Asset Pricing with Heterogeneous Agents and Long-Run Risk*

Walter Pohl  Karl Schmedders  Ole Wilms
University of Zurich  University of Zurich and Swiss Finance Institute  University of Zurich

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Abstract

This paper uses projection methods to solve the social planner problem for an endowment economy with heterogeneous agents with recursive preferences. We use this to examine the effect of agent belief heterogeneity on long-run risk models. We find that for the long-run risk explanation to adequately explain the equity premium, it is not sufficient for long-run risk to merely exist: agents must all agree that it exists. Agents who believe in a lower persistence level can come to dominate the economy, even if their belief is wrong. In the long run, this drives the equity down below the level observed in the data.

Keywords: asset pricing, long-run risk, recursive preferences, heterogeneous agents.

JEL codes: G11, G12.

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1 Introduction

Under the standard assumption of CRRA preferences, belief differences in asset pricing models with heterogeneous investors are well understood. If investors differ with regard to their beliefs, it is always the investor with the beliefs closer to the true distribution who dominates the economy in the long-run. However, a recent paper by Borovička (2015) shows that this simple conclusion does not hold true for the general case of Epstein-Zin preferences. Investors who systematically overestimate the mean growth rate of the economy can dominate in the long-run in a simple setting with Epstein-Zin preferences. Borovička (2015) finds four channels that determine the equilibrium outcomes. In this paper we pay particular attention to the risk premium channel, as it has striking implications for asset pricing models.

The risk premium channel describes the fact that—under the assumption that risk premia are high—investors who are more optimistic will invest into a portfolio with a higher average return and therefore accumulate wealth. To be consistent with the data, the risk premium in asset pricing models must be high (see Mehra and Prescott (1985)). As a consequence, the influence of the risk premium channel is also potentially high. So if there are investors with different beliefs about the state processes, the risk premium channel will shift the wealth towards the investors with more optimistic beliefs. But risk premia are lower for optimistic investors as they are less afraid of negative shocks. So if there are investors with different beliefs, the risk premium channel will shift wealth to the more optimistic investors and hence decrease risk premia.

As Borovička (2015) shows, the risk premium channel does not solely determine equilibrium outcomes, but there are several interacting effects. So it is not clear if the risk premium channel dominates, whether its influence significantly affects short-run dynamics or only matters in the very long-run. (Whether an investor holds most of the wealth in several thousand years is rather unimportant for asset pricing models as financial market data is only considered for the past century.)

To analyze the aggregate effects of the risk premium channel, we use a simple long-run risk asset pricing model, with optimistic and pessimistic investors. More precisely, investors have different beliefs about the long-run risk process. We find that investors who are more optimistic about long-run risks accumulate wealth on average. Even if they initially hold only very small consumption shares, their shares increases dramatically even after short time periods. As small differences in the beliefs about the long-run risk process have large effects on asset prices, we report a drop in the equity premium by 2% within a century. This result holds true for small differences in the beliefs and irrespectively of whether the optimistic investor has the correct beliefs or not.
Related Literature The influence of agent heterogeneity on market outcomes under the standard assumption of time-separable preferences is well-understood. For example [Sandroni (2000)] or [Blume and Easley (2006)] have analyzed the influence of differences in beliefs on market selection. The market selection hypothesis, as first described by [Alchian (1950)] and [Friedman (1953)], states that agents with systematically wrong beliefs about the distribution of future quantities will lose wealth on average and will be eventually driven out of the market. So in the long-run only the agents with rational expectations will survive. [Sandroni (2000)] and [Blume and Easley (2006)] find strong support for this hypothesis under the assumption of time separable preferences. [Yan (2008)] and [Cvitanić, Jouini, Malamud, and Napp (2012)] analyze the survival of investors in a continuous time framework when there are not only differences in the beliefs but also potentially differences in the utility parameters of the investors. They show that it is always the investor with the lowest survival index who survives in the long-run. However, the 'long-run' can be very long and hence, irrational investors can have significant effects on asset prices even under the assumption of time-separable preferences. [David (2008)] considers a similar model setup, where both agents have distorted estimates about the mean growth rate of the economy and shows, that—as agents with lower risk aversion undertake more aggressive trading strategies—the equity premium increases, the lower the risk aversion. [Chen, Joslin, and Tran (2012)] analyze how differences in the beliefs about the probability of disasters affect asset prices. They show that, even if there is only a small fraction of investors who are optimistic about disasters, they sell insurance for the disaster states and hence, eliminate most of the risk premium associated with disaster risk.

For non-time-separable utility equilibrium outcomes change fundamentally. However, there has been little research in this area, as solving such models is anything but trivial. [Borovička (2015)] shows that agents with fundamentally wrong beliefs can survive or even dominate in an economy with recursive utility. So the inferences about market selection and equilibrium

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1. Yan (2008) shows that the survival index increases with the belief distortion, risk aversion and subjective time discount rate of the investor.
2. Dumas, Uppal, and Wang (2000) show how to solve continuous time asset pricing models with heterogeneous investors and recursive utility. In particular, they show how to characterize the equilibrium by a single value function instead of one value function for each agent. Bhamra and Uppal (2014) show how to solve models with heterogeneous investors that have habit preferences. Closely related to the method we use in this paper is the approach described in Collin-Dufresne, Johannes, and Lochstoer (2015), who show how to solve discrete time economies with heterogeneous investors and recursive preferences. They derive similar expressions for the characterization of the equilibrium by equation the intertemporal marginal rates of substitutions of the investors. However, their numerical methods to solve for the equilibrium functions numerically fundamentally differs from our approach. While they transform the infinite horizon economy to a finite horizon and solve the model by a backward recursion, we propose a solution method based on projection methods to actually solve the infinite horizon problem.
3. Borovička (2015) describes four channels that affect equilibrium outcomes. We examine these channels in more detail in Section 4.1 and show how they affect equilibrium outcomes in the asset pricing model considered in this paper.
outcomes fundamentally differ under the assumption of general recursive utility compared to
the special case of standard time separable preferences. While Borovička (2015) concentrates
on the special case of i.i.d. consumption growth, Branger, Dumitrescu, Ivanova, and Schlag
(2011) generalize the results to a model with long-run risks as a state variable.

However, most papers with heterogeneous investors and recursive preferences only consider
an i.i.d. process for consumption growth. For example Gărleanu and Panageas (2015) analyze
the influence of heterogeneity in the preference parameters on asset prices in a two agent OLG
economy. Roche (2011) considers a model where the heterogeneous investors can only invest
in a stock but there is no risk-free bond. Hence, as there is no savings trade-off, the impact
of recursive preferences on equilibrium outcomes will be quite different.

Exceptions not relying on the i.i.d. assumptions are for example the papers by Branger,
papers reexamine the influence of belief differences about disaster risk with Epstein-Zin in
stead of CRRA preferences as in Chen, Joslin, and Tran (2012). Branger, Konermann, and
Schlag (2015) provide evidence that the influence of investors with more optimistic beliefs
about disasters is less profound, when disaster occur to the growth rate of consumption and
investors have recursive preferences. Collin-Dufresne, Johannes, and Lochstoer (2016) make
a similar claim but for a different reason. They show that, if the investors can learn about
the probability of disaster and if the investors have recursive preferences, the impact of the
optimistic investor on asset prices decreases. Optimists are uncertain about the probability of
disaster and hence, will provide less insurance to the pessimistic investors. Collin-Dufresne,
Johannes, and Lochstoer (2016) use an OLG model with two generations to model optimists
and pessimists. Hence—in contrast to the results in this study—the consumption shares of
the investors are fixed and the increasing influence of optimistic agents due to the risk aversion
channel over time are not captured.

The paper is organized as follows. In Section 2 we describe the general equilibrium for
the asset pricing model with heterogeneous investors and recursive preferences. A detailed
derivation of the equilibrium is shown in Appendix A. Section 3 describes the long-run risk
model with 2 investors and different beliefs about long-run risks. Results are shown in Section
4.

2 Model Setup and General Equilibrium

We consider a standard Lucas asset pricing model with heterogeneous agents and complete
markets. Time is discrete and indexed by \( t \in \mathbb{N}_0 \equiv \{0, 1, \ldots \} \). The economy is popu-
lated by a finite number of infinitely-lived agents of types \( \mathbb{H} = \{1, 2, \ldots H\} \) and we define
\( \mathbb{H}^{-} = \{2, 3, \ldots, H\} \). Log aggregate consumption growth \( \Delta c_{t+1} \) is exogenous and the individual consumption share of agent \( h \) in period \( t \) is given by \( c_{t}^{h} \). Agents have recursive utility and can differ with respect to their utility parameters and their beliefs about the state processes.

In the following we state the equilibrium conditions for the special case of recursive utility as in [Epstein and Zin (1989)] and [Weil (1989)] that we focus on in this paper. For an extensive derivation of the equations please refer to Appendix A.

