A Dynamic Theory of Learning and Relationship Lending

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Abstract

We introduce private learning into a banking model to study the dynamics of relationship lending. In our model, an entrepreneur chooses between bank and market financing. Bank lending facilitates learning over time, but it subjects the borrower to the downside of information monopoly. We construct an equilibrium in which the entrepreneur starts with bank financing and subsequently refines with the market, and we find conditions under which this equilibrium is unique. Our model generates several novel results. First, both information asymmetry and entrepreneur’s reputation are accumulated over time. As a result, the bank will roll over bad loans for entrepreneurs who have accumulated enough reputation for the prospect of future loan sales. Second, this incentive to extend and pretend gets mitigated when the entrepreneur faces financial constraint. We further endogenize learning as bank’s costly decision and show how it is affected by asymmetric information and financial constraint.

Keywords: private learning, experimentation, reputation, relationship banking, information monopoly, debt rollover, extend and pretend, adverse selection, dynamic games.

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1 Introduction

How do lending relationships evolve over time? How do firms choose dynamically between bank and market financing? How does a firm’s reputation interact with its financial constraint in determining its access to finance? Why do banks sometimes roll over loans that are known to be insolvent? To answer these questions, we introduce a dynamic framework in the context of relationship lending.

It has been widely documented that bank loans contain important information about borrowers that is not available to market-based lenders (Addoum and Murfin, 2017; James, 1987; Gustafson et al., 2017). Moreover, as suggested by Lummer and McConnell (1989), such information is not produced upon a bank’s first contact with a borrower, but, instead, through repeated interactions during prolonged lending relationships which involve substantive screening and monitoring. On the other hand, as shown in Rajan (1992), learning provides information advantage to the relationship bank and thus increases the information-monopoly cost so that ultimately, the borrower may switch to lenders in the financial market. When should borrowers switch from relationship lending to market financing? How do entrepreneurs balance the tradeoff between learning and information-monopoly cost? How does loan maturity affect these decisions?

To answer these questions, we introduce private learning into a dynamic model of relationship lending. Specifically, we model an entrepreneur investing in a long-term, illiquid project whose quality is either good or bad. Only a good project has positive net present value (NPV) and should be financed. A bad project should be liquidated immediately. Initially, the quality of the project is unknown to everyone, including the entrepreneur herself. She can raise funding from either the competitive financial market or a bank that will develop into a relationship. Market financing takes the form of arm’s-length debt so that lenders only need to break even given their beliefs on the project’s quality. Under market financing, no information is ever produced and therefore, the maturity of the market debt is irrelevant. In contrast, if the entrepreneur borrows from a bank, screening and monitoring will produce “news” about the project’s quality. We model news arrival as a Poisson event and assume this news is only observed by the entrepreneur and the bank once the lending relationship starts. In other words, the bank and the entrepreneur learn privately about the project’s quality as time goes by. Meanwhile, all agents, including lenders in the financial market can observe the time since the initialization of the project, which will turn out to be the state variable.

Given the structure of learning, the bank and the borrower possess one of the three types of private information after time 0: 1) news has arrived and implies the project is good – the
informed-good type \( g \); 2) news has arrived and implies the project is bad – the informed-bad type \( b \); and 3) no news has arrived yet – the uninformed type \( u \). Upon the maturing of the bank loan, the bank and the entrepreneur jointly determine whether to roll it over, to liquidate the project, or to switch to market-based financing. These decisions are modeled as a Nash Bargaining problem with the financial constraint that the entrepreneur has no personal wealth.

By solving the model in closed form, we characterize the equilibrium with two thresholds \( \{t_g, t_b\} \) in the time since project initialization. Consequently, the equilibrium is characterized into three stages. If the bank loan matures between 0 and \( t_b \), an informed-bad type’s project will be liquidated. All other types’ matured loans will be rolled over. During this period, the average quality of borrowers who remained with banks drifts up because the informed-bad types get liquidated and exit funding. Equivalently, remaining borrowers gain reputation from the liquidation decisions of others. These liquidation decisions are socially efficient and therefore we name this stage after \textit{efficient liquidation}. If the bank loan matures between \( t_b \) and \( t_g \), however, it will be rolled over irrespective of the quality of the project. In particular, the relationship bank will roll over the loan matured between \( t_b \) and \( t_g \) even if bad news has arrived. In other words, the bank keeps extending the loan to pretend no bad news has occurred yet. Clearly, this rollover decision is inefficient. This result on banks’ rolling over bad loans can be linked to zombie lending. Finally, after time passes \( t_g \), all entrepreneurs will refinance with the market upon their bank loans maturing – the \textit{market financing} stage.

The intuitions for these results can be best explained backwards in time. When time elapsed gets sufficiently long, all entrepreneurs will ultimately switch to market financing, driven by the assumption that market-based lenders are competitive and have lower costs of capital. This effect is captured by the threshold \( t_g \). Now, imagine a scenario that bad news arrives shortly before \( t_g \). The relationship-bank could liquidate the project, in which case it receives a fixed payoff. Alternatively, it can roll over the loan and pretend as if no bad news has arrived yet. Essentially, by hiding bad news \textit{today}, the bank helps the borrower accumulate reputation so that the loan could be sold to the market in the \textit{future}. Such “extending and pretending” incur relatively low costs since shortly afterwards, these bad loans will be sold to the lenders in the market, and the loss will be shared. On the other hand, if negative news arrives early on, “extending and pretending” are much more costly, due to both large time discounting and the high probability that before \( t_g \), the project may mature and the loss will be entirely born by the relationship bank. In this case, liquidating the project is the more profitable option. The threshold \( t_b \) captures the time at which an informed-bad bank is indifferent between liquidating and rolling over. Note that there is a significant gap between \( t_b \) and \( t_g \) so that the extend and pretend stage lasts for a significant
period. During this period, the average quality of borrowers stays unchanged. However, this period is necessary to incentivize informed-bad types to liquidate and exit before $t_b$, which leads to an improvement in the average quality during the efficient liquidation stage.

Related Literature

Our model builds on the literature on private learning, reputation and experimentation (Che and Hörner, 2017; Akcigit and Liu, 2015; Kremer et al., 2014; Grenadier et al., 2014; Martel et al., 2018; Daley and Green, 2012). Hwang (2018) is closely related but with different equilibrium dynamics, driven by assumptions. First, we model both an informed buyer (the bank) and also a competitive set of buyers (market-based lenders), whereas in Hwang (2018), buyers are competitive and uninformed. Second, there are “gains from trade” for bad types in Hwang (2018) so that separating equilibrium can be sustained. These differences lead to different equilibrium dynamics. Our paper is among the first set of papers that introduces learning to the context of banking (also see Halac and Kremer (2018) and Hu (2017)).

Our paper extends the literature to study the dynamics of relationship lending (Diamond, 1991; Rajan, 1992) and debt rollover (He and Xiong, 2012; He and Milbradt, 2016). In Diamond (1991), the lender’s decision is myopic, and therefore, a lender would never want to engage in extend and pretend. Rajan (1992) studies the tradeoff between relationship-based lending and arm’s length debt, without explicit role of borrower’s reputation. Our paper is also related to Parlour and Plantin (2008), which study the efficiency of a secondary market for loan sales. The literature on debt rollover has focused on a set of competitive financiers, and we introduce private information and the role of potentially an informed lender.

Existing explanations on zombie lending largely rely on either loan officers’ career concerns (Rajan, 1994) or additional regulatory capital triggered by writing off bad loans (Caballero et al., 2008; Peek and Rosengren, 2005). We offer a dynamic explanation based on the prospect of future loan sales. This result is also related to Kremer and Skrzypacz (2007) and Fuchs and Skrzypacz (2015) which study how suspension and delaying trading can promote efficiency in markets plagued by adverse selection.

2 Model

We consider a continuous-time model with an infinite horizon. An entrepreneur (she) invests in a long-term project with unknown quality. She borrows from either a bank that will develop into a relationship or the competitive financial market. Compared to market
financing, bank financing has the advantage of producing valuable information but with the downside of a higher cost of capital and the possibility of information monopoly. Below, we describe the model in detail.

2.1 Project

We consider a long-term project that generates a constant stream of interim cash flows \(cdt\) over a period \([t, t + dt]\). The project matures at a random time \(\tau_{\phi}\), which arrives at an exponential time with intensity \(\phi > 0\). Upon maturity, the project produces some random final cash flows \(\tilde{R}\), depending on its type. A good \((g)\) project produces cash flows \(\tilde{R} = R\) with certainty, whereas a bad \((b)\) project produces \(\tilde{R} = R\) with probability \(\theta < 1\). With probability \(1 - \theta\), a matured bad project fails to produce any final cash flows, i.e., \(\tilde{R} = 0\). Initially, no agent, including the entrepreneur herself knows the exact type of the project; all agents share the same public belief that \(q_0\) is the probability of the project’s type being good. At any time before the final cash flows are produced, the project can be terminated with a liquidation value \(L > 0\). Note that the liquidation value is independent of the project’s quality, so it shall be understood as the liquidation of the physical asset used in production. Let \(r > 0\) be the entrepreneur’s discount rate and therefore the fundamental value of the project to the entrepreneur is given by the discounted value of its future cash flows:

\[
NPV_r^g = \frac{c + \phi R}{r + \phi}, \quad NPV_r^b = \frac{c + \phi \theta R}{r + \phi}, \quad NPV_r^u = q_0 NPV_r^g + (1 - q_0) NPV_r^b. \tag{1}
\]

2.2 Agents and Debt Financing

The borrower has no initial wealth and needs to borrow through debt contracts. The use of debt contracts can be justified by non-verifiable final cash flows (Townsend, 1979). One can therefore think of the entrepreneur as a manager of a start-up venture who faces financial constraints. We consider two types of debt, offered by banks and market-based lenders, respectively. First, the entrepreneur can take out a loan from a banker (he), who has the same discount rate \(r\). Following Leland (1998), we assume a bank loan lasts for a random period and matures at a random time \(\tau_m\), upon the arrival of an independent Poisson event with intensity \(m > 0\). The assumption of exponentially maturing loan simplifies the analysis, since at any time before the loan matures, the expected remaining maturity is always \(\frac{1}{m}\). In Section 5, we study the case with deterministic maturity and show all the results carry over.

\(1\) We will derive the maximum amount that she can raise at the initial data after solving the model, in which case we can discuss the minimum net worth needed to finance a project with a fixed investment scale.
The second type of debt is provided by the market and thus can be thought as public bonds. In particular, we consider a competitive financial market in which lenders have discount rate $\delta$ satisfying $\delta \in (0, r)$. As a result, market financing is cheaper than bank financing so that if the project’s type were publicly known, the entrepreneur would strictly prefer to borrow from the market. Relate to (1), let us define the NPV of the project to the market as

$$\text{NPV}_g^\delta = \frac{c + \phi R}{\delta + \phi}, \quad \text{NPV}_b^\delta = \frac{c + \phi \theta R}{\delta + \phi}, \quad \text{NPV}_u^\delta = q_0 \text{NPV}_g^\delta + (1 - q_0) \text{NPV}_b^\delta. \quad (2)$$

The assumption $\delta < r$ captures the realistic feature that banks have higher cost of capital than the market, which can be justified by either regulatory requirements or the skin in the game needed to monitor borrowers (see Holmstrom and Tirole (1997) for example and Schwert (2018) for recent empirical evidence).\footnote{The entire model can be written as one with $r = \delta$ but there is transaction cost associated with rolling over bank loans.} As it will be clear shortly, the maturity of the public debt does not matter and for simplicity, we assume it only matures with the project.

