expected penalty

\[
\int_0^t e^{-\rho s} \lambda P_s^a ds
\]

larger than the corresponding expected gains

\[
\int_0^t e^{-\rho s} \left( d\tilde{C}_s^a - d\tilde{C}_s^1 \right) ds.
\]
This must hold regardless of whether \( d\tilde{C}_s = d\tilde{C}_s \) (diversion) or \( d\tilde{C}_s = dC_s \) (not). When there is diversion, we therefore need \( P_t \geq \eta \frac{\Delta \mu}{\sigma} X_t \) and when diversion is deterred too, we require \( P_t \geq W_m(X_t) \).

That second part is novel. Engaging in speculation generates an additional gain \( \Delta \mu X_t \), of which the agent may appropriate a fraction \( \eta \leq \beta_t/\sigma \). The incentives are strongest when \( P_t \equiv W_t \): the agent must be wiped out after a large adverse event. Any further penalty would violate limited liability, thus \( W_t \geq W_m(X_t) \) so as to preserve incentive compatibility.

**Remark 1** Condition (3.10) is noteworthy: it requires the continuation value to grow linearly with the size of the firm \( X_t \). This condition arises naturally here and may contribute to explain the increase in executive compensation at large firms (see Gabaix and Landier, 2008).

### 3.2 Optimal contract

We conjecture for now (and later verify) that the optimal contract is such that

\[
\beta_t^\alpha = \beta \quad \text{and} \quad W_t \geq W_m(X_t)
\]

Given that, by assumption, the losses \( L \) are catastrophically large, the principal never wants to allow speculation. Therefore, along the optimal path \( W_t \) evolves according to the dynamics

\[
dW_t = \rho W_t dt - dI_t + \beta X_t dZ_t. \tag{3.11}
\]

### 4 The value function and the optimal contract

The principal maximizes the discounted, expected cashflows

\[
V(X, W) := \sup_{(dX, dI, \tau)} \mathbb{E} \left[ \int_0^\tau e^{-r s} (dS_a - dI_s) \bigg| X_0 = X, W_0 = W \right] \tag{4.1}
\]

through his choice of contract \( \Xi \). Under an incentive compatible contract, the dynamics of the value function \( V \) are given by the Hamilton–Jacobi–Bellman equation

\[
rV(X_t, W_t) = \rho X_t dt + \sup_{dX_t, dI_t} \left\{ -dI_t + V_W(X_t, W_t)(\rho W_t dt - dI_t) + V_X(X_t, W_t) dX_t + \frac{\beta^2 X^2}{2} - V_{WW}(X_t, W_t) dt \right\}, \tag{4.2}
\]

subject to the incentive compatibility constraints (3.8) and (3.10). Since \( X \) is a decreasing process it has bounded variations. Therefore the cross–variation term \( \langle X, W \rangle_t \) and the quadratic variation
\langle X, X \rangle_t \text{ are both zero. As a consequence there are no } V_{XW} \text{ nor } V_{XX} \text{ terms in Equation (4.2). Handling this equation is not trivial because (i) the agent is subject to downsizing at any point in time to preserve incentive compatibility – Condition (3.10) – and (ii) if downsizing leads it to become too small, it may be terminated:}

\[ V(\bar{X}, W_m) = \Pi. \] (4.3)

**Result 1:** For any size \( X_t \) the sensitivity of the agent’s continuation value and the penalty should be set as low as possible:

\[ \beta_t = \beta \]

and

\[ P_\tau = W_\tau = W_m \]

The first part of this claim is well known and follow directly from inspection of (4.2), where \( V(X_t, W_t) \) is concave in \( W_t \) – so \( \beta \) should be as small as possible. Exposing the agent to risk is costly to the principal, who does it only just enough to generate the right incentives. The second part is new, but mirrors the claim in Rochet and Roger (2015) (where, however, the arithmetic Brownian motion is not scaled). Setting the termination threshold \( W_\tau \) any higher than \( W_m \) does not change the incentive problem by (3.9). Furthermore, \( W_m \) is an intervention threshold: setting \( W_\tau \) beyond \( W_m \) only increases the probability of intervention.