We solve the model as a social planner’s problem where each agent has a normalized pareto weight \( \lambda_{h}^{t} \in (0, 1) \). \( \lambda_{h}^{t} \) varies over time and is endogenously determined by the agent’s optimal consumption decisions. We begin the description of the equilibrium with the value functions of the individual agents. For Epstein-Zin (EZ) preferences the value functions are given by

\[
v_{h}^{t} = (1 - \delta^{h})(c_{t}^{h})^{\rho^{h}} + \delta^{h} R_{t}^{h} \left[ v_{t+1}^{h} e^{\rho^{h} \Delta c_{t+1}} \right], \quad h \in \mathbb{H}.
\]

where \( R_{t}^{h} [x] = \frac{1}{\rho^{h}} \left( E_{t}^{h} \left[ (\rho^{h} x)^{\alpha^{h}} \right] \right)^{\frac{1}{\alpha^{h}}} \). The parameter \( \delta^{h} \) is the discount factor, \( \rho^{h} = 1 - \frac{1}{\psi^{h}} \) determines the intertemporal elasticity of substitution \( \psi^{h} \) and \( \alpha^{h} = 1 - \gamma^{h} \) determines the relative risk aversion \( \gamma^{h} \) of agent \( h \). \( E_{t}^{h} \) denotes the time \( t \) conditional expectation of agent \( h \) under his subjective probability measure \( dP_{t,t+1}^{h} \). We also define the true probability measure by \( dP_{t,t+1} \). In Appendix A we show that the consumption share \( c_{t}^{h} \) of agent \( h \) is given by

\[
\Delta^{h} (1 - \delta^{h})(c_{t}^{h})^{\rho^{h}-1} = (1 - \Delta^{h}) (1 - \delta^{1})(c_{t}^{1})^{\rho^{1}-1}, \quad h \in \mathbb{H}^{-}
\]

and market clearing requires that

\[
\sum_{h=1}^{H} c_{t}^{h} = 1.
\]

Hence, for given values of \( \Delta_{t}^{h} \) we can solve for the individual consumption shares by solving the \( H - 1 \) nonlinear equations (CD) jointly with the market clearing condition (MC). The equilibrium is completed by the dynamics of the weights \( \Delta_{t}^{h} \) given by

\[
\Delta_{t+1}^{h} = \frac{\Pi_{t+1}^{h} \Delta_{t}^{h}}{(1 - \Delta_{t}^{h}) \Pi_{t+1}^{1} e^{(\rho^{1} - \rho^{h}) \Delta c_{t+1}} + \Delta_{t}^{h} \Pi_{t+1}^{h}}
\]

\[
\Pi_{t+1}^{h} = \delta^{h} \left( \frac{v_{t+1}^{h} e^{\rho^{h} \Delta c_{t+1}}}{R_{t}^{h} [v_{t+1}^{h} e^{\rho^{h} \Delta c_{t+1}}]} \right)^{\frac{1}{\rho^{h}}} \frac{dP_{t,t+1}^{h}}{dP_{t,t+1}^{h}}, \quad h \in \mathbb{H}^{-}.
\]

The dynamics of the weights \( \Delta_{t}^{h} \) depend on the value functions (VF) that in turn depend

\footnote{This implies \( E^{h}_{t}[f(x_{t+1})] = \int_{X} f(x_{t+1}) dP_{t,t+1}^{h} = E_{t}[f(x_{t+1}) \frac{dP_{t,t+1}^{h}}{dP_{t,t+1}^{h}}] \)}
on the consumption decisions \( (CD) \). Hence, to compute the equilibrium we need to jointly solve equations \( (VF) \), \( (CD) \), \( (MC) \) and \( (DA) \). As there are—to the best of our knowledge—no closed-form solutions for the general model, we present in Appendix A.2 a solution approach based on projection methods to compute for the equilibrium functions numerically.

To gain some intuition about equation \( (DA) \), consider the special case of CRRA utility where \( \rho^h = \alpha^h \) and hence \( \Pi_{t+1}^h = \delta^h \frac{dP_{t,t+1}}{dP_{t,t+1}} \). We observe that the weight \( \lambda_{t+1}^h \) and hence the survival of the agent only depends on the IES \( \rho^h \), the time discount factors \( \delta^h \) and the subjective beliefs of the agents about the growth rate of the economy. A higher time discount factor, a higher IES and beliefs closer to the true probability distribution increase the agent’s chances of survival. This result is in line with Yan (2008) who derives corresponding expressions for the survival indexes of the agents in continuous time.\(^5\)

\[\text{3 A Long-Run Risk Model with Different Beliefs}\]

We consider a standard long-run risk model as in Bansal and Yaron (2004) where log aggregate consumption growth \( \Delta c_{t+1} \) and log aggregate dividend growth \( \Delta d_{t+1} \) are given by

\[
\begin{align*}
\Delta c_{t+1} &= \mu + x_t + \sigma \eta_{c,t+1} \\
x_{t+1} &= \rho_x x_t + \phi_x \sigma \eta_{x,t+1} \\
\Delta d_{t+1} &= \mu_d + \Phi x_t + \phi_d \sigma \eta_{d,t+1} + \phi_{d,c} \sigma \eta_{c,t+1}.
\end{align*}
\]

\( x_t \) captures the long-run variation in the mean of consumption and dividend growth and \( \eta_{c,t+1}, \eta_{x,t+1} \) and \( \eta_{d,t+1} \) are i.i.d. normal shocks. A key feature of long-run risk models are highly persistent shifts in the growth rate of consumption. Together with a preference for the early resolution of risks \( \gamma > \frac{1}{\psi} \) investors will dislike shocks in \( x_t \) and require a large premium for bearing those risks. Hence, the results in the long-run risk literature rely on a highly persistent state process \( x_t \), or put differently \( \rho_x \) needs to be very close one \((0.975 in the original calibration of Bansal and Yaron (2004))\). But what if there are investors who either do not belief in long-run risks or simply estimate \( \rho_x \) to be smaller compared to the large benchmark values reported in the literature? Asset prices are determined by the beliefs of the agents and hence, different opinions about the long-run risk process potentially have large affects on prices and the wealth shares of the agents.

In this paper we analyze the equilibrium implications of differences in beliefs about the long-run risk process. As \( x_t \) is not directly observable from the data, it is reasonable to assume that investors disagree—at least slightly—about the calibration of \( x_t \). However, the

\(^5\)The concept of the survival index states that, in the long-run, only the agent with the lowest index will survive.
majority of investors needs to belief in a highly persistent long-run risk process, as otherwise asset prices would be determined by the investors not (or less) believing in long-run risks and hence, the model outcomes would not be consistent with the data. Therefore, we assume that a majority investors beliefs in a highly persistent long-run risk process. But what if there is a small fraction of investors with less persistent beliefs? Initially they will have no or only a very small impact on asset prices. But do they remain irrelevant, or are they gaining wealth shares over time and if yes, how fast do their shares increase? Asset pricing models consider only short data samples, as reliable financial market data is available only for the past century. Hence, investor behavior (or survival) in the long-run is rather irrelevant for asset prices. Therefore, we analyze the short-run effects on belief differences in the long-run risk model.

For this we consider a setup with \( H = 2 \) agents where the first agent beliefs that \( \rho_x \) is close to one while the second agent beliefs that \( \rho_x \) is slightly smaller. We do not make a specific assumption about which agent has the correct beliefs and we show in our results, that for small belief differences, the true distribution has a negligible influence on equilibrium outcomes. We denote by \( \rho_x^h \) the belief of agent \( h \) about \( \rho_x \). As \( x_{t+1} \) conditional on time \( t \) information is normally distributed with mean \( \rho_x x_t \) and variance \( \phi_x^2 \sigma^2 \), \( dP_t^{h, t+1} \) is given by

\[
dP_{t,t+1}^h = \frac{1}{\sqrt{2\pi\phi_x \sigma}} \exp \left( -\frac{1}{2} \left( \frac{x_{t+1} - \rho_x^h x_t}{\phi_x \sigma} \right)^2 \right).
\]

We can think of this model as an extension of Borovička (2015) who considers a two agent setup with different beliefs about the mean growth rate of the economy. For Epstein-Zin preferences, Borovička (2015) shows that the agent with the more optimistic beliefs will dominate the economy in the long-run as long as the risk aversion in the economy is large enough. This result stands in stark contrast to the case of CRRA preferences, where the agent with the more correct beliefs will always dominate independent of the choice of preference parameters (see for example Yan (2008)).

In the model with different beliefs about the persistence of long-run risks, the beliefs about the mean growth rate of the economy change over time. Consider the example where \( \rho_x = \rho_x^1 > \rho_x^2 \). The time \( t \) expectation of agent \( h \) about the mean growth rate is given by \( \rho_x^h x_t \). This implies that for a negative realization of \( x_t \), \( \rho_x^2 x_t > \rho_x^1 x_t \) and hence the second agent is more optimistic. For \( x_t > 0 \), we have that \( \rho_x^2 x_t < \rho_x^1 x_t \) and hence the first agent is more optimistic (the second agent is more pessimistic). Hence we can think of this model as a time-varying version of Borovička (2015) where the beliefs about the growth rate change over time.

\(^6\)We assume that the only difference between the agents is their beliefs about the state processes and they share the same utility parameter specifications.
Most long-run risk models calibrate the underlying cash-flow parameters in order to match asset pricing data. For example, Bansal and Yaron (2004) use a value of $\rho_x = 0.979$, Bansal, Kiku, and Yaron (2012a) use $\rho_x = 0.975$, and Drechsler and Yaron (2011) assume $\rho_x = 0.976$. They obtain high values of $\rho_x$ by construction, as otherwise the models would not be consistent with the high equity premium observed in the data. The study by Bansal, Kiku, and Yaron (2012b) uses cash flow and asset pricing data to estimate the long-run risk model parameters and reports a value of $\rho_x \approx 0.98$ with a standard error of 0.01.