Both types of debt share the same exogenously-specified face value: $F \in (L, R)$. $F > L$ guarantees that debt is risky, whereas $F < R$ captures the wedge between a project’s income and its pledgeable income (Holmström and Tirole, 1998).\footnote{The maximum pledgeable cash flow can be microfounded by some unobservable action taken by the entrepreneur (e.g. cash diversion) shortly before the final cash flows are produced (Tirole, 2010).} Note that we take $F$ as given: our paper intends to study the tradeoff between relationship borrowing and public debt, rather than the optimal leverage. At $t = 0$, the entrepreneur chooses between public debt and a bank loan that will develop into a relationship. Once the bank loan matures at time $\tau_m$, she can still replace it with public bond. Alternatively, she could roll over the loan with the same bank who may have information monopoly over the project’s quality.\footnote{We assume without loss of generality that the entrepreneur would never want to switch to a different banks upon the loan matures. Intuitively, the market has lower cost of capital as an outsider bank and they have the same information structure.} In this case, the two parties bargain over $y_{\tau_m}$, the rate of the loan that is prevalent from $\tau_m$ until the next rollover date. Specifically, we follow Rajan (1992) and model the decision of $y_{\tau_m}$ as a Nash Bargaining game with $(\beta, 1 - \beta)$ being the entrepreneur’s and the bank’s bargaining power. Due to the financial constraint, the entrepreneur cannot promise any rate $y_{\tau_m}$ above $c$, the maximum level of the interim cash flow. As we will show shortly, this constraint limits the size of the transfer that the entrepreneur can make to the bank at rollover dates and therefore, the Nash bargaining outcome is sometimes not the one that maximizes the joint surplus of the two parties.

Since market financing is competitive and market-based lenders have a lower cost of capital...
capital, the entrepreneur will always prefer to take as high leverage as possible. Therefore, the coupon payments associated with the public bond are cdt.

Remark 1. We have assumed that the entrepreneur is only allowed to take one type of debt. In other words, we have ruled out possibilities of the entrepreneur using more sophisticated capital structure to signal her type. See Leland and Pyle (1977) and DeMarzo and Duffie (1999) for these issues.

2.3 Learning and Information Structure

The quality of the project is initially unknown, with \( q_0 \in (0,1) \) being the belief that it is good. This belief is based on public information and is commonly shared by all agents in the economy. If the entrepreneur finances with the bank, i.e., if she takes out a loan, the entrepreneur-bank pair can privately learn the true quality of the project through “news”. News arrives at a random time \( \tau_\lambda \), modeled as an independent Poisson event with intensity \( \lambda > 0 \). Upon arrival, the news fully reveals the true type of the project. In practice, one can think of the news process as information learned during bank screening and monitoring, which includes due diligence and covenant violations. We assume that such news can only be observed by the two parties and there is no committable mechanism to share it with third parties such as credit bureaus and market participants. In this sense, the news can be understood as soft information on project quality (Petersen, 2004). For instance, one can think of this news as the information that banks acquire upon covenant violation, which includes details on the business prospect, collateral quality, and financial soundness of the borrower.

Remark 2. Note that learning and news arrival require joint input from both the entrepreneur and the bank. Therefore, we can think of learning as exploration and understanding of the underlying business prospect which require the entrepreneur’s experimentation and the bank’s previous experience in financing related businesses. In this regard, our model could also be applied to study venture capital firms. Alternatively, we can model learning as a process that solely relies on the entrepreneur’s input, whereas the bank simply observes the news content through monitoring. Put it differently, even without bank financing, the entrepreneur will still be able to learn news about the quality of her project over time. Our results are identical in this alternative setting.

Although the public market participants do not observe the news, they can observe \( t \) – the project’s time since initialization and therefore make inference about the project’s quality. Clearly, they form beliefs based on the time elapsed, as well as the decisions during the (random) rollover events. In the benchmark model, we assume the realization of each rollover
event $\tau_m$ is unobservable to market participants. In Section 5, we relax this assumption and show all results continue to hold qualitatively. Let $i \in \{u, g, b\}$ denote the type of the bank/entrepreneur, where $u$, $g$, and $b$ refer to the uninformed, informed-good and informed-bad types, respectively. We assume that any failure to rollover the loan is publicly observable because in this case either the project will be liquidated or the entrepreneur will seek market financing. In other words, the market cannot observe when the bank loan has been rolled over with the same bank but can observe whether the firm still has bank loans on its balance sheet. Throughout the paper, we assume the loan contract signed between the bank and the entrepreneur, i.e., the loan rate $y_t$ is not observable by the third party. One should therefore interpret $y_t$ not just as the interest rate payments made by the entrepreneur, but also include fees, administrative costs etc.\footnote{The equilibrium under which past loan rates are observable will involve players using mixed strategies. See Hörner and Jamison (2008) for a solution.}

Given the unique feature of Poisson learning, the \textit{private} belief process, i.e., the belief held by the bank and the entrepreneur, is straightforward. If news hasn’t arrived yet, the belief remains at $\mu_i^u = q_0$. In this case, no news is simply no news. Upon news arrival at $t_\lambda$, the private belief jumps to $\mu_i^g = 1$ in the case of good news and $\mu_i^b = \theta$ if bad. For the remainder of this paper, we will suppress the time subscripts for private belief and simply use $\{\mu^u, \mu^g, \mu^b\}$ without loss of generality. To characterize the public belief process, we introduce a belief system $\{\pi_i^u, \pi_i^g, \pi_i^b\}$, where $\pi_i^u$ is the public’s belief at time $t$ that news hasn’t arrived yet, and $\pi_i^g (\pi_i^b)$ is the public belief that the news has arrived and is good (bad). In any equilibrium where the belief is rational, $\pi_i^t$ is consistent with the actual probability that the bank and the entrepreneur are of type $i \in \{u, g, b\}$. Given $\{\pi_i^u, \pi_i^g, \pi_i^b\}$, the public belief that the project is good is

$$q_t = \pi_t^u q_0 + \pi_t^g.$$  \hspace{1cm} (3)

\subsection*{2.4 Rollover}

When the loan matures at $\tau_m$, the entrepreneur and the bank have a total of three options. They can liquidate the project for $L$, switch to market financing, or continue the relationship by rolling the loan over. Control rights are assigned to the bank if it is not fully repaid, and renegotiation could potentially be triggered. Let $O_{\tau_m}^i \equiv O_{E\tau_m}^i + O_{B\tau_m}^i$, $i \in \{u, g, b\}$ be the maximum joint surplus to the two parties in the case that the loan is not rolled over, where $O_{E\tau_m}^i$ and $O_{B\tau_m}^i$ are the value accrued to the entrepreneur and the bank, respectively. Since $F > L$, in the case of liquidation, the bank receives the entire liquidation value $L$
and the entrepreneur receives nothing, i.e., $O_{B_{\tau_m}}^i = L$, and $O_{E_{\tau_m}}^i = 0$. If the two parties choose to switch to market financing, the bank receives full payment $O_{B_{\tau_m}}^i = F$, whereas the entrepreneur receives the remaining surplus $O_{E_{\tau_m}}^i = \bar{V}_{\tau_m}^i - F$, where

$$\bar{V}_{\tau_m}^i = D_{\tau_m} + \frac{\phi \mu^i (R - F)}{r + \phi}.$$  \hspace{1cm} (4)

In (4),

$$D_{\tau_m} = \frac{c + \phi [q_{\tau_m} + (1 - q_{\tau_m}) \theta] F}{\delta + \phi},$$  \hspace{1cm} (5)

is the amount of proceeds that the entrepreneur raises from the market at time $\tau_m$ by issuing a bond with coupon $c_{dt}$ and face value $F$ due whenever the project matures. The two components, $\frac{c + \phi [q_{\tau_m} + (1 - q_{\tau_m}) \theta] F}{\delta + \phi}$ and $\frac{\phi [q_{\tau_m} + (1 - q_{\tau_m}) \theta] F}{\delta + \phi}$ correspond to the present value of the coupon payments and final payoff, respectively. The second term in (4) is the expected final cash flows that the entrepreneur receives upon the project matures. Because the entrepreneur is financially constrained, the bond price $D_{\tau_m}$ must be at least $F$, implying that

$$q_{\tau_m} \geq q_{\min} \equiv 1 - \frac{c - \delta F}{\phi F (1 - \theta)}.$$  \hspace{1cm} (6)

If the entrepreneur and the bank decide to roll over the loan at time $\tau_m$, the two parties bargain over the loan rate $y_{\tau_m}$ until the next roll-over date. To simplify notation, we sometimes drop the subscripts for loan rate unless it causes confusion. With some abuse of notation, let $B_{\tau_m}^i (y)$ and $E_{\tau_m}^i (y)$ be the continuation value for the bank and the entrepreneur if $y$ is the loan rate decided by the bargaining.\footnote{We assume the two parties do not bargain over the face value $F$, with the underlying microfoundation that $F$ is the maximum pledgeable income of the project.} Let $V_{\tau_m}^i \equiv B_{\tau_m}^i (y) + E_{\tau_m}^i (y)$. Specifically, the Nash Bargaining problem can be written as:

$$y_{\tau_m}^i = \arg\max_{y \leq c} \left\{ (B_{\tau_m}^i (y) - O_{B_{\tau_m}}^i)^{1-\beta} (E_{\tau_m}^i (y) - O_{E_{\tau_m}}^i)^{\beta} : B_{\tau_m}^i (y) \geq O_{B_{\tau_m}}^i, E_{\tau_m}^i (y) \geq O_{E_{\tau_m}}^i \right\}.$$  \hspace{1cm} (7)

Note that in principle, Nash Bargaining enables the entrepreneur and the bank to always pursue the option that maximizes their joint surplus. However, this result requires the solution to be implementable by some loan rate $y$ below $c$. The financial constraint $y \leq c$ therefore results in scenarios that the maximal joint surplus may not be implementable. If the solution is interior, that is $y_{\tau_m}^i < c$, the bank value at the rollover rate is given by the
conventional rule for the division of surplus

\[ B^i_{\tau_m}(y) = O^i_{B\tau_m}(y) + (1 - \beta)(V^i_{\tau_m} - O^i_{\tau_m}). \] (8)

If the solution is a corner one, i.e., \( y^i_{\tau_m} = c \), the entrepreneur is financially constrained from making a higher transfer to the bank. In this case, \( B^i_{\tau_m} < O^i_{B\tau_m} + (1 - \beta)(V^i_{\tau_m} - O^i_{\tau_m}) \), and the constraint limits the transfer from the entrepreneur to the bank. In both cases, it is convenient to write the continuation value of the bank in two parts

\[ B^i_{\tau_m}(y) = B^i_{\tau_m}(rF) + T(y), \] (9)

where \( B^i_{\tau_m}(rF) \) is the continuation value of the bank with loan rate \( y = rF \), and

\[ T(y) = \mathbb{E}\left[ \int_{0}^{\tau_m \land \tau_\phi} e^{-r(s-t)} (y - rF) \, ds \right] = \frac{y - rF}{r + m + \phi} \]

is the discounted value of all the interim payments in excess of \( rF \) until either the loan or the project matures. Clearly, (9) makes it clear that by negotiating the loan rate, the entrepreneur effectively makes a (possibly negative) transfer to the bank at the rollover date.