In the spirit of BMRV10 the firm’s scale \( X \) appears as a state variable in Equation (4.2), whereas \( dX_t \) is a control. This obeys the non–standard structure of our problem, where the firm’s current size matters for both incentives and continuation values. Indeed, \( X_t \) is present in the equations that determine the dynamics of \( V \) and \( W \), but its impact does not stop there. The next results are novel and highlights the role of the no-speculation constraint (3.10).

**Result 2:** Let \( \tilde{W}(X) \) be characterized by

\[ V_W(X, \tilde{W}) = -1, \ V_{WW}(X, \tilde{W}) = 0. \]

Let

\[ \overline{W}(X) \equiv \tilde{W}(X) \lor W_m(X), \]
and suppose that
\[ V(X, W) > \Pi \]
then
\[ dI_t \begin{cases} \equiv 0, & \text{if } W_t < \overline{W}(X); \\ > 0, & \text{if } W_t \geq \overline{W}(X). \end{cases} \]

As is common in the literature, payments to the agent will be postponed until the time when her increasing her continuation utility becomes too expensive for the principal. This is formalized by the existence of the payment barrier \( \overline{W}(X) \) that is the smallest \( W \) that satisfies the condition \( V_W(X, \overline{W}(X)) = -1 \). Then this reflecting barrier induces upper boundary conditions for the HJB equation (4.2) that are summarized as
\[ rV(X, \overline{W}) + \rho \overline{W} = \mu X, \]
almost as in [?]. However this is only true when \( W_m(X) < \overline{W}(X) \), i.e. when the reflecting barrier is not in conflict with the incentive constraint (3.10). Whether it is true here depends on the risk-taking problem too. The complementary case described by \( W_m(X) \geq \overline{W}(X) \) pits the reflecting barrier \( \overline{W}(X) \) with the no speculation constraint (3.10). Then of course there can be no payment at \( \overline{W}(X) \), the contract is not even incentive compatible at that point. In fact the principal must even reduce \( X_t \) to restore incentive compatibility.\(^4\) Then the upper boundary of \( V(X, W) \) induced by \( \overline{W} \) and the lower boundary induced by the termination condition (4.3) are confounded. This is when speculation is the dominant issue. Accommodating the speculation issue is clearly costly to the principal in that \( V_W < -1 \) on the interval when \( \overline{W}(X) < \overline{W}(X) \).

Whether cash diversion (effort provision) or speculation (risk-shifting) is the main problem is identified also in Biais and Casamatta (1999), who study a static problem of financial contracting. In the former case a debt contract is optimal, in the latter the optimal contract features both debt and equity.

In the sequel we distinguish between two cases. In the first one the continuation value \( W_t \) reflects at \( \overline{W} \) – speculation is a mild problem, yet not trivial. In the second case the upper and lower boundaries are, loosely speaking, confounded – speculation is the dominant issue.

\(^4\)In [?], [?] or [?] for example the unique reflecting barrier is \( \overline{W} \).
4.1 Reflection at $\widetilde{W}$

Assuming that $W_m(X) < \widetilde{W}(X)$, as long as $W_m(X_t) < W_t < \widetilde{W}(X_t)$, there is neither downsizing nor monetary transfers to the agent. In that region the continuation values for the principal and the agent are characterized by the equations

$$rV(X, W) = \mu X + \rho W V_W(X, W) + \frac{\beta^2 X^2}{2} V_{WW}(X, W)$$

and

$$dW_t = \rho W_t dt + \beta X_t dZ_t,$$

respectively. As long as the (lower) incentives constraint remains slack, no changes occur to $\widetilde{W}(X_t)$ and the cumulative payments to the agent are such that $W_t$ is reflected downwards at the payment barrier. Intuitively speaking $dI_t \text{ "equals" } \max \{0, W_t - \widetilde{W}(X_t)\}$, i.e. all the value in excess of the payment barrier is immediately paid out to the agent.\(^5\)