We take the estimation interval of Bansal, Kiku, and Yaron (2012b) and assume that both agents estimate $\rho_x$ to be in the range reported by Bansal, Kiku, and Yaron (2012b). For our baseline calibration we assume that the first agent believes that $\rho_x^1 = 0.985$. This implies an equity premium of 6.5% for the representative agent economy. The second agent has slightly smaller beliefs about the persistence with $\rho_x^2 = 0.975$ implying a premium of only 2.8%. Hence, even a small decrease in $\rho_x$ has large effects on asset prices. Both values lie well within the confidence interval provided by Bansal, Kiku, and Yaron (2012b). Except from the differences in beliefs, the two agents are the same and share the properties of the representative investor of Bansal and Yaron (2004) with $\psi^1 = \psi^2 = 1.5$, $\gamma^1 = \gamma^2 = 10$, $\delta^1 = \delta^2 = 0.998$. For the remaining parameters of the state processes (1) we also use the calibration from Bansal and Yaron (2004) with $\mu = 0.0015, \sigma = 0.0078, \Phi = 3, \phi_d = 4.5, \phi_{d,c} = 0$ and $\phi_x = 0.044$. (This calibration will be used for all results in this paper, unless otherwise stated.)

4 Results

We begin with the analysis of the equilibrium dynamics of the consumption shares of the individual agents. Figure 1 shows the consumption share of the second agent ($\rho_x^2 = 0.975$) over time for different initial shares $c_0^2 = \{0.01, 0.05, 0.5\}$. We show the median, 5% and 95% quantile paths using 1000 samples each consisting of 500 years of simulated data. The left panel shows the results for $\rho_x = \rho_x^1 = 0.985$ (the first agent has correct beliefs) and the right panel for $(\rho_x = \rho_x^2 = 0.975$ (the second agent has correct beliefs).

We observe, that in all cases, the consumption share of agent 2 strongly increases over time. While it occurs faster if agent 2 has the correct beliefs (right panel) the increase is almost as strong if agent 1 has the correct beliefs (left panel). Hence, given a small differences in the beliefs, independent of whether agent 1 or agent 2 has the correct beliefs, in the long-run the agent with the lower beliefs about $\rho_x$ will dominate the economy. Most importantly, even if the economy is initially almost entirely populated by agent 1 ($c_0^2 = 0.01$), his consumption share decreases sharply and he loses significant shares even in a short amount of time.
Table 1 reports the corresponding median consumption shares for different time horizons for $c_0^2 = \{0.01, 0.05, 0.5\}$. We observe that for $c_0^2 = 0.01$ the consumption share of agent 1 has decreased by 27% after 100 years, 63% after 200 years and almost 93% after 500 years. What does this imply for asset prices and aggregate financial market statistics? Initially the economy is almost entirely populated by agent 1 implying a high equity premium. But his consumption share decreases rapidly and so will his influence on asset prices.

In Table 2 we show the annualized equity premium for a given consumption share $c_t^2 = \bar{c}^2$. For $c_t^2 = 0.01$, where agent 1 dominates the economy, the aggregate risk premium is 6.42%. After 100 years, when the share of agent 1 has decreased from 99% to 72%, the premium decreases to 4.59%. Hence, even if agent 1 holds almost all wealth initially, which implies a high risk premium, the premium will drop by almost 2% within a hundred years. After 200 years, the premium decreases by almost 3% and after 500 years it is almost at the level of the representative agent economy populated only by agent 2 with a premium of 2.89%.

So if there are different investors that all believe in long-run risks but use slightly different estimates for the long-run risk process, the investor with the lower beliefs about $\rho_x$ will dominate the economy. The investor with larger $\rho_x$ will rapidly lose wealth, independent of whether his beliefs are correct or not. But a large $\rho_x$ is needed to obtain a high risk premium in the long-run risk model. Even if the investor with the high belief about $\rho_x$ almost entirely populates the economy initially, his consumption share decreases so fast, that the equity premium in the economy drops tremendously even in a short amount of time. Hence, with differences in beliefs—that are certainly present for real-world investors—there must be mechanisms that shift wealth to the investors with the higher beliefs about $\rho_x$ as otherwise risk premia in the economy collapse even in small samples.

### 4.1 Optimal Consumption Decisions and Equilibrium Dynamics

In this section we analyze the different effects that determine the optimal decisions of the agents. For this purpose we set our results in relation to the findings of Borovička (2015). Borovička (2015) considers a simple two-agent economy with identical preferences of Epstein-Zin type and different beliefs about the mean growth rate of the economy. Our model can be viewed as a generalized version of his model with time-varying beliefs about the mean growth rate in the economy. Borovička (2015) describes four channels through which the individual choices influence long-run equilibrium dynamics: the speculative bias channel, the risk premium channel, the savings channel and the speculative volatility channel. The speculative

\footnote{Note that it does not report the premium starting with a given value for $c_0^2$ and simulating a long time series, but we report the average one-period ahead premium over all $x_t$ for a given consumption share $c_t^2 = \bar{c}$.}

\footnote{In Borovička (2015) there is no long-run risk and log aggregate consumption growth is normally distributed.}
The figure shows the median, 5% and 95% quantile paths of the consumption share of agent 2 for 1000 samples each consisting of 500 years of simulated data. Agent 2 beliefs that $\rho_x = 0.975$ and agent 1 beliefs that $\rho_x = 0.985$. Results are shown for different initial consumption shares ($c_0^2 = \{0.01, 0.05, 0.5\}$). The left panel depicts the case where the pessimistic agent has the right beliefs about the long-run risk process ($\rho_x = 0.985$) and in the right panel, the optimistic agent has the right beliefs ($\rho_x = 0.975$).
The table shows the median and the standard deviation (in parenthesis) of the consumption share of agent 2 using 1000 samples each consisting of 500 years of simulated data. Agent 2 believes that $\rho^2_x = 0.975$ and agent 1 believes that $\rho^1_x = 0.985$. Summary Statistics are shown for different initial consumption shares ($c^2_0 = \{0.01, 0.05, 0.5\}$) and different time periods $T = \{100, 200, 500\}$ years. The left panel depicts the case where the pessimistic agent has the right beliefs about the long-run risk process ($\rho^2_x = 0.985$) and in the right panel, the optimistic agent has the right beliefs ($\rho^2_x = 0.975$).

Table 2: Equity Premium for Different Consumption Shares

<table>
<thead>
<tr>
<th>$c^2_t$ (rep. Agent)</th>
<th>$\rho^2_x = 0.985$</th>
<th>$\rho^2_x = 0.975$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c^2_0 = 0.01$</td>
<td>6.42</td>
<td>6.39</td>
</tr>
<tr>
<td>$c^2_0 = 0.2824$</td>
<td>4.59</td>
<td>4.57</td>
</tr>
<tr>
<td>$c^2_0 = 0.6376$</td>
<td>3.49</td>
<td>3.48</td>
</tr>
<tr>
<td>$c^2_0 = 0.9278$</td>
<td>2.89</td>
<td>2.89</td>
</tr>
<tr>
<td>$c^2_0 = 1$</td>
<td>2.76</td>
<td>2.76</td>
</tr>
</tbody>
</table>

The table shows the annualized equity premium for different consumption shares $c^2_t$. Results are reported under the assumption that the consumption share remains constant (i.e. conditional expected excess return given that $c^2_t = \bar{c}$). Agent 2 believes that $\rho^2_x = 0.975$ and agent 1 believes that $\rho^1_x = 0.985$. The left panel depicts the case where the pessimistic agent has the right beliefs about the long-run risk process ($\rho^2_x = 0.985$) and in the right panel, the optimistic agent has the right beliefs ($\rho^2_x = 0.975$).
**volatility channel** only influences equilibrium outcomes for small degrees of risk aversion and has therefore a negligible influence for the results obtained in this paper. In the following we show how the three other channels influence the equilibrium dynamics in the long-run risk model with different beliefs.

### 4.1.1 The Speculative Bias Channel

The *speculative bias channel* solely determines equilibrium outcomes in the special case of CRRA preferences. The investors assign different subjective probabilities to future states and buy assets that pay off in states they believe are more likely. Hence, for CRRA utility the agent with the more correct beliefs will survive in the long-run, as the investor with the more distorted beliefs bets on states that have a vanishing probability under the true probability measure.

To demonstrate how the *speculative bias channel* affects equilibrium outcomes in the long-run risk model with different beliefs, we first consider the special case of CRRA preferences. In Figure 2 we show the change in the Pareto weights $\lambda_{t+1}^2 - \lambda_t^2$ as a function of $\lambda_t^2$. The blue and yellow line depict the cases of a small shock ($x_t = -0.001$) and a large shock ($x_t = -0.001$) in $x_{t+1}$ respectively. The red line shows the expectation over all shocks. From left to right, the results are shown for $x_t = -0.008$, $x_t = -0.002$, $x_t = 0$, $x_t = 0.002$ and $x_t = 0.008$. All results in this section are computed for the case where agent 1 has the correct beliefs $\rho_1 = \rho_x = 0.985$ and agent 2 beliefs that $\rho_2 = 0.975$.