A total of two conditions must be satisfied for a loan to be rolled over. First, \( V^i_{\tau_m} \geq \max\{L, \bar{V}^i_{\tau_m}\} \) so that rolling over is indeed the decision that maximizes the joint surplus. Second, it must be that \( B^i_{\tau_m}(c) \geq L \) so that the bank prefers rolling over the loan and receiving the entire interim cash flow over liquidating the project and receiving \( L \). Otherwise, the bank, endowed with control rights over the asset once not fully repaid, will choose to liquidate the asset.

2.5 Strategies and Equilibrium

The public history \( H_t \) consists of time \( t \) and the entrepreneur’s and the bank’s actions up to \( t \). Specifically, it includes at any time \( s \leq t \), whether the entrepreneur borrows from the bank or the market and whether the project is liquidated. For any public history, the strategy of the market is summarized by the price of market debt \( D_{\tau_m} \). Given that the market is competitive, the price of debt at which it breaks even satisfies (5).\(^8\)

The private history \( h_t \) consists of the public history \( H_t \), the rollover event, as well as the Poisson event on news arrival and of course the content of news. Essentially, a strategy of

\(^8\)We offer a micro-foundation as follows. In each period, two short-lived market-based lenders simultaneously enter and make private offers to all entrepreneurs. Those whose loans have matured, i.e., \( t = \tau_m \) may choose to accept the offer. This microfoundation will give rise to the No-Deals condition as in Daley and Green (2012).
the entrepreneur is a stopping time that determines the time to refinance with the market. The strategy of the bank specifies whether to roll over the loan at each rollover date \( \tau_m \) or to liquidate it, in the case it does not receive the full payment \( F \). \( V_t^i \) – the joint value of the entrepreneur and the bank in the lending relationship – satisfies the following Bellman equations:\(^9\)

\[
V_t^u = \mathbb{E}_t \left\{ \int_t^\tau e^{-r(s-t)}cds + e^{-r(\tau-t)} \left[ 1_{\tau=\tau_\phi} \left[ q_0 + (1-q_0) \theta \right] R + 1_{\tau=\tau_\lambda} \left[ q_0 V^g_\tau + (1-q_0) V^b_\tau \right] + \max_{B^c_\tau(c) \geq L} \left\{ V^u_\tau, L, V^b_\tau \right\} \right] \right\}
\]

(10a)

\[
V_t^g = \mathbb{E}_t \left\{ \int_t^\tau e^{-r(s-t)}cds + e^{-r(\tau-t)} \left[ 1_{\tau=\tau_\phi} R + 1_{\tau=\tau_m} \max_{B^c_\tau(c) \geq L} \left\{ V^g_\tau, L, V^b_\tau \right\} \right] \right\}
\]

(10b)

\[
V_t^b = \mathbb{E}_t \left\{ \int_t^\tau e^{-r(s-t)}cds + e^{-r(\tau-t)} \left[ 1_{\tau=\tau_\phi} \theta R + 1_{\tau=\tau_m} \max_{B^c_\tau(c) \geq L} \left\{ V^b_\tau, L, V^b_\tau \right\} \right] \right\}.
\]

(10c)

With some abuse of notation, in the first equation we let \( \tau = \min\{\tau_\phi, \tau_\lambda, \tau_m\} \), while in the last two equations we let \( \tau = \min\{\tau_\phi, \tau_m\} \). The first term in all three equations, \( cds \), is the value of interim cash flows over time \([s, s + ds]\). The project matures and pays off the final cash flows if \( \tau = \tau_\phi \). If \( \tau = \tau_m \), the bank loan matures and the two parties choose among rolling over, liquidating, or switch to market financing. In the case that the pair is uninformed, news arrives at random time \( \tau_\lambda \), after which they become informed. The maximization at time \( \tau_m \) is subject to the additional constraint that the value of the bank at any rollover date has to be greater than \( L \). Otherwise, the continuation value at the rollover date is \( L \).

Recall that we have defined \( E^i_t(y_t) \) and \( B^i_t(y_t) \) as the continuation value of the entrepreneur and the bank at time \( t \), where \( y_t \) is the prevalent loan rate. Sometimes, we will also refer to \( E^i_t(y_t) \) as equity value. By definition, for type \( i \in \{g,b\} \)

\[^9\text{We use the standard notation } \mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|h_t], \text{ to indicate that the expectations is conditional on the history before the realization of the stopping time } \tau.\]
\[ E_i^t(y_t) = \mathbb{E}_t\left\{ \int_t^\tau e^{-r(s-t)}(c - y_t)ds + e^{-r(\tau-t)} \left[ \max \left\{ \tilde{R} - F, 0 \right\} + 1_{\tau=\tau_m} \left( 1_{\text{rollover}} E_{\tau_m}^i(y_{\tau_m}) \right) \right] \right\} \]

\[ B_i^t(y_t) = \mathbb{E}_t\left\{ \int_t^\tau e^{-r(s-t)}y_t ds + e^{-r(\tau-t)} \left[ 1_{\tau=\tau_a} \min(\tilde{R}, F) + 1_{\tau=\tau_m} \left( 1_{\text{rollover}} B_{\tau_m}^i(y_{\tau_m}) \right) \right] \right\}, \]

where \( E_{\tau_m}^i = V_{\tau_m}^i - F \) is the continuation value of the entrepreneur once she finances with the market, with \( V_{\tau_m}^i \) defined in (4). As before, the indicators variables imply whether the loan is rolled over, the project is liquidated, or the entrepreneur obtains market financing. The value of type \( u \) is similar, with the additional term of getting informed.

We look for a perfect Bayesian equilibrium of this game.

**Definition 1.** An equilibrium of the game satisfies

1. Optimality: the rollover decisions are optimal for the bank and the entrepreneur, given the beliefs \( \{ \pi_{i_t}, \mu_{i_t}, q_t \} \). The rate of the loan at rollover dates solves the Nash bargaining problem (7).

2. Belief Consistency: for any history on the equilibrium path, the belief process \( \{ \pi_u^t, \pi_g^t, \pi_b^t \} \) is consistent with Baye’s rule.


4. No (unrealized) Deals: for any \( t > 0 \) and \( i \in \{ u, g, b \} \), \( V_i^t \geq \mathbb{E} \left[ V_t | \mathcal{H}_t \right]. \)

The first three conditions are standard. The No Deals condition follows Daley and Green (2012), reflecting the requirement that the market cannot profitably deviate by making an offer that the entrepreneur and the bank will accept.

As standard in the literature, we use a refinement to rule out unappealing equilibria that arise only due to unreasonable beliefs off the equilibrium path. Specifically, we restrict the belief on the off-equilibrium to be non-decreasing. Effectively, this condition guarantees that once the good type refinances her loan with the market at some time \( t \), all other types will do so for any \( t > t \).
Definition 2. Belief monotonicity is satisfied if the public’s belief that the project is good conditional on the entrepreneur still borrowing from the bank is non-decreasing in $t$. An equilibrium that satisfies belief monotonicity is referred to as a monotonic equilibrium.

We will that show there is a unique monotone equilibrium. Moreover, we show that if the maturity of the loan is sufficiently long, the equilibrium is unique even without the refinement – that is, that any perfect Bayesian equilibrium satisfies belief monotonicity.

2.6 Parametric Assumptions

We make the following parametric assumptions to make the problem non-trivial.

Assumption 1 (Liquidation value).

$$NPV_b^\delta < L < NPV^g$$ \hspace{1cm} (11)

According to Assumption 1, the NPV of a good project to the bank and the entrepreneur is above its liquidation value, which is in turn above the NPV of a bad project to the market. Therefore, it is socially optimal to liquidate a bad project but to continue a good project.

Assumption 2 (Risky debt).

$$F > \max \{\theta R, L\}.$$ \hspace{1cm} (12)

Assumption 2 assumes the face value of the debt is above both the liquidation value and the expected repayment; otherwise both the bank loan and the public bond can be safe.

Finally, we assume the size of the interim cash flow $c$ to be weakly higher than $rF$.

Assumption 3 (interim cash flow).

$$c \geq rF.$$ \hspace{1cm} (13)

2.7 First-best Outcome

Before formally characterizing the equilibrium, we present the first-best outcome, which is achieved if news could be publicly observable and loans mature instantly. Assumption 1 guarantees that any good project will immediately receive financing from the market, whereas a bad project will be liquidated upon news coming out. Let $NPV^u_{r\rightarrow\delta}$ be the time-0 valuation of the unknown project if it is financed with the bank and switches to market/liquidation upon good/bad news.

$$NPV^u_{r\rightarrow\delta} = \frac{c + \phi [q_0 + (1 - q_0) \theta] R}{r + \phi + \lambda} + \frac{\lambda}{r + \phi + \lambda} \left[ q_0 \frac{c + \phi R}{\delta + \phi} + (1 - q_0) L \right].$$ \hspace{1cm} (14)
Proposition 1. In the first-best outcome, a good project is immediately financed by the market, whereas a bad project is immediately liquidated. If \( \{ NPV^u_\delta, NPV^u_{r\to\delta}, L \} = L \), an unknown project will be liquidated. If \( \{ NPV^u_\delta, NPV^u_{r\to\delta}, L \} = NPV^u_\delta \), it will be financed by the market. If \( \{ NPV^u_\delta, NPV^u_{r\to\delta}, L \} = NPV^u_{r\to\delta} \), it will be financed by the bank until news comes out.

3 Equilibrium

In this section, we solve the equilibrium. In subsection 3.1, we study a benchmark economy by ignoring the financial constraints \( y_{\tau m} \leq c \) and \( D_{\tau m} \geq F \).\(^{10}\) In subsection 3.2, we describe the equilibrium with a formal treatment of the financial constraints. The equilibrium will be similar to the one in subsection 3.1, except for the boundary conditions. Both subsection 3.1 and 3.2 assume learning as an exogenous process, whereas Subsection 3.3 analyzes the case that learning is a costly decision by banks.

3.1 Benchmark: No financial constraints

The economy is characterized by state variables in private and public beliefs \( \{ \mu^i_t, \pi^i_t \} \). It turns out that all public beliefs are deterministic functions of the time elapsed. Therefore, we use time \( t \) as the state variable. Specifically, the equilibrium will be characterized by two thresholds \( \{ t_b, t_g \} \), as illustrated by Figure 1. If \( t \in [0, t_b] \), the bank and the entrepreneur liquidate the project upon loan maturing if bad news has arrived – efficient liquidation region. Loans for other types (good and unknown) will be rolled over. If \( t \in [t_b, t_g] \), all types of loans will be rolled over, including the bad ones – extend and pretend region. Finally, if \( t \in [t_g, \infty) \), the two entities will always refinance with the market upon loan maturing – market financing.

![Figure 1: Equilibrium regions](image)

Given the equilibrium conjecture, the evolution of beliefs follow Lemma 1.

\(^{10}\)This case corresponds to an entrepreneur who is less financially constrained but still not deep-pocked enough to finance the entire loan \( F \). In other words, we assume she has enough funds to absorb the rollover losses but not enough to fully repay the bank debt. Alternatively, one can think of this in the traditional trade-off framework with a deep-pocked entrepreneur who uses debt to take advantage of the tax shields.
Lemma 1. In a monotone equilibrium with threshold \( \{t_b, t_g\} \), beliefs evolve as follows.