4.1.1 Incentive compatibility and downsizing

A key feature that distinguishes our model form that of He (2009) is the possibility of facultative downsizing by the principal. Downsizing may become necessary when the incentives constraint $W_t \geq W_m(X_t)$ becomes binding. Violating this constraint would induce termination, but as long as $X_t > \overline{X}$ incentive compatibility may be restored: the principal can simply downsize. That is, the incentive constraint (3.10) induces a floating (lower) boundary at $W_m(X_t)$ – which is clearly a function of $X_t$. Intuitively speaking, the floating boundary tracks the downward component of the agent’s continuation utility (the negative increments of $W_t$) once the constraint becomes binding.

Furthermore, when this occurs the dynamics of $W$ are also impacted by the firm’s downsizing since the corresponding volatility is linear in $X$. Let $\Lambda = \beta \Delta \mu/(\sigma \lambda)$ and define

$$R_t := \inf \{W_s, 0 \leq s \leq t\},$$  \hspace{1cm} (4.4)

the running infimum of $W$ up to time $t$, i.e. the smallest value that the agent’s continuation value has attained during her tenure as manager. As long as $R_t > \Lambda X_0 = \Lambda$ no downsizing has

\(^5\)In technical terms, the process $I(t)$ is the local time of $W$ at the level $\widetilde{W}(X_t)$. We elaborate on this in the Appendix.
been required to keep incentive compatibility. As soon as that barrier is reached, though, we have
$X_t = R_t \cdot \Lambda^{-1}$. This is formalized by setting

$$X_t = \min \{ R_t \cdot \Lambda^{-1}, 1 \}. \quad (4.5)$$

Then the cumulative downsizing process $X^d$ is given by

$$X^d_t = 1 - X_t. \quad (4.6)$$

Loosely speaking, $X^d$ counts the time that the agent’s continuation value spends at the boundary $W_m(X)$, i.e. the length of each downsizing period. In terms of the optimal (constrained) size of the firm we have the following:

**Result 3:** The optimal downsizing policy is given by $X_t = \min \{ R_t \cdot \Lambda^{-1}, 1 \}$.

Under the assumption that $W_m(X) < \tilde{W}(X)$, it is clear that there should be no downsizing as long as $dI_t > 0$. Instead of downsizing the principal could simply stop the process $dI_t$, which would increase $W_t$ one-to-one.

### 4.1.2 Homogeneity of $V(X, W)$

In order to study the solution to Equation (4.2) we make use of the fact that, by virtue of the cash–flow process $S$ being linear in $X$, the value function $V$ is homogeneous in $X$, as in He 2009 and BMRV. In other words, there exists a function $v : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$V(X, W) = X v \left( \frac{W}{X} \right). \quad (4.7)$$

If we let $y_t := W_t/X_t$ we may rewrite Equation (4.2) as

$$r v(y) = \mu dt + \sup_{d_i, dx_t} \left\{ - d_i + v'(y)(\rho y dt - d_i) + (v(y) - y v'(y)) dx_t + v(y) dx_t + \frac{\beta^2}{2} v''(y) \right\}, \quad (4.8)$$

where $dx_t := dX_t / X_t$, subject to the no-diversion constraint (3.8) as before, together with the size-adjusted no-speculation constraint

$$y_t \geq y_m := \frac{\beta \Delta \mu}{\sigma \lambda}. \quad (4.9)$$

The HJB equation (4.8) resembles that of He 2009 – less the cross–derivatives for the reasons presented previously – and to some extent BMRV. It does not, however, feature a discontinuous
downsizing process, which BMRV must use to preserve incentive compatibility after observing losses. Instead, our downsizing process is continuous, and importantly, is used preemptively only to preserve incentive compatibility. That is, it is used only at the lower boundary.