The second agent believes that $x_t$ converges faster to its long-run mean compared to agent 1. Hence, if $x_t < 0$ he assigns larger probabilities to large $x_{t+1}$ and bets on those states as $\rho_2^2 x_t > \rho_1^2 x_t$ (left panels). The opposite holds true for $x_t > 0$. So agent 2 loses wealth if $x_t$ is low and the shock in $x_t$ is negative (blue line in the left figures) or if $x_t$ is high and the shock in $x_t$ is also high (yellow line in the right figures). Taking the expectation over all future realization of $x_{t+1}$ (red line), agent 2 loses wealth on average (red line). For $x_t = 0$ both agents have the same beliefs ($\rho_2^2 x_t = \rho_1^2 x_t$) and hence they assign the same probabilities to $x_{t+1}$ (red and blue line coincide with the red line). As agent 2 loses wealth on average for all $x_t$ except for $x_t = 0$ he will eventually diminish in the long-run. Note that the influence of the *speculative bias channel* becomes stronger, the larger $|x_t|$, as the belief dispersion grows the more $x_t$ deviates from its unconditional mean $E(x_t) = 0$.

The *speculative bias channel* can be directly related to the two sets of results in Section 4. Results are shown for the case where agent 1 has the correct beliefs ($\rho_x = \rho_1^2$) as well as for the case where agent 2 has the correct beliefs ($\rho_x = \rho_2^2$). In the first case, the *speculative bias channel* works in favor of agent 1, while in the second case, it works in favor of agent 2. Hence, in case two, the consumption share of agent 2 increases more rapidly, as the *speculative bias channel*...
The speculative **bias channel** entirely determines the equilibrium in the standard case of CRRA preferences. Under the general assumption of Epstein-Zin preferences equilibrium dynamics become more complex. In the following we first describe the general effects of the **risk premium channel** and then analyze how the two effects interact and influence equilibrium outcomes.

**Figure 2: Changes in the Wealth-Distribution—The CRRA Case**

The figure shows the change in the optimal weights $\lambda^2_{t+1} - \lambda^2_t$ as a function of $\lambda^2_t$. From left to right, the change is shown for $x_t = -0.008$, $x_t = 0$ and $x_t = +0.008$ (± 4 standard deviations). The blue line depicts the case of a negative shock in $x_{t+1}$ ($x_{t+1} - \rho_x x_t = -0.001$) and the yellow line of a positive shock in $x_{t+1}$ ($x_{t+1} - \rho_x x_t = 0.001$). The red line shows the average over all shocks. Baseline calibration with $\rho_x = \rho^1_x$ and CRRA preferences.

### 4.1.2 The Risk Premium Channel

With Epstein-Zin preferences, risk-return trade-offs are not the same among agents and optimistic agents are willing to take larger risks (see Borovička (2015)). So if aversion, and hence risk premia, are high, the more optimistic agent will profit from investing in a portfolio with a higher average return. **Borovička (2015)** calls this the **risk premium channel**.

In Figures 3 we show the corresponding results to Figure 2 but for the general case of Epstein-Zin preferences. As risk premia in the economy are high due to the combination of high risk aversion, the preference of early resolution of risks and highly persistent shocks to $x_t$, the **risk premium channel** has a strong influence on equilibrium outcomes. Agent 2 has a lower estimate of $\rho_x$ and hence assigns lower probabilities to extreme values of long-run risks. Or put differently, agent 2 is less pessimistic (more optimistic) about long run risks compared...
to agent 1. Therefore, he is willing to invest in a riskier portfolio that potentially suffers large losses in bad times, bad has a higher return on average. We observe this pattern in Figure 3, where for all \( x_t \in \{-0.002, 0, 0.002\} \), \( \Delta_t^2 - \Delta^2 \) is positive for large positive shocks in \( x_{t+1} \) (yellow line) and negative for large negative shocks in \( x_{t+1} \) (blue line). As the portfolio of agent 2 has a higher average return, on average agent 2 accumulates wealth (red line, average over all shocks). Furthermore—as we showed in the previous section—risk premia are high, if the consumption share of agent 2 is small and low, if his share is large. So the influence of the risk premium channel decreases for large \( \Delta^2_t \). (The red line converges to zero for \( \Delta^2_t \to 1 \)).

An increase in \( \Delta^2_{t+1} - \Delta^2_t \) for positive shocks (yellow line) and an decrease for negative shocks (blue line) for positive \( x_t \), is opposed to what we would expect from the speculative bias channel. However, for very large \( x_t \) (right panel) the results reverse and hence are in line with the speculative bias channel. As noted before, the belief difference and hence the difference between the subjective probabilities becomes more pronounced for large \( |x_t| \). So the influence of the speculative bias channel becomes stronger, the further \( x_t \) is away from its unconditional mean. So the risk premium channel dominates the speculative bias channel for \( x_t \) close to its unconditional mean, but the larger \( |x_t| \) the higher becomes the influence of the speculative bias channel. For very large \( |x_t| \) the channel dominates in our benchmark calibration and agent 2 potentially loses wealth on average. However, values of \( x_t = \pm 0.008 \) (\( \pm 4 \) standard deviation of \( x_t \)) occur only very rarely and most of the time, the process stays within the \( \pm 1 \) standard deviation band. So on average, the risk premium channel dominates the speculative bias channel and hence, the consumption share of agent 2 increases on average.

### 4.1.3 The Savings Channel

The third channel that influences equilibrium outcomes for Epstein-Zin preferences is the savings channel. It states that agents with high subjective beliefs about expected returns will choose a high (low) savings rate if the IES is large (small). In the long-run risk model the IES needs to be significantly larger than 1 in order to model a strong preference for the early resolution of risks. Hence, the agent with the higher subjective expected returns chooses a higher savings rate and therefore—all other things being equal—his consumption share increases relative to the agent with the lower expected returns.

Figure 4 shows the subjective expected risk premia of the two agents as a function of the states (Figure 4a) as well as the difference between the two (Figure 4b). Agent 2 has higher subjective risk premia for small \( x_t \) and the opposite is true for large \( x_t \). Therefore, for small (large) \( x_t \), agent 2 will choose a higher (lower) savings rate compared to agent 1 that in turn increases (decreases) his consumption share. However, we find that in the aggregate, the
The figure shows the change in the optimal weights $\frac{\lambda_{t+1}^2 - \lambda_t^2}{\lambda_t^2}$ as a function of $\lambda_t^2$. From left to right, the change is shown for $x_t \in \{-0.008, -0.002, 0, 0.002, 0.008\}$. The blue line depicts the case of a negative shock in $x_{t+1}$ ($x_{t+1} - \rho_x x_t = -0.001$) and the yellow line of a positive shock in $x_{t+1}$ ($x_{t+1} - \rho_x x_t = 0.001$). The red line shows the average over all shocks. Baseline calibration with $\rho_x = \rho_x^1$.

The influence of the savings channel is rather small compared to the risk premium channel.

### 4.2 Examination of the Risk Premium Channel (Robustness of the Results)

In this section we examine the influence of the risk premium channel in more detail. We have argued that, if risk premia are high, optimistic agents invest in a portfolio with a higher expected return and therefore their consumption share increases on average. In Figure 5a we show the median consumption share of agent 2 (as in Figure 1) for different degrees of risk aversion $\gamma^h = \{2, 5, 10\}$. For $\gamma^1 = \gamma^2 = 10$ the equity premium for the representative agent economies either populated only by agent 1 or 2 are 6.53% and 2.76% respectively (see Table 2). For $\gamma^1 = \gamma^2 = 5$ they decrease to 2.71% and 0.72% and for $\gamma^1 = \gamma^2 = 2$ the premia are only -0.61% and -0.68%. So for $\gamma^h = 5$ and $\gamma^h = 2$, we expect the impact of the risk premium channel to decrease significantly. For $\gamma^h = 10$ (yellow line) the influence of the risk premium channel is strong. Hence, agent 2 profits from the investment in a portfolio with the higher expected return and rapidly accumulates wealth. For $\gamma^h = 5$ (red line) this effect becomes less severe and his consumption share increases less quickly. For $\gamma^h = 2$ (green line) this effect is almost negligible.

In Appendix B, Figure 7 we show the corresponding results to Figure 3 but with $\psi^1 = \psi^2 = 1.1$ and hence, a smaller influence of the savings channel channel. We observe that the quantitative change is rather small and the qualitative conclusions stay the same.
The figure shows the expected subjective risk premium of the two agents as a function of the states $\lambda^2_t$ and $x_t$. Panel (a) show the absolute values for the two agents and Panel (b) shows the difference between the subjective risk premium of agent 2 and agent 1.

$\gamma^h = \pi^4$ (blue line), where risk premia are negative the risk premium channel has no influence and the speculative bias channel becomes more important. As $\rho_x = \mu_1^1$ the speculative bias channel works in favor of agent 1 (agent 2 bets on states, that have a vanishing probability under the true probability measure) and agent 1 dominates the economy in the long-run. If agent two has the correct beliefs $\rho_x = \mu_2^2$, the speculative bias channel works in favor of agent 2. We show this case in Figure 5b. The blue line shows the consumption shares for $\rho_x = \mu_1^1$ and the red line for $\rho_x = \mu_2^2$. So in the absence of the risk premium channel the speculative bias channel determines equilibrium outcomes.