1. Without liquidation, the public beliefs \((\pi_t^u, \pi_t^g, \pi_t^b)\) satisfy the following differential equation:

\[
\dot{\pi}_t^u = -\lambda \pi_t^u + 1_{t \leq t_b} m \pi_t^u \pi_t^b \\
\dot{\pi}_t^g = \lambda \pi_t^u q_0 + 1_{t \leq t_b} m \pi_t^g \pi_t^b \\
\dot{\pi}_t^b = \lambda \pi_t^u (1 - q_0) - 1_{t \leq t_b} m \pi_t^b \left(1 - \pi_t^b\right) \tag{15c}
\]

2. With liquidation, \(\pi_t^b\) jumps to 1, whereas \(\pi_t^u\) and \(\pi_t^g\) jump to 0.

3. Initially, \(\pi_0^u = 1\) and \(\pi_0^g = \pi_0^b = 0\).

4. For any \(t < t_b\),

\[
q_t = \frac{q_0 \left(1 - q_0 + q_0 e^\lambda t\right)^{\frac{1}{\lambda}} - 1}{1 + m \int_0^t \left(1 - q_0 + q_0 e^\lambda s\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds} \tag{16}
\]

5. For \(t > t_b\), \(q_t = \bar{q}\) is a constant.

Figure 2 provides a graphical illustration to the public belief systems for \(t < t_g\). The left panel shows \(\pi_t^u\), which decreases monotonically due to the arrival of news over time. In contrast, \(\pi_t^g\) keeps increasing, since the informed good type gets discovered over time and keeps rolling over the loan. Finally, \(\pi_t^b\) evolves non-monotonically. During \([0, t_b]\), it increases initially as bad types get revealed (note that they don’t exit immediately due to the finite maturity of the loan). Ultimately, it starts to decline as more and more of the informed bad types get liquidated and exit funding. After \(t\) passes \(t_b\) (the dashed line), since no bad type will further liquidate their projects, \(\pi_t^b\) starts to increase again.

![Figure 2: Public Beliefs when \(t < t_g\)](image)

This figure plots the public beliefs process with the following parameter values: \(r = 0.1, \delta = 0.05, m = 10, F = 1, \phi = 1, R = 2, c = 0.2, \theta = 0.1, L = 1.2 \times NPV_r^b, \lambda = 2, q_0 = 0.1, \beta = 0.2\).
The evolution of $q_t$ is also straightforward. During $[0, t_b)$, $q_t$ drifts up because bad entrepreneurs’ projects get liquidated over time. After $t$ goes above $t_b$, the average quality of borrowers remains unchanged from the market’s perspective, that is

$$\dot{q}_t = \dot{q}_t^b + q_0 \dot{q}_t^u = 0.$$ 

For the remainder of this subsection, we treat the bank and the entrepreneur as one entity and solve for $t_b$ and $t_g$, the optimal choice of timing when they liquidate the project and when they switch to market finance. By considering the changes in valuation $V_t^i$, $i \in \{u, g, b\}$ over a small interval $[t, t + dt]$, we are able to derive the following Hamilton-Jacobi-Bellman (HJB) equation system:

\begin{align}
(r + \phi) V_t^u &= \dot{V}_t^u + c + \phi [q_0 + (1 - q_0) \theta] R \\
&\quad + \lambda [q_0 V_t^g + (1 - q_0) V_t^b - V_t^u] + m \mathcal{R}(V_t^u, \bar{V}_t^u) \tag{17a} \\
(r + \phi) V_t^g &= \dot{V}_t^g + c + \phi R + m \mathcal{R}(V_t^g, \bar{V}_t^g) \tag{17b} \\
(r + \phi) V_t^b &= \dot{V}_t^b + c + \phi \theta R + m \mathcal{R}(V_t^b, \bar{V}_t^b), \tag{17c}
\end{align}

where

$$\mathcal{R}(V_t^i, \bar{V}_t^i) \equiv \max \left\{0, \bar{V}_t^i - V_t^i, L - V_t^i \right\} \tag{18}$$

The first term on the right hand side $\dot{V}_t^u$ is the change in valuation; the second term captures the benefits of interim cash flow, and the third term corresponds to the project maturing, with arrival rate $\phi$. In the latter case, the bank and the entrepreneur receive a payoff of $R$ with probability $q_0 + (1 - q_0) \theta$. The fourth term stands for the arrival of news at rate $\lambda$. Following the news, the bank and the entrepreneur become informed. Finally, upon loan maturing which happens with an arrival rate $m$, the bank and the entrepreneur choose between rolling over the debt ($0$ in Equation (18)), replacing the loan with the market bond ($\bar{V}_t^i - V_t^i$ in (18)), and liquidating the project ($L - V_t^i$ in (18)). Note that we have assumed a project will be rolled over if $0 = \arg \max \mathcal{R}(V_t^i, \bar{V}_t^i)$, which will no longer be the case with explicit account for financial constraint. Equation (17b) and (17c) can be interpreted in the similar vein.

The three equilibrium regions will differ in $\mathcal{R}(V_t^i, \bar{V}_t^i)$, i.e., decision when the loan matures. To better explain the economic intuition, we describe the equilibrium backwards in the time elapsed.
Market Financing: \([t_g, \infty)\). In this region, \(R(V_t^i, \bar{V}_t^i)\) is maximized by letting it equals \(\bar{V}_t^i - V_t^i\). Also, \(\dot{\bar{V}}_t^i, \ i \in \{u, g, b\}\) is dropped in equations (17a)-(17c) because the belief \(q_t\) stays unchanged.

Ultimately, market financing is cheaper because market lenders have lower discount rates \(\delta < r\), and all types will therefore replace their loans with public bonds. Note that the bond is set at a price that reflects the average quality of the project \(\bar{q}\), which exceeds the initial quality \(q_0\). This is because in equilibrium, some bad types would have liquidated their projects in the efficient liquidation region.

Extend and Pretend: \([t_b, t_g)\). Working backward, we now consider the region \([t_b, t_g)\) during which all loans, including bad ones, are rolled over. When time is close to \(t_g\), the bank finds optimal to wait until \(t_g\). Mathematically, on the right-hand-side of equations (17a)-(17c), \(R(V_t^i, \bar{V}_t^i)\) is maximized by letting it equals 0. Intuitively, rolling over bad loans allows the bank to get fully repaid and transfer the losses from a bad loan to market lenders. When time is closed to \(t_g\), this decision can be optimal for the bank and the entrepreneur.

During \([t_b, t_g)\), the off-equilibrium belief for any entrepreneur who seeks financing from the market will be treated as bad for sure. Otherwise, the equilibrium is no longer viable. To see this, note that as in standard dynamic signaling games, the good type does not mix, and if the good type switches to the market, so will the other two types.\(^{11}\) Therefore, if the good type refinances with the market right after \(t_b\), the bad type will never liquidate before \(t_b\). As a result, \(q_t\) will not increase up to \(\bar{q}\), and the equilibrium is no longer viable.

Equilibrium in this region is clearly inefficient. A bad project should be liquidated but instead, the bank and the entrepreneur roll it over in the hope of sharing the losses with the market lenders after \(t_g\). By not liquidating between 0 and \(t_b\), they have accumulated “good” reputation and as a result, “extend and pretend” can be sustained in equilibrium.

Efficient Liquidation: \([0, t_b)\) Finally, we focus on the initial region \([0, t_b)\), where bad loans are not rolled over but instead liquidated. The equilibrium is socially efficient in this region. Mathematically, on the right-hand-side of equations (17a)-(17c), \(R(V_t^b, \bar{V}_t^b)\) is maximized by letting it equals \(L - V_t^b\), whereas \(R(V_t^g, \bar{V}_t^g)\) and \(R(V_t^u, \bar{V}_t^u)\) are still maximized by letting it equals 0. At the early stage of the lending relationship, only the uninformed and informed-good types roll over maturing loans. By contrast, banks who have learned that the project is bad choose to liquidate. Assumption 1 guarantees that liquidation possesses a higher value than continuing the project until the final date \(t_\phi\). By continuity, liquidation

\(^{11}\)See Lemma 5.1 of Daley and Green (2012) for a proof.
still has a higher payoff if type \( b \) needs to wait for a long time (until \( t_g \)) to refinance. In this region, extend and pretend are suboptimal since \( t_g \) is far away: it is likely that the firm could default before it receives the opportunity of market financing.

**Boundary Conditions:** The following two boundary conditions are needed to pin down \( \{t_b, t_g\} \)

\[
V_{t_b}^b = L \tag{19a}
\]
\[
\dot{V}_{t_g}^g = 0. \tag{19b}
\]

(19a) is the indifference condition for the bad type to liquidate at \( t_b \). This is the traditional value matching condition in optimal stopping problems. In this case, rolling over brings exactly the same payoff \( L \) and thus by continuity and monotonicity, she prefers liquidating when \( \tau_m < t_b \) and rolling over when \( \tau_m > t_b \). The second condition, smooth pasting, comes from the No-Deals condition. We show in Lemma 4 of Appendix A.1.2 that if this conditions fails then the type \( g \) will have strictly higher incentives to switch to market financing before \( t_g \), which constitutes an arbitrage opportunity for market participants. Essentially, the No-Deals guarantees the equilibrium will ultimately be one with pooling, and given so, the smooth-pasting condition solves the optimal-stopping time problem for the good types. The smooth-pasting condition picks the earliest \( t_g \) for the good entrepreneur to switch to refinance with the market. With the boundary conditions, we can uniquely pin-down \( \{t_b, t_g\} \), which is given by the following proposition.

**Proposition 2.** In absence of financial constraints, there is a unique monotone equilibrium characterized by rollover thresholds \( t_b \) and \( t_g \):

\[
t_b = \min\{t : q_t = \bar{q}\}, \tag{20}
\]

and

\[
t_g = t_b + \frac{1}{r + \phi} \log \left( \frac{(r + \phi)V_{t_g}^b - (c + \phi \theta R)}{(r + \phi)L - (c + \phi \theta R)} \right), \tag{21}
\]

where

\[
\bar{q} = \frac{1}{(1 - \theta)} \left( \delta + \frac{c + \phi F}{r + \phi} - \frac{c}{\phi F} - \theta \right) \tag{22}
\]
\[
V_{t_g}^b = \frac{c + \phi R}{r + \phi} - \frac{\phi R (1 - \theta) + m \phi (R - F) (1 - \theta)}{r + \phi + m}. \tag{23}
\]
Numerical Example  Figure 3 plots the value function of all three types. In this example, the equilibrium $t_b = 2.5806$ and $t_g = 4.6861$. In all three panels, the blue solid lines stand for the value function, whereas the red dashed line shows the levels of $L$. Clearly, all three value functions stay constant after $t$ passes $t_g$. In fact, as proven in Lemma 5 of Appendix A.1.2, $V_g$ stays a constant throughout the entire range. In other words, the informed-good types always expect the same continuation value. By contrast, the value of informed-bad types (right panel) exceeds $L$ only after $t$ passes $t_b$ and then increases sharply until $t = t_g$.

![Figure 3: Value Functions](image)

This figure plots the value function with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.2 \times NPV^b_r$, $\lambda = 2$, $q_0 = 0.1$, $\beta = 0.2$.

We conclude this subsection by discussing the equilibrium uniqueness without the monotone belief refinement.

Proposition 3. There exists a unique pair $\{m, \bar{m}\}$ satisfying $m < \bar{m}$ such that the equilibrium described in Proposition 2 is unique if and only if $m \in (m, \bar{m})$. If $m > \bar{m}$, the equilibrium is not unique without the monotone belief refinement. If $m < m$, the unique equilibrium is one in which $t_b = t_g$.