In size–adjusted terms, the floating barrier \( W_m(X_t) \) becomes a sticky one at \( y_m \) for the size–adjusted continuation value \( y_t \) of the agent. More precisely, the process \( y \) may remain constant at the lower bound \( y_m \) at the principal’s will, as long as \( X_t > \bar{X} \) and \( v(y_t = y_m) \geq \Pi / \bar{X} \). If we define \( \bar{x}_t := \bar{X}/X_t \), then the termination time \( \tau \) is becomes

\[
\tau := \inf \{ t \geq 0 | y_t = y_m, \bar{x}_t = 1 \}
\]

We also know there exists \( \bar{y} \) such that

\[
v'(\bar{y}) = -1, v''(\bar{y}) = 0
\]

and both \( dx_t \) and \( dx_t \) are inactive on \((y_m, \bar{y})\). In this domain Equation (4.8) becomes

\[
rv(y) = \mu + \rho y v'(y) + \frac{\beta^2}{2} v''(y),
\]

which is a well–known problem (see [?]). Observe that \((v(y) - y v'(y))dx_t \) is only a boundary term, since \( dx_t = 0 \) whenever \( y > y_m \). In order to pin down the boundary condition that this imposes on \( v \) we use the relation

\[
V_X(X_t, y_m X_t) = v(y_m) - y_m v'(y_m).
\]

This relation holds for all \( X_t \), in particular for \( X_t = \bar{X} \). If we let \( K := V_X(\bar{X}, y_m \bar{X}) \), then we have

\[
\frac{K + y_m v'(y_m)}{v(y_m)} = 1,
\]

with \( K \), yet to be determined. Since \( \Pi = V(\bar{X}, y_m \bar{X}) = \bar{X} v(y_m) \) the above expression yields the boundary condition

\[
v'(y_m) = \frac{1}{y_m} \left[ \frac{\Pi}{\bar{X} y_m} - K \right] =: \nu
\]

(4.11)

By now we have a fair description of the problem, which we summarize in the following result.

**Proposition 2 (Description of the problem)** The function \( v \) is the solution on \((y_m, \bar{y})\) to the differential equation

\[
rv(y) = \mu + \rho y v'(y) + \frac{\beta^2}{2} v''(y)
\]

subject to the Neumann boundary conditions \( v'(y_m) = \nu \) and \( v'(\bar{y}) = -1 \). The size–adjusted payment barrier is characterized by the super–contact condition \( v''(\bar{y}) = 0 \).
Once we have found $v$, we can recover $V$ using the homogeneity property (4.7). In particular, the payment barrier is given by

$$\widetilde{W}(X_t) = \tilde{y}X_t$$

(4.12)

and along the boundary $W = y_m X$ we have

$$V(X, y_m X) = \Pi \frac{X}{X}.$$

**Proposition 3** When $\widetilde{W}(X)W_m(X)$, there exists a unique solution $V(X, W)$ to equation (4.2). The function $V(X, W)$ is concave in $W$ and increasing in $X$.

### 4.2 Confounded boundary conditions

When $W_m(X) > \widetilde{W}(X)$ the upper and lower boundaries are, loosely speaking, confounded. More precisely, $\widetilde{W}(X)$ cannot identify an upper boundary (and so cannot be a payment barrier). The principal must preserve incentive compatibility: $W_t \geq W_m$. Simultaneously optimality requires that cash be paid out. This results in a degenerate case where over any time interval $[t, t + \Delta t]$ the agent is paid and the firm is downsized infinitely often. This works as follows:

- The project is initiated with size $X_0$ and the agent’s continuation utility is $W_0$.
- If $W_0 > W_m(X_0)$ then an exceptional lump sum payment $I_0 = W_0 - W_m(X_0)$ is made to the agent because her continuation utility exceeds the payment barrier $\widetilde{W}(W_0)$.
- We then have $W_{0+} = W_m(X_0)$ and the incentives constraint is binding. At this point there are counteracting forces at work: $W_t$ should be reflected downwards because $W_t > \widetilde{W}(X_t)$. Simultaneously, the reflections of $W_t$ must be accompanied by downsizing, so as to preserve incentive compatibility. In other words the aforementioned property that downsizing never occurs if $dI_y > 0$ no longer holds.
- The contract that results in these dynamics is the following:

$$I_t = \max \left\{ 0, \max_{0 \leq s \leq t} \{ W_t - \widetilde{W}(X_t) \} \right\}$$