But long-run risk models require a high degree of risk aversion in order to obtain an equity premium consistent with the data. Consequently, the impact of the risk premium channel will be strong and the investor with the more optimistic belief will dominate the economy. To demonstrate the robustness of this result, we show in Figure 5c the consumption shares for different values of $\rho_x^2 = \{0.97, 0.975, 0.98\}$. For all cases we assume that $\rho_x = \rho_1^1$, so the speculative bias channel works in favor of the first agent. We observe that the smaller $\rho_x^2$, the faster is the increase in the consumption share initially. A smaller $\rho_x^2$ implies more optimism about long-run risks and hence a stronger influence of the risk premium channel. Simultaneously, risk premia decrease with $c_t^2$ (see Table 2) and hence, the influence of the risk premium channel decreases over time (or with $c_t^2$). This effect is stronger for small $\rho_x^2$ compared to large $\rho_x^2$ as equity premia are increasing in $\rho_x^2$ (the equity premium of the representative agent economy populated only by agent 2 with $\rho_x^2 = 0.97$ is only 1.82% compared to a value of 4.24% for $\rho_x^2 = 0.98$). So in the long-run (for large $c_t^2$) the consumption share of agent 2 grows more slowly for small $\rho_x^2$ compared to larger $\rho_x^2$. After about 500 years the second agent
has more than 80% consumption share for all three cases. So the magnitude of the belief difference influences the speed of convergence but not the qualitative implications of our main results\textsuperscript{10}.

5 Conclusion

We demonstrate that projection methods are an effective method for solving the social planner problem for an endowment economy with heterogenous agents with recursive preferences. This permits us to perform a detailed study of heterogeneity in agent beliefs for the long-run risk model of Bansal and Yaron (2004). In particular, we consider agents with different beliefs about the level of persistence in long-run risk. We find that as long as the level of heterogeneity is not too large, agents who believe in a lower level of persistence come to dominate the economy relative to agents who believe in a higher level of persistence. This holds even if the agent with the higher level of persistence holds the correct belief. This suggests that for long-run risk to work as an explanation of the equity premium, it is not sufficient for consumption suffer from long-run risk – agents must also agree on the amount of long-run risk the economy experiences.

\textsuperscript{10}For very large belief differences (i.e. very small $\rho^2$) the results may change, as the influence of the speculative bias channel becomes more important and could dominate the risk premium channel. In this case, whether agent 1 or agent 2 has the correct beliefs will become more important for equilibrium outcomes.
The figure shows the median consumption share of agent 2 for 1000 samples each consisting of 500 years of simulated data. The Panel (a) shows the time series for different degrees of risk aversion $\gamma^h = \{2, 5, 10\}$. Agent 2 believes that $\rho^2_x = 0.975$ and agent 1 has the correct beliefs with $\rho^1_x = \rho_x = 0.985$. Panel (b) shows the time series for $\gamma^h = 2$, $\rho^1_x = 0.985$, $\rho^2_x = 0.975$ for the two cases where either agent 1 (blue line) or agent 2 (red line) has the correct beliefs. Panel (c) shows the time series for $\gamma^h = 10$, $\rho^1_x = \rho_x = 0.985$ and different values for $\rho^2_x = \{0.97, 0.975, 0.98\}$. 
Appendix

A Solution Method for Asset Pricing Models with Heterogeneous Agents and Recursive Preferences

We consider a standard Lucas asset pricing model with heterogeneous agents and complete markets. Time is discrete and indexed by $t \in \mathbb{N}_0 \equiv \{0, 1, \ldots\}$. Let $s_t \in \mathbb{R}$, $l \geq 1$ denote the state of the economy which is populated by a finite number of infinitely-lived agents of types $\mathcal{H} = \{1, 2, \ldots H\}$. We also define $\mathcal{H}^- = \{2, 3, \ldots H\}$. Aggregate consumption $C(s_t)$ is exogenous and only depends on the state of the economy. Let $C^h(s_t)$ denote individual consumption in period $t$ and $\{C^h\}_t = \{C^h(s_t), C^h(s_{t+1}), \ldots\}$ denotes the consumption stream of agent $h = 1, \ldots H$. Market clearing requires that

$$\sum_{h=1}^H C^h(s_t) = C(s_t). \quad (2)$$

Each agent has recursive utility $U^h(\{C^h\}_t)$ specified by an aggregator $F^h(c, x)$ and a certainty-equivalence function $G(x)$:

$$U^h(\{C^h\}_t) = F^h(C^h(s_t), R^h_t[U^h(\{C^h\}_{t+1})]) \quad (3)$$

where

$$R^h_t(x_{t+1}) = G^{-1}_h(E^h_t[G^h(x_{t+1})]). \quad (4)$$

$E^h_t$ denotes the time $t$ conditional expectation of agent $h$ under his subjective probability measure $dP^h_{t,t+1}$. We also define the true probability measure by $dP_{t,t+1}$. To solve the model we write it as a social planner’s problem. The social planner maximizes a weighted sum of the individual agent’s utilities in $t = 0$:

$$SP(\{C\}_0, \lambda) = \sum_{h=1}^H \lambda^h U^h(\{C^h\}_0) \quad (5)$$

where each agent has a weight $\lambda = \{\lambda^1, \ldots, \lambda^H\}$, known as the Negishi weight $\{C\}_0 = \{\{C^1\}_0, \ldots, \{C^H\}_0\}$. The optimal decision of the social planner in the initial period takes into account all future consumption streams of the individual agents and the optimal decisions must satisfy the market clearing condition (2). For the ease of notation we abbreviate the state dependence in the following, so we use $C^h_t$ for $C^h(s_t)$ and $U^h_{(t)}$ for $U^h(\{C^h\}_t)$.

To derive the first-order conditions, we borrow a technique from the calculus of variations.
For any function $f_t$ we can vary the consumption of two agents by

$$
C^h_t \rightarrow C^h_t + \epsilon f_t \\
C^l_t \rightarrow C^l_t - \epsilon f_t.
$$

(6)

It is sufficient to consider the variation with agent $l = 1$. Since we have an optimal allocation it must be true that

$$
\left. \frac{dSP \{ \{C \} \}, \lambda} {d\epsilon} \right|_{\epsilon = 0} = 0.
$$

(7)

This gives us

$$
\lambda^h \hat{U}^h_{0,t} = \lambda^1 \hat{U}^1_{0,t}, \quad h \in \mathbb{H}^-,
$$

(8)

where $\hat{U}_{t,t+k}^h$ is defined as

$$
\hat{U}_{t,t+k}^h = \left. \frac{dU^h(C^h_t, \ldots, C^h_{t+k} + \epsilon f_{t+k}, \ldots)} {d\epsilon} \right|_{\epsilon = 0}.
$$

(9)

$\hat{U}_{t,t+k}^h$ satisfies a recursive equation with the initial condition

$$
\hat{U}_{t,t}^h = \left. \frac{dU^h(C^h_t + \epsilon f_t, \ldots)} {d\epsilon} \right|_{\epsilon = 0} = F^h_1 \left( C^h_t, R^h_t[U^h_{t+1}] \right) \cdot f_t
$$

(10)

where $F^h_k \left( C^h_t, R^h_t[U^h_{t+1}] \right)$ denotes the derivative of $F^h \left( C^h_t, R^h_t[U^h_{t+1}] \right)$ with respect to its $k$th argument. The recursive step is given by

$$
\hat{U}_{t,t+k}^h = \left. \frac{dF^h \left( C^h_t, R^h_t[U^h_{t+1}], \ldots + f_{t+k}, \ldots \right)} {d\epsilon} \right|_{\epsilon = 0}
$$

$$
= F^h_2 \left( C^h_t, R^h_t[U^h_{t+1}] \right) \cdot \left. \frac{dR^h_t[U^h(t)]} {d\epsilon} \right|_{\epsilon = 0}
$$

$$
= F^h_2 \left( C^h_t, R^h_t[U^h_{t+1}] \right) \cdot \left. \frac{dG^{-1}_h \left( E^h_t G^h_t \left[ U^h(t) \right] \right)} {dE^h_t G^h_t \left[ U^h(t) \right]} \right|_{\epsilon = 0}
$$

$$
= F^h_2 \left( C^h_t, R^h_t[U^h_{t+1}] \right) \cdot \left. \frac{1} {G^h_t(\cdot) \left( E^h_t G^h_t \left[ U^h_{t+1} \right] \right)} \right|_{\epsilon = 0}
$$

$$
= F^h_2 \left( C^h_t, R^h_t[U^h_{t+1}] \right) \cdot \left. \frac{G^h_t(\cdot) \cdot E^h_t \left( G^h_t(U^h_{t+1}) \cdot \hat{U}_{t+1,t+k}^h \right)} {G^h_t(R^h_t[U^h_{t+1}])} \right|_{\epsilon = 0}
$$

$$
= F^h_2 \left( C^h_t, R^h_t[U^h_{t+1}] \right) \cdot \left. \frac{E^h_t \left( G^h_t(U^h_{t+1}) \cdot \hat{U}_{t+1,t+k}^h \right)} {G^h_t(R^h_t[U^h_{t+1}])} \right|_{\epsilon = 0}
$$

(11)
where we use \( \frac{\partial G^{-1}(x)}{\partial x} = \frac{1}{G'(G^{-1}(x))} \) and abbreviate \( U^h(C_{t+1}^h, \ldots, C_{t+k}^h + \epsilon f_{t+k}, \ldots) \) by \( U^h(\cdot) \).