Intuitively, when $m$ gets sufficiently low and equivalently the maturity of the loan gets long enough, an entrepreneur will need to wait for a long time after $t_g$ until her loan matures, upon which she could refinance with the market. The prolonged waiting period resulted from loan maturity will be sufficient to deter bad types from mimicking others.

Remark 3. The results will stay unchanged if we allow the entrepreneur to renegotiate and prepay the bank loans. During $[0, t_g)$, renegotiation is never triggered. After $t_g$, all bank loans will immediately be renegotiated. As a result, for all three types, $V^i_{t_g} = \bar{V}^i_{t_g}$. According to Proposition 2, $t_b$ stays unchanged, whereas $t_g$ gets even higher.
3.2 Binding Financial Constraint

By relaxing financial constraints, we have assumed all rollover decisions are made to maximize the joint surplus of the bank and the entrepreneur. Specifically, at each rollover date, a loan will be rolled over if the joint surplus is above the liquidation value $L$. In this subsection, we formally analyze the model with two financial constraints. First, at rollover dates before $t_y$, the negotiated loan rate $y$ cannot exceed the rate of interim cash flow $c$. This constraint limits the transfer from the entrepreneur to the bank. Second, at rollover dates after $t_y$ when the entrepreneur intends to refinance with the market, the price of the market debt $D_{\tau_m}$ must be sufficient to cover the face value of the loan $F$. As we will see shortly, the bank will sometimes liquidate the project to get $L$ even though the joint surplus is higher.

The HJBs for the value function $\{V_t^i, i \in \{u, g, b\}\}$ are the same as those in subsection 3.1. Again, we can use two thresholds $\{t_b, t_y\}$ to characterize the equilibrium solutions. One may wonder whether the financial constraint could always be slack. Lemma 2 shows this is never the case.

**Lemma 2.** The financial constraint $y \leq c$ always binds at $t_b$.

We offer a heuristic proof as follows. Suppose instead the constraint is always slack, the boundary condition at $t_b$, characterized by (19a), immediately shows that $B_{t_b}^b(y_{t_b}) = L + (1 - \beta)(V_{t_b}^b - L) = L$. This implies that if a bad loan matures at $t_b$, the bank will receive a continuation value $L$, and the bad entrepreneur receives a continuation value 0. This result constitutes a violation, as the bad entrepreneur could always wait until $t_y$ and finance with the market, which guarantees a strictly positive payoff.

3.2.1 Equilibrium Boundaries

Let us now turn to the the boundary conditions under financial constraints. First, the smooth-pasting condition $\dot{V}_{t_y}^g = 0$ continues to hold. As explained in subsection 3.1, this condition follows from the No Deals condition, which essentially selects the equilibrium in which a good-type entrepreneur chooses to refinance with the market as early as possible. Note that the smooth-pasting condition pins down $\bar{q}$, the average quality of entrepreneurs during $[t_b, t_y)$. Therefore, the average quality of entrepreneur after $t_b$ is identical to the case without financial constraint, and $t_b = \min \{t: q_t \geq \bar{q}\}$ also remains unchanged. Given $\bar{q}$, we can easily verify the second financial constraint that after $t_y$, the entrepreneur can borrow more than $F$ from from the market to repay the bank.

**Lemma 3.** Under Assumption 3, $\bar{q} \geq q_{\text{min}}$, and $D_{\tau_m} \geq F$. In other words, the entrepreneur can raise more than $F$ from the market after time passes $t_y$.
For the remainder of this subsection, let us focus on the first financial constraint \( y \leq c \). According to Lemma 2, this constraint must alter the second boundary condition – value matching condition. In particular, since the entrepreneur is financially constrained and cannot repay its loan before \( t_g \), it is the bank that decides whether to liquidate the project. Therefore, the value-matching condition at \( t_b \) becomes

\[
B^b_{t_b} (c) = L. \tag{24}
\]

Depending on parameters, the constraint \( y \leq c \) may or may not bind at \( t = t_g \) for the bad type. Proposition 4 summarizes the equilibrium that it also binds at \( t_g \). The other case is described and proved in the appendix.\(^{12}\)

**Proposition 4.** If

\[
L - c + \left( \frac{\phi \theta + m}{r + \phi + m} \right) F + (1 - \beta) \left[ \frac{c}{r + \phi} + \frac{\phi}{r + \phi} \left( 1 - \theta \right) \frac{mF}{r + \phi + m} + \theta R \right] - L > 0,
\]

the equilibrium is characterized by two thresholds \( \{t_b, t_g\} \), where \( \{\bar{q}, t_b\} \) are identical to those in Proposition 2, and

\[
t_g = t_b - \frac{1}{r + \phi} \log \left( \frac{r + \phi) L - (c + \phi \theta F)}{r + \phi + m(c + \phi \theta F + mF) - (c + \phi \theta F)} \right). \tag{25}
\]

In subsection 3.1, we show that the development of private information over time gives rise to an equilibrium region in which bad types pretend and extend, which is socially suboptimal. Our next result shows that financial constraint mitigates this inefficiency.

**Corollary 1.** The length of the extend and pretend period \( t_g - t_b \) gets shorter under the financial constraint.

*Proof.** Since \( E^b_{t_b} (c) > 0 \), (24) implies at \( t = t_b \), the joint surplus \( V^b_{t_b} > L \).

\( \square \)

This result has implications for the relationship between financial constraint and credit quality. In particular, it highlights the role of financial constraint and its interaction with asymmetric information across lenders. While the benchmark case shows asymmetric information per se will endogenously generate incentives for banks to roll over bad loans. Results in this subsection show that financial constraint can mitigate this concern. Intuitively, the ability for bad types to mimic others is compromised under the financial constraint, since

\(^{12}\)In the appendix, we show that the constraint binds monotonically. If \( y \leq c \) binds at \( t_g \), it binds everywhere at \( t \in [t_b, t_g] \). Otherwise, there exists a \( t_c \in [t_b, t_g] \) such that the constraint binds on \( t \in [t_b, t_c] \) but not \( t \in [t_c, t_g] \).
there is a limit to the transfer that a bad entrepreneur can use to “bribe” the bank from not liquidating the project.

Moreover, note that the financial constraint does not affect $\bar{q}$, the average quality of entrepreneur that will be ultimately financed with the market. This result follows because entrepreneurs will not refinance with the market until they have accumulated enough reputation such that their financial constraint no longer binds once they switch to the market. It relies on Assumption 3 that the interim cash flow needs to be sufficiently high. Otherwise, the financial constraint will result in a higher $\bar{q}$, so that the credit quality financed by the market is higher under financial constraint.

**Numerical Example** Under the same set of parameter values, $t_b = 2.5806$ and $t_g = 3.8418$. A comparison with the benchmark case shows that $t_b$ stays unchanged, whereas $t_g$ gets smaller, confirming our theoretical findings. Moreover, we can compute $B_0^u(c)$, the maximum amount of bank borrowing at time 0, which equals 0.4401. By contrast, in the benchmark case, the entrepreneur is able to borrow up to $B_0^u(c) = 0.5242$. As expected, the financial constraint during debt rollovers in the future decreases the amount of upfront borrowing.

3.2.2 Entrepreneur and Bank Value

In this subsection, we study how the joint surplus of the entrepreneur and the bank is distributed between the two parties. We will mainly describe the results graphically and leave the analytical details to Appendix A.2, including the HJB equations.

Figure 4 plot the value functions at loan rate $y = rF$. A first prominent feature is the kink in the entrepreneur’s value function at $t_b = 2.5806$. This is not surprising since it is the bank that makes liquidation decision. Second, while most of the value functions are monotonically increasing in time, a good-bank’s value function decreases with time. Intuitively, there are two forces at work here. First, the value of a project is (weakly) increasing, as time gets closer and closer to the market financing stage. As a result, the surplus of rolling over the loan $V_t^i - L$ gets (weakly) larger. Ceteris paribus, both the entrepreneur’s and the bank’s value function should increase. However, there is a second, countervailing force. During the market financing stage, the disagreement point in the Nash bargaining game is $(0, L)$: if the bargaining does not reach an agreement, the bank only receives the liquidation value $L$. During the market financing stage, however, the bank will always gets fully repaid and thus receives $F$. As time $t$ gets closer to the market financing stage, the bank’s ability to “exploit” the entrepreneur gets more limited because it is increasingly likely that the next roll-over event will occur during the market financing stage. Therefore, the entrepreneur’s
value function increases whereas the bank’s one decreases. For the good type, the first effect is muted, as the value function $V^g_t$ is a constant over time. The second effect in this case dominates and leads to the monotonically decreasing pattern. For the other two types, the first effect dominates. Given the opposite effects of these two forces, the overall effect can be non-monotonic.\footnote{As we show in Lemma 7 in the Appendix, the sign of $E^b_t$ can only change sign at most once though.}

3.3 Endogenous Learning

Our analysis has so far assumed learning as an exogenous process, which happens as long as the entrepreneur borrows from the bank. In this subsection, we consider the situation in which learning is endogenously chosen by the bank as a costly decision. In particular, we assume the learning rate is chosen as $a_t \in [0, 1]$ by the bank, and for a given rate, the news arrives at Poisson intensity $\lambda a_t$. In the meantime, learning incurs a flow cost $\psi a_t$. Let us continue to look for a monotone equilibrium with rollover thresholds $\{t_b, t_g\}$.

Obviously, the bank never learns after $t > t_g$, as the entrepreneur will always refinance the loan with the market whenever the loan matures, i.e., $a_t \equiv 0$, $\forall t \geq t_g$. At any time
\( t \in (0, t_g) \), the bank’s continuation value satisfies the HJB equation

\[
(r + \phi + m) B^u_t = y_t F + \phi[q_0 + (1 - q_0)\theta] F + \dot{B}^u_t \\
+ \max_{a_t \in [0,1]} \left\{ \lambda a_t \left[ q_0 B^g_t + (1 - q_0) B^b_t - B^u_t - \frac{\psi}{\lambda} \right] \right\} + m[L + (1 - \beta)(V^u_t - L)].
\]

Note that we have used the result that at any rollover date, the bank’s continuation value jumps to \( m[L + (1 - \beta)(V^u_t - L)] \) following Nash bargaining. Clearly, the net benefit of learning is positive, which implies \( a_t \equiv 1 \) if and only if

\[
q_0 B^g_t + (1 - q_0) B^b_t - B^u_t > \frac{\psi}{\lambda}.
\]

**Proposition 5.** In an equilibrium characterized by \( \{t_b, t_g\} \), the bank never learns after \( t_b \).

1. If

\[
\frac{\psi}{\lambda} < \frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left( L - \frac{c + \phi R}{r + \phi} \right),
\]

there exists an equilibrium in which the bank learns during \([0, t_a]\), where \( t_a \equiv \frac{1}{\lambda} \log \left( \frac{\psi}{1 - q_0 q_0} \right) < t_a < t_b \). In this equilibrium, beliefs on \((0, t_a)\) are still given by (15a) - (15c) while on \((t_a, t_b)\), beliefs evolve as

\[
\dot{\pi}^u_t = m\pi^a_t \pi^b_t, \ \dot{\pi}^g_t = m\pi^a_t \pi^b_t, \ \dot{\pi}^b_t = -m\pi^b_t (1 - \pi^b_t).
\]

2. If

\[
\frac{\psi}{\lambda} > \frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left( L - \frac{c + \phi R}{r + \phi} \right),
\]

the bank never learns.