(4.13)

and

$$X_t = \Lambda \min_{0 \leq s \leq t} \{ W_t \}.$$

(4.14)

Homogeneity of the function $V(X, W)$ continues to hold.
Result 4: Let
\[ \bar{y} = \tilde{y} \lor y_m \]
When \( \bar{y} = y_m \) the agent is exposed to “excessive” volatility.

Volatility is excessive in that sense that it is beyond what is required to solve the diversion problem. It is necessary to preserve the no speculation condition. The agent must be compensated for it, which is costly to the principal since \( V_W < -1 \) on that range. If he could he’d rather pay the agent cash, but instead he must offer an “inefficiently” high continuation value \( W_m \). Immediately to the right of \( W_m \), \( V \) attaches to a linear segment with slope -1.

4.3 Solutions

In this section we collect results that are common to both cases. To complete the analysis of the value function we compute to optimal payment rate when \( W_t \) reflects at \( W \). The principal releases just enough cash to maintain the agent at \( \bar{W} \). As in DS, He or BMRV, this level of \( W_t \) also corresponds to the highest possible value \( V(X, W) \) for the principal and the largest social value – here subject to the additional no-speculation condition.

Result 5: Let
\[ \bar{y} = \tilde{y} \lor y_m \]
The cumulative transfers to the agent are computed as
\[ I_t = \int_0^t X_t d\bar{i}_t, \text{ where } \bar{i}_t = \max \left\{ 0, \max_{0 \leq s \leq t} \{ y_t - \bar{y} \} \right\} \]
Collecting all these results we may state the following result:

Proposition 4 There exists a unique solution \( v(y) \) to the differential equation (4.2).

- When \( y_m < \tilde{y} \), the solution is identified by Proposition 3 and there is termination at \( \bar{X} \) with strictly positive probability.

- When \( y_m \geq \tilde{y} \), the solution \( v(y) \) reduces to a single point \( v(y_m) \), where
  \[ v'(y_m) = -1, \quad v''(y_m) = 0, \]
  and \( y_m \) is absorbing. There is termination at \( \bar{X} \) with probability 1.
The value function is depicted in Figure 1 where the left panel is the reflecting case and the right panel the confounded case.

5 Implementation

At this point we broadly outline an implementation for the regular case; the complementary one is still to be developed. Because we use a scaled arithmetic Brownian motion we can implement the contract using standard securities as in DS or Rochet and Roger (2015).

If we suppose that the principal represents the shareholders of a firm who contract with a CEO (the agent), the contract may be implemented with a mix of equity debt and credit line, plus a requirement that the agent’s inside equity \( W_t \) be at least equal to \( W_m \), as well as the downsizing process \( X^d_t \). When \( W_t \) reaches \( \tilde{W} \) the agent receives a bonus—short-term incentives. But when \( W_t \) reaches \( W_m \) the principal retains the right to downsize. In consequence all other (debt) contracts are also contingent on size. We interpret this as a contractual covenant.

If contemplating a regulation contract where the principal is a regulator and the agent a regulated entity, we can let \( \omega = \eta y \) denote the agent’s claim on the franchise value of the entity, and \( (1-\eta)y \) that of the principal. The agent also issues riskless debt (since she is not speculating) with face value \( \mu/\rho \). The value \( y_m \) is an intervention threshold that is interpreted as a leverage ratio, together with a downsizing policy. The fraction \( 1-\eta \) of the entity’s cash flows is a licensing fee; for example, it may be a banking licensing fee.
6 Extensions

To complete later

6.1 Upsizing: costly investment

In the main analysis we only consider downsizing, i.e. $X_t$ in a non-increasing process. But of course firms to grow as well as shrink, and given that $dS_t$ is linear in $X_t$, the principal may want to invest in the firm sometimes. In this section we consider exactly that option.