We can recast this recursion into a useful form. Therefore we define a second recursion \( U_{t,t+k}^h \) by

\[
U_{t,t}^h = F_1^h \left( C_t^h, R_t^h[U_{t+1}^h] \right)
\]

and

\[
U_{t,t+k}^h = \Pi_{t+1}^h U_{t+1,t+k}^h
\]

where

\[
\Pi_{t+1}^h = F_2^h \left( C_t^h, R_t^h[U_{t+1}^h] \right) \cdot \frac{G_h'(U_{t+1}^h)}{G_h(R_t^h[U_{t+1}^h])} \, dP_{t+1}^h.
\]

A simple induction shows that

\[
\hat{U}_{t,t+k}^h = E_t(U_{t,t+k}^h, f_t).
\]

Plugging (15) into the optimality condition (8) we get

\[
E_0((\lambda^h U_{0,t}^h - \lambda^1 U_{0,t}^1) f_t) = 0, \quad h \in \mathbb{H}^-.
\]

The expression above has to hold for any function \( f_t \). Hence, by the fundamental lemma of the calculus of variations (Gelfand and Fomin (1963)), it must be true that

\[
\lambda^h U_{0,t}^h = \lambda^1 U_{0,t}^1, \quad h \in \mathbb{H}^-.
\]

We can then split expression (17) into two parts. First define \( \lambda_0^h \equiv \lambda^h \) to obtain

\[
\frac{\lambda_0^h}{\lambda_1^h} = \frac{U_{0,t}^1}{U_{0,t}^0} = \frac{\Pi_0^1 U_{1,t}^1}{\Pi_0^0 U_{1,t}^0} = \frac{\Pi_1^1 \lambda_0^h}{\Pi_1^0 \lambda_1^h}, \quad h \in \mathbb{H}^-,
\]

where \( \lambda_1^h \) denotes the Negishi weight of the social planner’s optimum in \( t = 1 \). Generalizing this equation for any period \( t \), we obtain the following dynamics for the optimal weight \( \lambda_{t+1}^h \)

\[
\frac{\lambda_{t+1}^h}{\lambda_{t+1}^1} = \frac{\Pi_{t+1}^h \lambda_t^h}{\Pi_{t+1}^1 \lambda_t^1}, \quad h \in \mathbb{H}^-.
\]

Note that we can either solve the model in terms of the ratio \( \frac{\lambda_t^h}{\lambda_t^1} \) (this is equal to setting \( \lambda_t^1 = 1 \forall t \) as done in Judd, Kubler, and Schmedders (2003)) or we can normalize the weights so that they are bounded between \((0, 1)\). We later propose a solution method that uses the

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latter approach as it obtains better numerical properties.

The second expression is obtained by inserting the initial condition (12) into (17) for \( t = 0 \) and generalizing it for any social planner’s optimum at time \( t \):

\[
\lambda^h_t F^h_1 \left( C^h_t, R^h_t [U^h_{t+1}] \right) = \lambda^1_t F^1_1 \left( C^1_t, R^1_t [U^1_{t+1}] \right), \quad h \in \mathbb{H}^-. \tag{19}
\]

Equation (19) states the optimality conditions for the individual consumption choices at any time \( t \). Note that for time separable utility, \( F^h_1 \left( C^h_t, R^h_t [U^h_{t+1}] \right) \) is simply marginal utility of agent \( h \) at time \( t \), and so we obtain the same optimality condition as for example in Judd, Kubler, and Schmedders (2003) (compare equation (7) on page 2209). In this special case the Negishi weights can be pinned down in the initial period and thereafter remain constant. For general recursive preferences this is not true. The optimal weights vary over time following the law of motion described by equation (18).

We can use the two equations (18) and (19) together with the market clearing condition (2) to compute the social planner’s optimum. We therefore define \( \lambda^- = \{ \lambda^2_1, \lambda^3_1, \ldots, \lambda^H_1 \} \) and let \( V^h(\lambda^-, s_t) \) denote the value function of agent \( h \in \mathbb{H} \). We are looking for model solutions of the form \( V^h(\lambda^-, s_t) \). So additional to the exogenous states \( s_t \), the model solution depends on the time varying Negishi weights \( \lambda^- \). An optimal allocation is then characterized by the following four equations:

1. the market clearing condition (2)
   \[
   \sum_{h=1}^{H} C^h(\lambda^-, s_t) = C(s_t) \tag{20}
   \]

2. the optimality conditions (19) for the individual consumption decisions
   \[
   \lambda^h_t F^h_1 \left( C^h(\lambda^-, s_t), R^h_t [V^h(\lambda^-_{t+1}, s_{t+1})] \right) = \lambda^1_t F^1_1 \left( C^1(\lambda^-_{t+1}, s_t), R^1_t [V^1(\lambda^-_{t+1+1}, s_{t+1})] \right) \tag{21}
   \]
   for \( h \in \mathbb{H}^- \)

3. the value functions (3) of the individual agents
   \[
   V^h(\lambda^-, s_t) = F^h \left( C^h(\lambda^-, s_t), R^h_t [V^h(\lambda^-_{t+1}, s_{t+1})] \right), \quad h \in \mathbb{H} \tag{22}
   \]
the equations (18) for the dynamics of $\lambda_t^-$

\[ \frac{\lambda_{t+1}^h}{\lambda_t^1} = \frac{\Pi_{t+1}^h}{\Pi_{t+1}^1} \frac{\lambda_t^h}{\lambda_t^1}, \quad h \in \mathbb{H}^- \tag{23} \]

where

\[ \Pi_{t+1}^h = F_2^h \left( C^h(\lambda_t^-, s_t), R_t^h[V_t^h(\lambda_{t+1}^-, s_{t+1})] \right) \cdot \frac{G'_h(V_t^h(\lambda_{t+1}^-, s_{t+1}))}{G'_h(R_t^h[V_t^h(\lambda_{t+1}^-, s_{t+1})])} \frac{dP_{t,t+1}^h}{dP_{t,t+1}} \tag{24} \]

This concludes the general description of the equilibrium obtained from the social planner’s optimization problem.

A.1 The Case of Epstein-Zin Preferences

In this section we provide the specific expressions for $V^h$, $F_1^h$, $F_2^h$ and $\Pi^h$ when the heterogeneous investors have recursive preferences as in Epstein and Zin (1989) and Weil (1989). The value function for Epstein-Zin (EZ) preferences is given by\[ V_t^h = (1 - \delta^h)(C_t^h)^{\rho^h} + \delta^h R_t^h[V_{t+1}^h] \tag{25} \]

where

\[ R_t^h[V_{t+1}^h] = G_{h}^{-1}(E_t^h[G_h(V_{t+1}^h)]) \]
\[ G_h(V_{t+1}^h) = (\rho^h V_{t+1}^h)^{\frac{\alpha^h}{\rho^h}}. \]

The parameter $\delta^h$ is the discount factor, $\rho^h = 1 - \frac{1}{\psi^h}$ determines the intertemporal elasticity of substitution $\psi^h$ and $\alpha^h = 1 - \gamma^h$ determines the relative risk aversion $\gamma^h$ of agent $h$.

Using this notation the derivatives of $F^h \left( C_t^h, R_t^h[U_{t+1}^h] \right) = V_t^h$ with respect to its first and second argument are then given by

\[ F_{1,t}^h = (1 - \delta^h)(C_t^h)^{\rho^h-1} \tag{26} \]

and

\[ F_{2,t}^h = \delta^h. \tag{27} \]

\[ ^{11}\text{For the ease of notation we again abbreviate the dependence on the exogenous state } s_t \text{ and the endogenous state } \lambda_t^-. \text{ Hence we write } V_t^h \text{ for } V^h(\lambda_t^-, s_t) \text{ or } C_t^h \text{ for } C^h(\lambda_t^-, s_t). \]
Also we have

\[ G'_{h}(V_{t+1}^{h}) = \alpha^{h} \left( \rho^{h} V_{t+1}^{h} \right)^{\alpha^{h} - \rho^{h}}, \]

and for \( \Pi_{t+1}^{h} \) we obtain (see equation (24))

\[ \Pi_{t+1}^{h} = \delta^{h} \left( \frac{V_{t+1}^{h}}{F_{t}^{h} \left[ V_{t+1}^{h} \right]} \right)^{\alpha^{h} - \rho^{h}} \frac{dP_{t,t+1}^{h}}{dP_{t,t+1}}. \] (28)

In this paper we will mainly focus on growth economies. Therefore we need to introduce the following normalization to obtain a stationary formulation of the model. We define the consumption share of agent \( h \) by

\[ c_{t}^{h} = \frac{C_{t}^{h}}{C_{t}}, \quad v_{t}^{h} = \frac{V_{t}^{h}}{C_{t}}. \]

We also detrend \( \lambda_{t}^{h} \) and normalize it so that \( \lambda_{t}^{h} \in (0, 1) \):

\[ \lambda_{t}^{h} = \frac{\lambda_{t}^{h}}{C_{t}^{\rho^{h}}} + \frac{\lambda_{t}^{1}}{C_{t}^{\rho^{1}}}. \] (29)

Using simple algebra we then obtain the following system for the first-order conditions (20)-(24):
The market clearing condition:
\[ \sum_{h=1}^{H} c_t^h = 1. \]  
(MC)

The optimality condition for the individual consumption decisions:
\[ \Delta^h_t (1 - \delta^h_t)(c_t^h)^{\rho^h - 1} = (1 - \Delta^h_t)(1 - \delta^h_t)(c_t^1)^{\rho^1 - 1}, \quad h \in \mathbb{H}^- \]  
(CD)