According to Proposition 5, the relationship bank stops learning even before time reaches \( t_b \). Intuitively, when banks do not liquidate the bad project in equilibrium, the value of an uninformed bank is a linear combination between an informed-good and an informed-bad. In this case, the benefits of getting informed is zero. If the bank liquidates bad projects in equilibrium, however, the value of an informed-bad bank is at least \( L \) so that the overall payoff is convex in the type of information (see Figure 5 for a graphical illustration.). In this case, information is valuable and learning will be endogenously chosen if the cost is small enough.

**Numerical Example** Under the same set of parameters, with the additional parameter that \( \psi = 0.06 \), we can get \( t_a = 2.4690, t_b = 3.4825, \) and \( t_g = 5.5879 \). Note that since
the bank stops learning after $t_a$, it takes longer for the average quality $q_t$ to reach $\bar{q}$, which explains why $t_b$ is higher than the benchmark case. The length of extend and pretend period, $t_g - t_b$, stays unchanged at 2.1055.

Finally, we look at the equilibrium with endogenous learning once we incorporate the constraint that $y_t \leq c$. To keep the analysis simple, we only consider the case in which the constraint is always binding ($y_t = c$) for all types. This is the case if the bank’s bargaining power is high enough (equivalently $\beta$ is close enough to zero).

**Proposition 6.** Suppose that

$$\frac{c + \phi((1 - \theta)F + \theta R)}{r + \phi} > \frac{c + (\phi + m)F}{r + \phi + m},$$

then if

$$\frac{\psi}{\lambda} \leq \frac{m(1 - q_0)}{r + \phi + m} \left( \frac{c + \phi \theta F}{r + \phi} \right),$$

there is $\bar{\beta}$ such that, for any $\beta \leq \bar{\beta}$, there exists an equilibrium in which $y_i^* = c$ for $i \in \{u, g, b\}$ and the bank learns during $[0, t_a]$, where $t_a \equiv \frac{1}{\lambda} \log \left( \frac{\bar{q}}{1 - \bar{q} q_0} \right) < t_a < t_b$. In this equilibrium, beliefs follow Proposition 5.

Finally, it is instructive to compare the equilibrium in the case with financial constraints vis--vis the case without financial constraints. To guarantee the conditions in Proposition 6 always hold, we consider the special case where $\beta = 0$. All other parameters stay unchanged. Without the financial constraints, $t_a = 2.4691$, $t_b = 3.2101$, and $t_g = 5.3156$. With financial constraints, $t_a = 2.4739$, $t_b = 2.7983$, and $t_g = 4.0594$. Clearly, financial constraint increases the region in which banks choose to learn. The reason goes back to our previous result that financial constraint reduces the time length of extend and pretend (from 2.1055 to 1.2611.
under given parameters). As a result, bad projects are liquidated more often under financial constraints. Since the role of learning for a risk-neutral bank is to liquidate bad projects. Therefore, the information is more valuable, and learning persists for a longer period of time under financial constraints.

Figure 6 shows how $t_a$, $t_b$, and $t_g$ vary with loan maturity $\frac{1}{m}$. The left panel clearly shows for all maturities, the bank learns for a longer period of time under financial constraints, and the difference tends to increase with the maturity of loans. Moreover, in general, the length of loan maturity could have a non-monotonic effect in learning, due to the tradeoff between the value of information and strategic rollover. On one hand, longer maturity reduces the value of information, as the bank has to wait longer to liquidate after learning that project is bad. On the other hand, longer maturity reduce the incentives to roll over bad loans and increases the incentives to monitor. The overall effect therefore depends on which one of the two effects dominates.

![Graph showing $t_a$, $t_b$, and $t_g$](image)

Figure 6: Endogenous Learning with and without financial constraints

This figure plots $t_a$, $t_b$, and $t_g$ with the following parameter values: $r = 0.1$, $\delta = 0.05$, $F = 1$, $\phi = 1$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.2 \times NPV^b$, $\lambda = 2$, $q_0 = 0.1$, $\beta = 0$.

4 Special Case: Instantly-Maturing Loan

Next we consider a special case in which loans mature instantly. This case corresponds to the limit that the intensity of the debt maturing $m$ goes to infinity. By doing so, we obtain simple and intuitive closed-form solutions in primitives. The results follow naturally by taking limits to the propositions in the previous section and therefore, we omit the proofs. Proposition 7 is the counterpart of Proposition 2, where we study the case without financial constraints.
Proposition 7. If bank loans mature instantly and there is financial constraint, the equilibrium is given by

\[ q_t = \frac{q_0}{q_0 + (1 - q_0)e^{-\lambda t}} \quad \forall t < t_b \quad (26a) \]

\[ t_b = \frac{1}{\lambda} \left[ \log \left( \frac{\bar{q}}{1 - \bar{q}} \right) - \log \left( \frac{q_0}{1 - q_0} \right) \right] \quad (26b) \]

\[ t_g = t_b + \frac{1}{r + \phi} \log \left( \frac{\phi (1 - \theta) F}{(r + \phi)L - (c + \phi \theta R)} \right) \quad (26c) \]

\[ \bar{q} = \frac{1}{(1 - \theta)} \left( \frac{\delta + \phi c + \phi F}{r + \phi} - \frac{c}{\phi F - \theta} \right). \quad (26d) \]

When bank loans mature instantly, a project is immediately liquidated once it is known as bad before \( t_b \). In this case, \( t_b \), the length of this efficient liquidation region only depends on the speed of learning \( \lambda \) and the ultimate credit quality \( \bar{q} \). \( t_g - t_b \), the length of pretend and extend period can be written equivalently as \( \frac{1}{r + \phi} \log \left( \frac{\phi (1 - \theta) F}{L - NPV_b} \right) \). Indeed, if \( L - NPV_b \) gets higher so that liquidation becomes relatively more profitable, bad types find it less attractive to mimic others. As a result, \( t_g - t_b \) gets shorter. The numerator term, \( \frac{\phi (1 - \theta) F}{L - NPV_b} \), is the discounted net present value of the loss to the bad-type bank if it finances the borrower until the final cash flows are produced. If this value gets higher, mimicking other types and sharing the loss with market lenders become more appealing, and therefore \( t_g - t_b \) needs to be longer.

Next, we consider the equilibrium in Proposition 4 for the case in which the loan rate \( y \) is constrained by the interim cash flow \( c \). In the case that the financial constraint always binds for the bad type on \( [t_b, t_g] \), we get the following results.

Proposition 8. If

\[ L - F + (1 - \beta) \left[ \frac{c}{r + \phi} + \frac{\phi}{r + \phi} ((1 - \theta)F + \theta R) - L \right] > 0, \]

the equilibrium is characterized by two thresholds \( \{t_b, t_g\} \), where \( \{\bar{q}, t_b\} \) are identical to those in Equation (26a), and

\[ t_g = t_b + \frac{1}{r + \phi} \log \left( \frac{F - c + \phi \theta F}{r + \phi} \right) \quad (27) \]

Note that we can write equivalently \( t_g - t_b = \frac{1}{r + \phi} \log \left( \frac{F - NPV_b}{L - NPV_b} \right) \). Once again, the length of extend and pretend period decreases with the liquidation value of project, whereas it increases with the amount of loan payments due to the bank.
Finally, we consider the equilibrium in Proposition 5 under endogenous learning without financial constraints.

**Proposition 9.** In an equilibrium characterized by \( \{t_b, t_g\} \), the bank never learns after \( t_b \).

1. If

\[
\frac{\psi}{\lambda} < (1 - \beta)(1 - q_0) \left( L - \frac{c + \phi \theta R}{r + \phi} \right),
\]

there exists an equilibrium in which the bank learns during \([0, t_a]\). The belief threshold \( \bar{q} \) and the length of the extend and pretend period \( t_g - t_b \) are the same as in Proposition 7. The thresholds \( t_a, t_b \) and \( t_g \) are given by

\[
t_a = \frac{1}{\lambda} \left[ \log \left( \frac{\bar{q}}{1 - \bar{q}} \right) - \log \left( \frac{q_0}{1 - q_0} \right) \right],
\]

\[
t_b = t_a + \frac{1}{r + \phi} \log \left( \frac{(1 - \beta)(1 - q_0) \left( L - \frac{c + \phi \theta R}{r + \phi} \right)}{(1 - \beta)(1 - q_0) \left( L - \frac{c + \phi \theta R}{r + \phi} \right) - \frac{\psi}{\lambda}} \right),
\]

\[
t_g = t_a + \frac{1}{r + \phi} \log \left( \frac{\phi}{r + \phi} \left( 1 - \beta \right)(1 - q_0) \left( 1 - \theta \right) \frac{F}{r + \phi} \left( L - \frac{c + \phi \theta R}{r + \phi} \right) - \frac{\psi}{\lambda} \right).
\]

2. If

\[
\frac{\psi}{\lambda} > (1 - \beta)(1 - q_0) \left( L - \frac{c + \phi \theta R}{r + \phi} \right),
\]

the bank never learns.

Note that the right-hand side of (28) can be rewritten as \((1 - \beta)(1 - q_0) \left( L - NPV^{b}_r \right)\), where \((1 - \beta)\) is the bank’s bargaining power, \((1 - q_0)\) the ex-ante probability of a bad project, and \((L - NPV^{b}_r)\) the benefit from learning a project is bad. This expression makes it clear that the expected benefit of learning to the bank is to liquidate bad projects and receive a liquidation value, rather than continue to offer financing.

5 **Observable Rollover and Deterministic Maturity**

In the baseline model, we have assumed rolling over a loan is not observed by the market participants. Moreover, the maturing event has been modeled as a Poisson event to simplify the solution. Both assumptions have been made for simplicity. In this section, we introduce two modifications to the model. First, whenever a loan matures, we assume it is observable
whether the bank decides to rollover or liquidate. Second, the maturity of loans is publicly known to be fixed at $1/m$. We will show that the model is essentially identical to one in a discrete-time framework. Thus, as in most discrete-time models with binary-type of asymmetric information, the equilibrium will in general involve mixed strategies due to the integer problem. For the remainder of this section, we will construct an equilibrium which has features to those described in Proposition 2.

Let $n \in \{1, 2, 3 \cdots\}$ be the sequence of rollover events. The date associated to the $n$-th rollover event is $t_n = n/m$ and the time between two rollover dates is $1/m$. As before, we can construct an equilibrium with two thresholds: $t_b$ and $t_g$. However, with deterministic rollovers it is notationally more convenient to specify the two thresholds in term of the rollover events: $n_b$ and $n_g$. The following proposition describes the equilibrium.

**Proposition 10.** If loan maturity is fixed at $1/m$ and rollover is observable, then there exists \{n_b, n_g\} such that

1. Efficient liquidation
   
   (a) For $n < n_b$, bad projects are liquidated, whereas other projects are rolled over;

2. Extend and pretend
   
   (a) For $n = n_b$, a fraction $\alpha_b \leq 1$ of the bad projects are liquidated, whereas other projects are rolled over.

   (b) When $n \in (n_b, n_g - 1)$, all loans are rolled over.

3. Market Financing
   
   (a) When $n = n_g$, market lenders make an offer at $D_{t_m} (q_{t_m} = \bar{q})$ with probability $\alpha_g \leq 1$.

   (b) When $n = n_g + 1$, all entrepreneurs refinance with the market.