6.2 No liquidation at boundary

So far we simplified the analysis by pinning the lower boundary via the termination condition with an exogenous liquidation value. In many case liquidation may not be desirable, nor even possible. For example, liquidating Lehman Brothers proved to be very disruptive and socially costly. Likewise a large utility company (e.g. electricity transmission) can hardly be liquidated; ditto for a clearinghouse.

In such cases continuation of service prevents liquidation. Instead the contract must be terminated and the agent replaced, and a new contract with a new agent must be initialised. This modifies the termination condition, which becomes

$$V(\bar{X},W_m) = V(X_0 = 1, W_0) - \kappa$$

where $\kappa$ is the restructuring cost. This condition is a fixed point problem (at a floating boundary).

7 Discussion

Leverage and scale. The term $y_t = W_t/X_t$ may be interpreted in several ways. In the context of executive compensation it may be understood as the stake of the executive in the firm. Then this model suggests that this stake must be significant enough compared to the wealth of the agent and it must increase linearly in the size of the firm.\footnote{\textit{In this model the agent’s continuation value is her only wealth.}}

An alternative interpretation is that of leverage. Then $W_t$ is the equity market value of the firm and the model (i) explains the origin of leverage regulation, and roots it squarely in the opportunity
to speculate, (ii) shows that a firm has incentive gamble for resurrection when the market value of its equity is too low and (iii) claims that intervention is necessary on the equilibrium path.

Indeed, as argued by Diamond and Rajan (2000) in the context of banking, equity is not required to buffer credit losses. Deposit holders (and other kinds of creditors) may be shielded from credit losses by subordinated debt; what matters is that claim be junior to deposits. Here we show that equity is necessary – to prevent excessive risk-taking.

This interpretation is particularly appealing at the lower boundary, where maintaining adequate incentives requires some adjustment of the continuation value through downsizing for incentive reason. When the boundaries are confounded, that adjustment is motivated both by incentive compatibility by paying out for efficiency reasons. This maps well into the permanent adjustments that bank equity is the object of.

**Monitoring.** Monitoring is a standard remedy to moral hazard. Here one has to be careful as to what is monitored. Monitoring that somehow results in reducing the change $\Delta \mu$ in the drift is uniformly positive: it reduces $y_m$ by curtailing the incentives to engage in risky activities. However monitoring to reduce the probability $\lambda$ of catastrophic losses is uniformly bad(!); it increases $y_m$. It is a license to speculate: a large loss is even less likely. Hence under the lens of risk management, it is better to reduce the magnitude of losses ($L$) than their frequency $\lambda$.

**Rents.** In this model we require $y_m > 0$ for incentive reasons even though the agent’s outside option is 0; the agent earns rents. Observationally this corresponds to providing executives with seemingly too generous incentive packages, especially if some cash pay out is necessary for subsistence. Likewise, it may appear that some banks are made out hold too much equity; here it is justified as the only means to deter speculation.

### 8 Conclusion

This paper proposes a contracting model in continuous time with a scaled arithmetic process and the option to speculate. Speculation, or excessive risk taking, improves the drift of the stochastic process but also introduces the risk of large losses governed by a Poisson process.

Incentive compatibility requires that the agent hold large enough a continuation value at all
times. This justifies equity holding as a regulatory instrument. Because of the scaled ABM process, that continuation value must exceed a threshold that is linear in the size of the project. This implies that size becomes an instrument of incentive compatibility. This justifies leverage regulation, as in banking for example.

How exactly the continuation value is used to steer the agent depends on the severity of each problem. When diversion is comparatively severe, the optimal contract sees the continuation value grow to a (now) standard reflecting barrier \( \hat{W} \). In this case speculation is not the major concern and is dealt with at the lower boundary only; it takes the form of a leverage ratio. If speculation is the relatively more severe problem, the optimal contract requires that the continuation value grow to a new reflecting barrier \( \bar{W} > \hat{W} \). Doing so is costly to the principal, but it is necessary.