The value functions of the individual agents:
\[ v_t^h = (1 - \delta^h_t)(c_t^h)^{\rho^h} + \delta^h_t R_t^h \left[ v_{t+1}^h e^{\rho^h \Delta c_{t+1}} \right], \quad h \in \mathbb{H}. \]  
(VF)

The equation for the dynamics of \( \lambda_t^h \):
\[ \lambda_{t+1}^h = \frac{\Pi_{t+1}^h \Delta_{t+1}^h}{(1 - \lambda_t^h) \Pi_{t+1}^1 e^{(\rho^1 - \rho^h) \Delta c_{t+1}} + \lambda_t^h \Pi_{t+1}^h} \]
\[ \Pi_{t+1}^h = \delta^h \left( \frac{v_{t+1}^h e^{\rho^h \Delta c_{t+1}}}{R_t^h \left[ v_{t+1}^h e^{\rho^h \Delta c_{t+1}} \right]} \right)^{\frac{\rho^h - \rho}{\rho^h}} \frac{dP_{t,t+1}^h}{dP_{t,t+1}^1}, \quad h \in \mathbb{H}^- \]  
(D\( \lambda \))

Note that equation (CD) and hence the individual consumption decisions \( c_t^h \) only depend on time \( t \) information and there is no intertemporal dependence. This allows us to first solve for \( c_t^h \) given the current state of the economy and in a second step solve for the dynamics of the Negishi weights. Hence, we can separate solving the optimality conditions (MC)-(D\( \lambda \)) into two steps in order to reduce the computational complexity. In the following section we describe this approach in detail.

### A.2 Computational Procedure - A Two Step Approach

For the ease of notation the following procedures are described for \( H = 2 \) agents and a single state variable \( s_t \in \mathbb{R}^1 \). However, the approach can analogously be extended to the general case of \( H > 2 \) agents and multiple states. We solve the social planner’s problem using a collocation projection. For this we perform the usual transformation from an equilibrium described by the infinite sequences (with a time index \( t \)) to the equilibrium being described by functions of some state variable(s) \( x \) on a state space \( X \). We denote the current exogenous state of the economy by \( s \) and the subsequent state in the next period by \( s' \) with the state space \( S \in \mathbb{R}^1 \). \( \lambda_2 \) denotes the current endogenous state of the Negishi weight and \( \lambda'_2 \) denotes the
corresponding state in the subsequent period with $\lambda_2 \in (0, 1)$.

We approximate the value functions of the two agents $v^h(\lambda_2, s), h = \{1, 2\}$ by a set of Chebychev basis functions $\{\Upsilon_i\}_{i \in \{0, 1, \ldots, n\}}$:

$$\hat{v}^h(\lambda_2, s; \alpha^h) = \sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_{i,j}^h \Upsilon_i(\lambda_2) \Upsilon_j(s), \quad h = \{1, 2\}$$

(30)

where $\alpha^h$ are $n \times m$ matrices of unknown coefficients. For the collocation projection we have to choose a set of collocation nodes $\{\lambda_{2k}\}_{k=0}^{n}$ and $\{s_l\}_{l=0}^{m}$ at which we evaluate $\hat{v}^h(\lambda_2, s; \alpha^h)$. In the following we show how to first solve for the individual consumption shares at the collocation nodes $c_{k,l}^h = c^h(\lambda_{2k}, s_l)$ that are then used to solve for the value functions $v^h$ and the dynamics of the endogenous state $\lambda_2$.

**Step 1: Computing Optimal Consumption Allocations**

Equation \((CD)\) has to hold at each collocation node $\{\lambda_{2k}, s_l\}_{k=0, l=0}^{n,m}$:

$$\lambda_{2k}(1 - \delta^2) \left( c_{2k,l}^2 \right)^{\rho^2 - 1} = (1 - \lambda_{2k})(1 - \delta^1) \left( c_{1k,l}^1 \right)^{\rho^1 - 1}.$$  

Together with the market clearing condition \((MC)\) we get

$$\lambda_{2k}(1 - \delta^2) \left( c_{2k,l}^2 \right)^{\rho^2 - 1} = (1 - \lambda_{2k})(1 - \delta^1) \left( 1 - c_{2k,l}^2 \right)^{\rho^1 - 1}.$$  

(31)

So for each node $\{\lambda_{2k}, s_l\}_{k=0, l=0}^{n,m}$ the optimal consumption choice $c_{k,l}^2$ can be computed by solving equation \((31)\) and $c_{k,l}^1$ is obtained by the market clearing condition \((MC)\).\(^{12}\)

\(^{12}\)Note that in the case of $H$ agents we have to solve a system of $H - 1$ equations that pin down the $H - 1$ individual consumption choices $c^h \in \mathbb{H}^\times$. 

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Step 2: Solving for the Value Function and the Dynamics of the Negishi Weights

Solving for the value function is not straightforward as it depends on the dynamics of the endogenous state $\lambda_2$ that are unknown and follow equation $[D\lambda]$. We compute the expectation over the exogenous state by a Gauss-Quadrature with $Q$ quadrature nodes. This implies that the values for $s'$ at which we evaluate $v^h$ are given by the quadrature rule. We denote the corresponding quadrature nodes by $\{s'_{l,g}\}_{l=0, g=1}^{m,Q}$ and the weights by $\{\omega_g\}_{g=1}^Q$\textsuperscript{13} We can then solve equation $[D\lambda]$ for a given pair of collocation nodes $\{\lambda_{2k}, s_l\}_{k=0, l=0}^{n,m}$ and the corresponding quadrature nodes $\{s'_{l,g}\}_{l=0, g=1}^{m,Q}$ to compute a vector $\lambda'$ of size $(n + 1) \times (m + 1) \times Q$ that consists of the corresponding values $\lambda'_{2k,l,g}$ for each node. For each $\lambda'_{2k,l,g}$ equation $[D\lambda]$ then reads

$$
\lambda'_{2k,l,g} = \frac{\lambda_{2k} \Pi^2}{(1 - \lambda_{2k}) \Pi e^{(\rho^l - \rho^2) \Delta c(s'_{l,g})} + \lambda_{2k} \Pi^2}
$$

$$
\Pi^h = \delta^h \left( R_h \left[ (\lambda'_{2k,l,g}, s'_{l,g}) e^{\rho^h \Delta c(s'_{l,g})} \right] \left| (\lambda_{2k}, s_l) \right. \right)
$$

where

$$
R_h \left[ (\lambda'_{2k,l,g}, s') e^{\rho^h \Delta c(s')} \left| \lambda_{2k}, s_l \right. \right] = G_h^{-1} \left( E \left[ G_h \left( (\lambda'_{2k,l,g}, s') e^{\rho^h \Delta c(s')} \right) \frac{dP^h(s')}{dP(s')} \left| \lambda_{2k}, s_l \right. \right. \right).
$$

Note that $\lambda'_{2k,l,g}$ depends on the full distribution of $\lambda_2$ through the expectation operator. By applying the Gauss-Quadrature to compute the expectation we get

$$
E \left[ G_h \left( (\lambda'_{2k,l,g}, s') e^{\rho^h \Delta c(s')} \right) \frac{dP^h(s')}{dP(s')} \left| \lambda_{2k}, s_l \right. \right. \right] \approx \sum_{g=1}^Q G_h \left( (\lambda'_{2k,l,g}, s_l) e^{\rho^h \Delta c(s_l)} \right) \cdot \omega_g.
$$

So by computing the expectation with the quadrature rule, we do not need the full distribution of $\lambda'$ but only have to evaluate $V^h$ at the values $\lambda'_{2k,l,g}$ that can be obtained by solving $[32]$ for each pair of collocation nodes $\{\lambda_{2k}, s_l\}_{k=0, l=0}^{n,m}$ and the corresponding quadrature nodes $\{s'_{l,g}\}_{l=0, g=1}^{m,G}$. So at the end we have a square system of equations with $(n + 1) \times (m + 1) \times G$ unknowns $\lambda'_{2k,l,g}$ and as many equations $[32]$ for each $\{k, l, g\}$.

The value function is in general not known so we have to compute it simultaneously when solving for $\lambda'_{2k,l,g}$. Plugging the approximation $[30]$ into the value function $[VF]$ yields

\textsuperscript{13}Note that the quadrature nodes $\{\{s'_{l,g}\}_{g=1}^m\}_{l=0}$ depend on the state today $\{s_l\}_{l=0}^m$. 
$v^h(\Lambda_2, s, t^h, \alpha^h) = (1 - \delta^h) \left( \frac{c^h_{2,t^h}}{\rho^h} \right)^{\nu^h} + \delta^h R^h \left[ v^h(\Lambda_2', s', \alpha^h) e^{\delta^h \Delta c(s')} \right] \Lambda_2^h, s_t].$ (33)

The collocation projection conditions require that the equation has to hold at each collocation node $\{\Lambda_2, s_t\}_{k=0, l=0}^{n, m}$. So we obtain a system of equations with $(n + 1) \times (m + 1) \times 2$ unknown coefficients $\alpha^h, h \in \{1, 2\}$ and as many equations (33) for each collocation node that we solve simultaneously with the system for $\Lambda_2^{'k, l, g}$ described above.