4. If $m \to \infty$, the equilibrium converges to the one in Proposition 7.

Under observable rollovers, the public belief for the loan quality $q_t$ stays unchanged during any two roll-over events. At the roll-over event $n < n_b$ or equivalently $t < n_b/m$, the belief will experience a discrete jump. If the project is liquidated, clearly $q_t$ jumps to 0. If the loan is rolled over, $q_t$ jumps upwards to the level described in the instantly-maturing debt, as shown in (26a). Note there is an equivalence in beliefs under fixed maturity and instantly-maturing exponential debt, because the event of maturing is occurring with certainty at $t = n/m$. $q_t$ stays unchanged at $\bar{q}$ after $t > n_b/m$. 

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6 Conclusion and Empirical Relevance

In this paper, we introduce private learning into a banking model and study the dynamic tradeoffs of relationship-based lending. Compared to market financing, bank financing enables learning about the quality of the project being financed, but is also subject to the downside of information monopoly cost. We construct an equilibrium in which an entrepreneur starts with bank financing and subsequently refinanced with the market. The novel result is, under information asymmetry, banks will endogenously roll over bad loans after some time. We characterize conditions for this “extend and pretend” and show how it is affected by factors such as entrepreneur’s financial constraint. We also study banks’ incentives in learning over time.

Our paper is closely related to the empirical literature on relationship lending and loan sales. In practice, 60% of the loans are first sold within one month of loan origination and nearly 90% are sold within one year (Drucker and Puri, 2008). As argued in Gande and Saunders (2012), a special role of banks is to create an active secondary loan market while still producing information. A key assumption in our paper is that private information between a relationship bank and other credit suppliers is built up over time, which is consistent with the observation in Dahiya et al. (2003) and Jiang et al. (2013).

A main prediction in our paper is banks have endogenous incentives to roll over bad loans. This results on “extend and pretend” largely remind the popular discussions on how loan sales and securitization induce agency conflicts (Purnanandam, 2010; Keys et al., 2010). As documented by existing studies (Agarwal et al., 2011), mortgage lenders and loan servicers rarely wrote off losses shortly after borrowers got financially distressed. Dahiya et al. (2003) find firms that file bankruptcy after the loan sales are not necessarily the worst-performing firms at the time of loan sales. Jiang et al. (2013) find that at the time of loan origination, loans with high probability of sale have higher delinquency rates subsequently, whereas ex-post loans sold by banks have lower delinquency rates. Our paper implies that reputation may not reduce the related agency conflicts. Griffin et al. (2014) show that highly-reputable banks may produce complicated assets (CLO, MBS, ABS, and CDOs) that underperform subsequently, implying that bank reputation may contribute negatively to the quality of its assets. To some extent, our results is related to “evergreening” loans and zombie lending, which in the literature has mostly been attributed to banks’ capital requirements when they

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14Interestingly, while Dahiya et al. (2003) find a negative stock price reaction to the announcement of loan sales using data from 1995-1998, whereas Gande and Saunders (2012) find the reaction to be positive using data post 2000s. The main difference is, as argued by Gande and Saunders (2012), many lenders terminated their lending relationships in the sample of Dahiya et al. (2003). Therefore, they conclude the role of secondary loan market has evolved from informed lenders off-loading troubled loans to banks creating liquidity.
write off bad loans (Peek and Rosengren, 2005; Caballero et al., 2008).

Our paper has unique prediction on how the rates on bank loans vary with the length of lending relationships. López-Espinosa et al. (2017) using Spanish data and find loan rates do not drop until the relationship extends beyond two years. In contrast, Ioannidou and Ongena (2010) find loan rates initially decrease but eventually ratchet up, which encourages firms to switch to other banks. Moreover, our model predicts that the concerns for extend and pretend would be less severe for less-reputable borrowers who are more financially constrained and specifically, borrowers with longer relationship with banks are more likely to receive waivers upon covenant violation. These tests will be interesting to conduct with careful empirical studies.
References


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A Appendix

A.1 Proofs

A.1.1 Proof of Lemma 1

Proof. The proof relies on the filtering formula for counting processes in Lipster and Shiryaev (Chapter 19). Let $\chi^i_t$ be the probability that a type $i \in \{g, b, u\}$ firm looks for external financing at time $t$ and let $\ell^i_t$ be the probability that a type $i$ firm liquidates at time $t$. Let $L_t$ be the counting process associated to the liquidation time and let $M_t$ be the counting process associated with going to the market. If we denote the type of the firm at time $t$ by $i(t)$ then $L_t$ has intensity $m^i(t)$ while $M_t$ has intensity $m\chi^i(t)$. The process $i(t)$ has transitions governed by the infinitesimal generator

$$\Lambda \equiv \begin{pmatrix} -\lambda & \lambda q_0 & \lambda (1 - q_0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using Theorem 19.6 (and following similar calculations to the ones in Examples 2 and 3 therein) we get that

$$d\pi^u_t = -\lambda \pi^u_t dt + \pi^u_t \left( (\ell^u_t - \ell^b_t)(1 - \pi^u_t) - (\ell^g_t - \ell^b_t)\pi^g_t \right) \cdot [dL_t - m(\pi^u_t \ell^u_t + \pi^g_t \ell^g_t + \pi^b_t \ell^b_t)dt]$$
$$+ \pi^u_t \left( (\chi^u_t - \chi^b_t)(1 - \pi^u_t) - (\chi^g_t - \chi^b_t)\pi^g_t \right) \cdot [dM_t - m(\pi^u_t \chi^u_t + \pi^g_t \chi^g_t + \pi^b_t \chi^b_t)dt]$$

From here, we get that in absence of liquidation and market financing beliefs are given by

$$\dot{\pi}^u_t = -\lambda \pi^u_t - m\pi^u_t \left( (\ell^u_t + \chi^u_t - \ell^b_t - \chi^b_t)(1 - \pi^u_t) - (\ell^g_t + \chi^g_t - \ell^b_t - \chi^b_t)\pi^g_t \right)$$

Suppose that $\ell^b_t = 1$ and $\ell^u_t = \ell^g_t = \chi^u_t = \chi^g_t = 0$, then we have

$$\dot{\pi}^u_t = -\lambda \pi^u_t + m\pi^u_t \alpha_t$$

Similarly, we get

$$\dot{\pi}^g_t = \lambda q_0 \pi^u_t - m\pi^g_t \left[ (\ell^g_t + \chi^g_t - \ell^b_t - \chi^b_t)(1 - \pi^g_t) - (\ell^u_t + \chi^u_t - \ell^b_t - \chi^b_t)\pi^u_t \right]$$
$$\dot{\pi}^b_t = \lambda (1 - q_0) \pi^u_t - m\pi^b_t \left[ (\ell^b_t + \chi^b_t - \ell^g_t - \chi^g_t)(1 - \pi^b_t) - (\ell^u_t + \chi^u_t - \ell^g_t - \chi^g_t)\pi^u_t \right]$$
so in the particular case that $\ell_t^b = 1$ and $\ell_t^u = \ell_t^g = \chi_t^u = \chi_t^g = \chi_t^b = 0$, then we get

\[
\begin{align*}
\dot{\pi}_t^g &= \lambda q_0 \pi_t^u + m \pi_t^g \pi_t^b \\
\dot{\pi}_t^b &= \lambda (1 - q_0) \pi_t^u - m \pi_t^b (1 - \pi_t^b)
\end{align*}
\]

A.1.2 Proof of Proposition 2

We start with two lemmas that will be useful for later proofs. Throughout, we sometimes use $\bar{V}_i$ to represent (4) evaluated at $q_{rm} = \bar{q}$.

Lemma 4. The No Deals condition implies the good type’s value function must satisfy smooth-pasting at $t = t_g$. That is

\[\dot{V}_{t_g}^g = 0.\]

Proof. We prove by contradiction. Suppose $\dot{V}_{t_g-}^g < 0$, then Equation (17b) in $[t_b, t_g]$ implies $V_t^g < \frac{c + \phi R}{(r + \phi)}$. However, this is impossible because $\frac{c + \phi R}{(r + \phi)}$ is the continuation value of the good types if they never finance with the market.

Next, let us assume $\dot{V}_{t_g-}^g > 0$. Under the constructed equilibrium, $\dot{q}_t = 0$ for any $t > t_b$. As a result, $\dot{V}_t^g$ – the continuation payoff when the good type financed with the market at time $t$ also stays at a constant after $t_b$, which is denoted as $\bar{V}^g$. If $\dot{V}_{t_g-}^g > 0$, that implies that for $\epsilon$ sufficiently small, $V_{t_g-\epsilon}^g < \bar{V}^g$ so that the No Deals condition fails. Note that this step relies on the fact that $V_t^g$ stays a constant for $t \in [t_b, t_g]$. In the equilibrium without the zombie lending stage ($m < m^*$), this condition no longer holds so that in general, $\dot{V}_{t_g}^g \geq 0$.

Lemma 5. $V_t^g$ stays at a constant in any equilibrium that is constructed under $t_b$ and $t_g$.\textsuperscript{15}

Proof. We write out the HJB for the good types in all regions and the proof directly follows by plugging (30) into (29).

\[
\begin{align*}
(r + \phi) V_t^g &= \dot{V}_t^g + c + \phi R \quad t \in [0, t_g] \\
(r + \phi + m) V_t^g &= \dot{V}_t^g + c + \phi R + m \bar{V}^g \quad t \in [t_g, \infty).
\end{align*}
\]

\textsuperscript{15}This is true under any equilibrium that we construct, which consists of thresholds $\{t_b, t_g\}$. However, it may not hold under any arbitrary equilibrium, which could exist when $m$ gets very large.
Proof. By applying the smooth pasting condition

\[ V_{t_g}^g = \frac{c + \phi R}{r + \phi} = \frac{c + \phi R + m\bar{V}^g}{r + \phi + m}, \]

we get

\[ \bar{q} = \frac{1}{1 - \theta} \left( \delta + \phi \frac{c + \phi F}{r + \phi} - \frac{c}{\phi F} - \theta \right) \]

after some derivations. Clearly, the equation system in the last region shows

\[ V_{t_g}^g - V_{t_g}^b = \frac{\phi R (1 - \theta) + m (\bar{V}^g - \bar{V}^b)}{r + \phi + m} = \frac{\phi R (1 - \theta) + m \frac{\phi (R-F)(1-\theta)}{r+\phi}}{r + \phi + m}. \]

In that case, using the same smooth pasting condition, we get

\[ V_{t_g}^b = \frac{c + \phi R}{r + \phi} - \frac{\phi R (1 - \theta) + m \frac{\phi (R-F)(1-\theta)}{r+\phi}}{r + \phi + m}. \]

Given that, let us solve for \( t_g - t_b \) using the ODE system in region 2. In particular, for any \( t \in [t_b, t_g] \),

\[ V_t^b = e^{(r+\phi)(t-t_g)} V_{t_g}^b + \frac{c + \phi \theta R}{r + \phi} \left[ 1 - e^{(r+\phi)(t-t_g)} \right]. \]

Using the boundary condition \( V_{t_b}^b = L \), we can get

\[ t_g - t_b = -\frac{1}{(r+\phi)} \log \left( \frac{L - \frac{c + \phi \theta R}{r + \phi}}{V_{t_g}^b - \frac{c + \phi \theta R}{r + \phi}} \right). \]

The threshold \( t_b \) is determine by the condition

\[ t_b = \min \{ t : q_t = \bar{q} \}. \]