A point of interest that this paper does not address is whether the contract is fully optimal in the unrestricted class of contracts. What is characterized here is a the optimal incentive compatible contract, where speculation is never engaged into. But this comes at an additional cost on the range \([\bar{W}, y_{\mu}, X_t] \) (when \( \Delta \mu > \lambda \)). It is not immediate that the principal always wants to offer such as contract. That question is left to further research.
Appendix

A Technical background

A.1 Martingale representation

Theorem 1 (Martingale Representation) Let \((\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F} = (\mathcal{F}_t, t \geq 0))\) be a filtered probability space carrying a Poisson process \(N = (N_t, t \geq 0)\) with intensity \(\lambda = (\lambda_t, t \geq 0)\) as well as a standard Wiener process \(Z = (Z_t, t \geq 0)\). Assume that the filtration \(\mathcal{F}\) is the internal one, i.e.

\[ \mathcal{F}_t = \sigma(N_s, Z_s, s \leq t). \]

Then, for every \(\mathcal{F}\)-martingale \(X\), there exist measurable processes \(h = (h_t, t \geq 0)\) and \(g = (g_t, t \geq 0)\) such that

\[ X_t = x_0 - \int_0^t h_s [dN_s - \lambda_s ds] + \int_0^t g_s dZ_s. \]

The result is well known so the proof is omitted.

A.2 Other

The main text spares the reader some technicalities. Underlying the choice of whether to speculate is an action: \(a \in \{0, 1\}\) that alters the drift \(\mu(a)dt\) and introduces the Poisson process \(dN_t\) of losses \(L\). That action generates a probability distribution over the paths of both \(\mu(a)\) and \(N\); so (implicitly) all expectations are taken with respect to that distribution.

B Proofs

Proof of Proposition 1: We deal first with diversion, in the spirit of the aforementioned papers. Using Expression (2.2), we may rewrite the dynamics of \(W^a\) as

\[ dW^a_t = \rho W^a_t dt + \frac{\beta^a}{\sigma} (dX^a_t - \mu(a)X^a_t dt) - d\Tilde{C}^a_t - P^a_t [dN_t - \lambda dt]. \] (B.1)

Furthermore, we have that

\[ \frac{\beta^a}{\sigma} (dS^a_t - \mu(a)X_t dt) - d\Tilde{C}^a_t = \left[ \frac{\beta^a}{\sigma} - \eta \right] [dS^a_t - d\Tilde{S}^a_t] + \frac{\beta^a}{\sigma} [d\Tilde{S}^a_t - \mu(a)X_t dt]. \] (B.2)
Since \( \mathbb{E} \left[ P_t^a \left[ dN_t - \lambda dt \right] \right] = 0, \)
\[
\mathbb{E} \left[ dW_t^a - \rho W_t^a dt \right] \geq 0 \iff \left[ \frac{\beta_t}{\sigma} - \eta \right] \left[ dS_t^a - d\tilde{S}_t^a \right] \geq 0 \tag{B.3}
\]
and because \( dS_t^a - d\tilde{S}_t^a \), the latter inequality requires \( \frac{\beta_t}{\sigma} - \eta \), which is our first constraint. We now turn our attention to speculation. Recall the strategies \( a \equiv 1 \) and \( \tilde{a} = \{ a_s = 0, s < t; a_s = 1, s \geq t \} \); the agent’s total utility under strategy \( \tilde{a} \) satisfies
\[
\tilde{\psi}_t^\tilde{a} = \tilde{\psi}_t^1 + \int_0^t e^{-\rho s} (d\tilde{C}_s^\tilde{a} - d\tilde{C}_s^1) = \tilde{\psi}_0^1 - \int_0^t e^{-\rho s} P_s^a [dN_s(a_s) - \lambda(a_s)ds] + \int_0^t e^{-\rho s} \beta_s^a dZ_s + \int_0^t e^{-\rho s} (d\tilde{C}_s^\tilde{a} - d\tilde{C}_s^1).
\]
where the second line exploits the MRT and follows from the fact that \( \tilde{\psi}_t^\tilde{a} \) is a martingale. Let \( P^a \) be the probability distribution induced by the strategy \( a \) on the paths of \( N(\tilde{a}) \). Under \( P^a \) the above equation becomes
\[
\tilde{\psi}_t^\tilde{a} = \tilde{\psi}_t^1 - \int_0^t e^{-\rho s} P_s^a [dN_s(\tilde{a}_s) - \lambda(\tilde{a}_s)ds] - \int_0^t e^{-\rho s} \lambda P_s^a ds + \int_0^t e^{-\rho s} \beta_s^\tilde{a} dZ_s + \int_0^t e^{-\rho s} (d\tilde{C}_s^\tilde{a} - d\tilde{C}_s^1).
\]
In order to guarantee that the strategy \( a \equiv 1 \) is preferable to \( \tilde{a} \), it suffices to to make sure that the drift of the semimartingale \( \tilde{\psi}^\tilde{a} \) is negative, which holds iff
\[
\int_0^t e^{-\rho s} (d\tilde{C}_s^\tilde{a} - d\tilde{C}_s^1) \leq \int_0^t e^{-\rho s} \lambda P_s^\tilde{a} ds. \tag{B.4}
\]
If the agent chooses to speculate he may divert \( \eta \Delta \mu X_t \) or he may report truthfully to earn a higher payment. In the first case (B.4) implies
\[
\eta \Delta \mu X_t \leq \lambda P_t
\]
and in the second one,
\[
\frac{\beta_t}{\sigma} \Delta \mu X_t \leq \lambda P_t, \tag{B.5}
\]
which is our second constraint (the no-speculation constraint).