### A.3 Properties of the Value Function

In the case of heterogeneous agents the approximation of the value function is a delicate computational task as an agent can die out over time. Marginal utility of the agent at this limiting case is infinity which makes it difficult to obtain accurate approximations for the value function close to the singularity. To obtain information about the properties of the singularity, we formally derive the limiting behavior of the value function for the special case of an economy with no uncertainty. We then include this information in the value function approximation for the stochastic economy. This leads to significant improvements in the accuracy of the approximation of the value function as we demonstrate thereafter.

From equation (CD) we know that

$$c^2(\Lambda_2, s) = \left( \frac{1 - \delta^1}{1 - \delta^2} \right)^{\nu^2} (\Lambda_2)^{\nu^2} (1 - \Lambda_2)^{-\nu^2} (C^1(\Lambda_2, s))^{{\nu^2} \over \nu^1}. \quad (34)$$

We are interested in the properties of $c^2(\Lambda_2, s)$ for $\Lambda_2$ close to 0. For $\Lambda_2 \approx 0$ agent 1 obtains all consumption so $c^1(\Lambda_2, s) \approx 1$ and the Negishii weight of the first agent becomes 1. Therefore we obtain

$$c^2(\Lambda_2, s) \approx \left( \frac{1 - \delta^1}{1 - \delta^2} \right)^{\nu^2} (\Lambda_2)^{\nu^2} \quad (35)$$

for $\Lambda_2$ close to 0. The value function (25) for the deterministic economy at the steady state $s = s', \Lambda_2 = \Lambda_2'$ is given by

$$v^h(\Lambda_2, s) = \frac{(c^h(\Lambda_2, s))^{1 - {s^h \over \nu^h}}}{1 - {s^h \over \nu^h}}. \quad (36)$$

Inserting the behavior of $c^2(\Lambda_2, s)$ for $\Lambda_2$ close to 0, we obtain

$$v^1(\Lambda_2, s) \approx \frac{(1 - \delta^1)^{\nu^1 - 1}(1 - \Lambda_2)^{\nu^1 - 1}}{1 - {s^h \over \nu^h}}. \quad (37)$$

For the first agent we obtain a similar expression for $\Lambda_2$ close to 1 given by

$$v^1(\Lambda_2, s) \approx \frac{(1 - \delta^1)^{\nu^1 - 1}(1 - \Lambda_2)^{\nu^1 - 1}}{1 - {s^h \over \nu^h}}. \quad (38)$$

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\[ v^2(\lambda_2, s) \approx \left( \frac{1-\delta_1}{1-\delta_2} \right)^{\psi_2-1} \frac{(\lambda_2)^{\psi_2-1}}{1 - \frac{1}{\psi^2}} \equiv \Upsilon^0(\lambda_2, s). \] (37)

We denote by \( \Upsilon^0(\lambda_2, s) \) the zero basis functions which we add to the value function approximation (30) to obtain accurate approximations close to the singularity:

\[ \hat{v}^2(\lambda_2, s; \alpha^2) = \sum_{i=0}^{n} \sum_{j=0}^{m} \alpha^2_{i,j} \Upsilon_i(\lambda_2) \Upsilon_j(s) + \Upsilon^0(\lambda_2, s). \] (38)

In the following we provide an example how including the zero basis function can improve the polynomial approximation of the value function.

Consider a deterministic economy with no growth and two agents that differ with respect to their EIS. Also assume for the moment that \( \lambda_2 \) is constant over time. The value function of agent \( h \) for a given \( \lambda_2 \) is then given by

\[ v^h(\lambda_2) = \left( c^h(\lambda_2) \right)^{1-\frac{1}{\psi^h}} \] (39)

where \( c^2(\lambda_2) \) is obtained by solving

\[ \lambda_2(c^2(\lambda_2))^{-\frac{1}{\psi^2}} = (1 - \lambda_2)(1 - c^2(\lambda_2))^{-\frac{1}{\psi^2}} \] (40)

and \( c^1(\lambda_2) = 1 - c^2(\lambda_2) \). In Figure 6 we compare the accuracy of approximating the value function with and without the zero basis function for \( \psi^1 = 1.5 \) and \( \psi^1 = 1.2 \).

The first row shows closed-form solution of the value function (39) as well as the consumption choices of the two agents obtained from (40) for \( \lambda_2 \in (0, 1) \). We find that the value function increases strongly close to the singularities of the two agents that might introduce difficulties when polynomial approximations are used. In the second row we show the value function approximation using a standard polynomial of degree 4. The left panel shows the approximated function and the right panel the absolute difference to the true value function. We find that there are large approximation errors close to the singularities at \( \lambda_2 = 0 \) and \( \lambda_2 = 1 \). In the third row we show the corresponding 4-degree approximation including the zero basis function \( \Upsilon^0(\lambda_2, s) \). We find that the approximation errors are several orders of magnitudes smaller. Also the errors are approximately equally distributed over the approxi-

15 We use Chebychev nodes for the interpolation of the value function where the first node is fixed at 0 and the last node is fixed at 1.

16 The errors for the value function of agent 1 close to \( \lambda_2 = 1 \) are significantly smaller in absolute values compared to the errors for the approximation of the value function of agent 2 close to \( \lambda_2 = 0 \). But the errors for agent 1 are by far largest close to the singularity \( \lambda_2 = 1 \).
The figure shows the closed-form solution of the value function (39) as well as the consumption choices of the two agents (40) for $\lambda_2 \in (0, 1)$. There is no growth and $\psi^1 = 1.5$ and $\psi^2 = 1.2$. The figure also shows 4-degree polynomial approximations of the value function and corresponding errors $\hat{v} - \hat{v}^h$ with and without the zero basis function $\Upsilon^0(\lambda_2, s)$. 
mation interval, suggesting that the approximation adequately captures the properties of the singularities.

This concludes the description of the methodology for solving the heterogeneous agent model with recursive preferences. In the following section we apply the approach to solve the long-run risk model of Bansal and Yaron (2004) with heterogeneous agents.

A.4 Computational Details

For the projection method outlined above, we need to choose certain collocation nodes. In this paper we use 17 uniform nodes for the $\lambda^2$ dimension and 13 uniform nodes for the $x_t$ dimension. For $\lambda^2$ the minimum and maximum values are given by 0 and 1. For $x_t$ we choose the approximation interval to cover $\pm 4$ standard deviations around the unconditional mean of the process. We approximate the value functions using two-dimensional cubic splines with not-a-knot end conditions. We provide the solver with additional information that we can formally derive for the limiting cases. For example we know that for $\lambda^{2*,t} = 1$ ($\lambda^{2*,t} = 0$) agent 2 (1) consumes everything, so it corresponds to the representative agent economy populated only by agent 2 (1). Hence, we require that the value function for these cases equals the value function for the corresponding representative agent economy. We also know that for $\lambda^{2*,t} = 0$ ($\lambda^{2*,t} = 1$) consumption of agent 2 (1) is 0 and hence the value function is zero as long as $\psi^h > 1$. In the case of $\psi^h < 1$ the value function becomes $-\infty$ which is an unpleasant property when polynomials are used for the function approximation. Therefore, instead of $V^h$, we approximate $\frac{1}{V^h}$ in the case of $\psi^h < 1$. As the shocks in the model are normally distributed, we compute the expectations over the exogenous states by Gauss-Hermite quadrature using 5 nodes for the shock in $x_{t+1}$ and 3 nodes for the shock in $\Delta c_{t+1}$. Euler errors for the value function approximations evaluated on a $100 \times 100$ uniform grid for both states are less than $1 \times 10^{-4}$ suggesting a high accuracy of our results. We double checked the accuracy by increasing the approximation interval as well as the number of collocation nodes with no significant change is the results.

B Additional Figures and Tables

C Additional Stuff (remove later)

The Stochastic Discount Factor $M_{t+1}$ is given by
The figure shows the change in the optimal weights $\lambda^2_{t+1} - \lambda^2_t$ as a function of $\lambda^2_t$. From left to right, the change is shown for $x_t \in \{-0.008, -0.002, 0, 0.002, 0.008\}$. The blue line depicts the case of a negative shock in $x_{t+1}$ ($x_{t+1} - \rho x_t = -0.001$) and the yellow line of a positive shock in $x_{t+1}$ ($x_{t+1} - \rho x_t = 0.001$). The red line shows the average over all shocks. Baseline calibration with $\rho = \rho^*_x$ and $\psi^1 = \psi^2 = 1.1$ (instead of 1.5 in the baseline model).

For EZ preferences we get

$$M_{t+1} = \frac{\partial V^h_t}{\partial C_t^{h,t+1}}$$

plugging in asset pricing equation

$$1 = E_t \left( \frac{C^{h,t+1}}{C^h_t} \right)^{-\frac{1}{\psi^h}} \left( \frac{V^h_{t+1}}{R_t[V^h_{t+1}]} \right)^{\frac{1}{\psi^h} - \gamma^h} \left( \frac{W_{t+1}^{h}}{W^{h}_t C_t} + 1 \right) e^{\Delta c_{t+1}}$$

And in terms of detrended variables

$$1 = E_t \left[ \delta e^{(1 - \frac{1}{\psi^h}) \Delta x_{t+1}} \left( \frac{C^{h,t+1}}{C^h_t} \right)^{-\frac{1}{\psi^h}} \left( \frac{V^h_{t+1}}{R_t[V^h_{t+1}]} \right)^{\frac{1}{\psi^h} - \gamma^h} \left( \frac{W_{t+1}^{h}}{W^{h}_t C_t} + 1 \right) e^{c_{t+1} - c_t} \right]$$
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