Material supporting Results 1-6:
Result 1: sensitivity.

**Lemma 1** The value function $V(X, W)$ is concave in $W$.

This reflects the diminishing power of incentives as $W_t$ grows. So the term $V_W dW_t$ should be as large as possible, that is, $W_t$ and $dW_t$ as small as possible while still preserving incentive compatibility. Likewise with the quadratic term $V_W W^2/2$.

**Result 2: reflecting barrier.** Substitute the definition of the dynamics of $W_t$ (3.11) in the HJB equation (4.2) and maximise with respect to $dI_t$

$$-1 - V_W(X, \tilde{W}) = 0,$$

which characterizes $\tilde{W}$. At that point the continuation value $W_t$ remains unchanged: $dW_t = 0$. Reading from the HJB equation (4.2) $V_{dI_t} = -1$, and we attach that linear segment to $V_W(X, \tilde{W})$ at $\tilde{W}$. The second condition then requires differentiability at that point, that is, $V_W W(X, \tilde{W}) = V_{dI_t dI_t} = 0$ (super contact).

**Result 3: downsizing.** To do.

**Result 4: volatility.** When $y_t \geq y_m$, the volatility that the agent faces is

$$y_t \sigma X_t dZ_t \geq y_m \sigma X_t dZ_t = \eta \frac{\Delta \mu}{\lambda} \sigma X_t dZ_t > \beta dZ_t,$$

since $\Delta \mu > \lambda$.

**Result 5: payment flow.** At $\tilde{W}$, $dW_t = 0$; using the dynamics of $dW_t$ given by (3.11) and substituting the definition of $i_t$ gives the result. ■

**Proof of Proposition 2:** The boundary conditions and downsizing policies are reiterations of Results 2 and 3. ■

**Proof of Proposition 3:** From the work of DeMarzo and Sannikov (2006) we know that $v(y)$ exists, is unique and is concave on $(m, \bar{y})$. The Neumann boundary

$$v'(y) = \nu$$
is uniquely defined too since it involves only the constants $K$, $\Pi$ and $y_m$. Since $V(X, W) = Xv(y) = Xv(W/X)$, $V(X, W)$ is also concave in $W$. Last by homogeneity of $V(X, W)$ in $X$, $V(X, W)$ is increasing in $X$. ■