The final step is to find the solution for \( q_t \) in the interval \([0, t_b]\). The ODE system in region 1 is

\[ \dot{\pi}_t^u = -\lambda \pi_t^u + m \pi_t^u \pi_t^b, \]

\[ \dot{\pi}_t^g = \lambda \pi_t^u q_0 + m \pi_t^g \pi_t^b, \]

\[ \dot{\pi}_t^b = \lambda \pi_t^u (1 - q_0) - m \pi_t^b \left( 1 - \pi_t^b \right). \]
Let us define \( z_t = \frac{\pi^g_t}{\pi^u_t} \), then,
\[
\dot{z}_t = \frac{\frac{\pi^g_t}{\pi^u_t} - \frac{\pi^g_t}{\pi^u_t}}{(\frac{\pi^u_t}{\pi^u_t})^2} = \frac{\pi^g_t}{\pi^u_t} - z_t \frac{\pi^u_t}{\pi^u_t} = \lambda q_0 + m z_t (1 - \pi^g_t - \pi^u_t) - z_t (-\lambda + m (1 - \pi^g_t - \pi^u_t)) = \lambda (q_0 + z_t).
\]

Therefore, we have the solution
\[
z_t = q_0 (e^{\lambda t} - 1) \Rightarrow \pi^g_t = q_0 (e^{\lambda t} - 1) \pi^u_t.
\] (31)

Since \( \pi^u_t + \pi^g_t + \pi^b_t = 1 \), we also have
\[
\pi^b_t = 1 - (q_0 e^{\lambda t} + 1 - q_0) \pi^u_t.
\] (32)

Substituting (31) and (32) into the ODE system, we get a first-order ODE for \( \pi^u_t \)
\[
\dot{\pi}^u_t = (m - \lambda) \pi^u_t - m (q_0 e^{\lambda t} + 1 - q_0) (\pi^u_t)^2,
\]

which corresponds to a continuous-time Riccati equation. This equation can be transformed it into a second-order ODE. Let \( v_t = -m (q_0 e^{\lambda t} + 1 - q_0) \pi^u_t \) and \( Q_t = q_0 e^{\lambda t} + 1 - q_0 \),
\[
\dot{v}_t = v_t^2 + \frac{v_t}{Q_t} [q_0 e^{\lambda t} \lambda + Q_t (m - \lambda)].
\] (33)

Further, if we let \( v_t = -\dot{\nu}_t \Rightarrow \dot{v}_t = -\frac{\dot{\nu}_t}{\nu_t} + (v_t)^2 \), then we can transform equation (33) into the following second-order ODE
\[
\ddot{\nu}_t = \frac{\dot{\nu}_t}{Q_t} [q_0 e^{\lambda t} \lambda + Q_t (m - \lambda)]
\] From here, we get that
\[
\dot{\nu}_t = \dot{\nu}(0) e^{\int_0^t \frac{1}{Q_s} \frac{1}{q_0 e^{\lambda s} - 1} ds + (m - \lambda) t}
\]
Moreover,
\[
\int_0^t \frac{1}{1 + \frac{q_0 e^{\lambda s}}{1 - q_0 e^{-\lambda s}}} ds = \frac{1}{\lambda} \log (1 - q_0 + q_0 e^{\lambda t})
\]
so
\[
\dot{\nu}_t = \dot{\nu}(0) \left(1 - q_0 + q_0 e^{\lambda t}\right)^{-\frac{1}{\lambda}} e^{(m - \lambda) t}
\]
Integrating one more time, we get
\[ \nu_t = \nu(0) + \dot{\nu}(0) \int_0^t (1 - q_0 + q_0 e^{\lambda s})^{\frac{1}{2}} e^{(m-\lambda)s} ds. \]

Using the definition of \( v_t \) and \( \nu_t \), we have
\[ \dot{\nu}_0 = -v_0 \nu(0) = m \nu(0) \]
so
\[ \nu_t = \nu(0) \left( 1 + m \int_0^t (1 - q_0 + q_0 e^{\lambda s})^{\frac{1}{2}} e^{(m-\lambda)s} ds \right). \]

Using the definition of \( v_t \) we get
\[ v_t = -m \frac{(1 - q_0 + q_0 e^{\lambda t})^{\frac{1}{2}} e^{(m-\lambda)t}}{1 + m \int_0^t (1 - q_0 + q_0 e^{\lambda s})^{\frac{1}{2}} e^{(m-\lambda)s} ds} \]
so
\[ \pi_t^u = \frac{(1 - q_0 + q_0 e^{\lambda t})^{\frac{1}{2}} e^{(m-\lambda)t}}{1 + m \int_0^t (1 - q_0 + q_0 e^{\lambda s})^{\frac{1}{2}} e^{(m-\lambda)s} ds} \quad (34) \]

Thus, substituting (34) in the definition for \( q_t \), we get
\[ q_t = \pi_t^q + q_0 \pi_t^u \]
\[ = q_0 \left( e^\lambda - 1 \right) \pi_t^u + q_0 \pi_t^u \]
\[ = q_0 e^{\lambda t} \pi_t^u \quad (35) \]
\[ = \frac{q_0 \left( 1 - q_0 + q_0 e^{\lambda t} \right)^{\frac{1}{2}} e^{mt}}{1 + m \int_0^t (1 - q_0 + q_0 e^{\lambda s})^{\frac{1}{2}} e^{(m-\lambda)s} ds} \]

Because \( q_t \) is monotone, the solution for \( t_b \) and \( t_g \) is unique.

Finally, we examine \( t_g - t_b \), which equals
\[ \frac{1}{r + \phi} \log \left( \frac{(r + \phi) V_{t_g}^b - (c + \phi \theta R)}{(r + \phi) L - (c + \phi \theta R)} \right). \]

A necessary condition for the equilibrium to be true is \( t_g - t_b > 0 \). However, if \( m = 0 \), this is clearly violated because Assumption 1 guarantees \( V_{t_g}^b < L \). If \( m \to \infty \),
\[ V_{t_g}^b \to \frac{c + \phi R}{r + \phi} - \frac{\phi (R - F) (1 - \theta)}{r + \phi} \]
so that it exceeds $L$. Finally, a quick comparative static analysis shows that $\frac{dV^b_t}{dm} > 0$. Therefore, there exists a unique $m$ so that such an equilibrium exists if and only if $m > m$.

A.1.3 Proof of Proposition 3

Proof. We have already shown that the unique monotone equilibrium is the one in Proposition 2. It is only left to show that any equilibrium must be monotone if $m$ is low enough. The proof for monotonicity follows the traditional skimming property in bargaining models. In our case, the skimming property is satisfied only if $m$ is low enough. In particular, we show that

$$V^g_t - \bar{V}^g > V^u_t - \bar{V}^u > V^b_t - \bar{V}^b.$$  \hspace{1cm} (36)

Let $x^i_t \in \{0, 1\}$ and $\ell^i_t \in \{0, 1\}$ be the rollover and liquidation decision, respectively. The expected payoff, given strategy $(x^i, \ell^i)$ is

$$V^u_t = \mathbb{E}_t\left\{ \int_t^\tau e^{-r(s-t)}cFds + e^{-r(\tau-t)} \left[ 1_{\tau=\tau^\phi} [q_0 + (1-q_0) \theta] R 
+ 1_{\tau=\tau^m} [x^u V^u_T + \ell^u L + (1-x^u - \ell^u) \bar{V}^u] \right] \right\}$$

$$V^g_t = \mathbb{E}_t\left\{ \int_t^\tau e^{-r(s-t)}cFds + e^{-r(\tau-t)} \left[ 1_{\tau=\tau^\phi} R + 1_{\tau=\tau^m} [x^g V^g_T + \ell^g L + (1-x^g - \ell^g) \bar{V}^g] \right] \right\}$$

$$V^b_t = \mathbb{E}_t\left\{ \int_t^\tau e^{-r(s-t)}cFds + e^{-r(\tau-t)} \left[ 1_{\tau=\tau^\phi} \theta R + 1_{\tau=\tau^m} [x^b V^b_T + \ell^b L + (1-x^b - \ell^b) \bar{V}^b] \right] \right\}.$$  

A good type can always mimic the strategy of a low type, hence the continuation payoff of a good type must at least as high as the payoff of mimicking the strategy of the bad type.

$$V^g_t \geq \mathbb{E}_t\left\{ \int_t^\tau e^{-r(s-t)}cFds + e^{-r(\tau-t)} \left[ 1_{\tau=\tau^\phi} R + 1_{\tau=\tau^m} [x^g V^g_T + \ell^g L + (1-x^g - \ell^g) \bar{V}^g] \right] \right\}.$$  

Hence, for all $t < \tau$, we have

$$V^g_t - V^b_t \geq \mathbb{E}_t\left\{ e^{-r(\tau-t)} \left[ 1_{\tau=\tau^\phi} (1-\theta) R + 1_{\tau=\tau^m} [x^b (V^g_T - V^b_T) + (1-x^b - \ell^b) (\bar{V}^g - \bar{V}^b)] \right] \right\}.$$  

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Because the time $\tau = \min\{\tau_\phi, \tau_m\}$ is exponentially distributed with mean arrival rate $m + \phi$, we can write the previous expression as

$$V^g_t - V^b_t \geq \frac{\phi(1 - \theta)R}{r + \phi + m} + \int_0^\infty e^{-(r + m + \phi)(s-t)}m \left[ x_s^b(V^g_s - V^b_s) + (1 - x_s^b - \ell_s^b)(\bar{V}^g - \bar{V}^b) \right] ds,$$

where

$$\bar{V}^g - \bar{V}^b = \frac{\phi(1 - \theta)(R - F)}{r + \phi} > 0.$$

Letting $\Delta_t \equiv (V^g_t - \bar{V}^g) - (V^b_t - \bar{V}^b)$, we can write the previous inequality as

$$\Delta_t = A + \int_t^\infty e^{-(r + m + \phi)(s-t)}m \left[ x_s^b \Delta_s + (1 - \ell_s^b) (\bar{V}^g - \bar{V}^b) \right] ds,$$

where

$$A \equiv \frac{\phi(1 - \theta)}{(r + \phi + m)(r + \phi)} [(r + \phi)F - m(R - F)]$$

Differentiating $\Delta_t$, we get the differential equation

$$\dot{\Delta}_t = (r + m(1 - x_t^g) + \phi) \Delta_t + m(1 - \ell_t^b) (\bar{V}^g - \bar{V}^b) - (r + m + \phi)A.$$

The solution to this equation is given by

$$\Delta_t = \int_t^\infty e^{-(r + \phi)(s-t) + \Psi(s,t)} \left[ \phi(1 - \theta) \left( F - \frac{m}{r + \phi}(R - F) \right) + \frac{m}{m + \phi} (1 - \ell_s^b)(\bar{V}^g - \bar{V}^b) \right] ds$$

where

$$\Psi(s,t) \equiv \int_t^s m(1 - x_u^b)du$$

From here we get that if

$$m < \bar{m} = (r + \phi)\frac{F}{R - F}$$

then $\Delta_t > 0$ for any policy $\ell_t^b$. Hence, in any equilibrium we have that

$$V^g_t - \bar{V}^g > V^b_t - \bar{V}^b,$$

which means that if there is a time $\tilde{t}_g$ at which the good type chooses market financing with positive probability, then the bad type chooses market financing for sure. Repeating the same calculations for the pairs $\{g, u\}$ and $\{u, b\}$, we can conclude that the skimming property (36) holds.

Using the definition $q_t = q_0 \pi^u_t + \pi^g_t$ and the evolution of beliefs in the proof of Lemma 1,
we get that the evolution of \( q_t \) is given by the following differential equation.

\[
\dot{q}_t = m q_t \left[ (\ell^b_t + x^b_t)(1 - \pi^u_t - \pi^g_t) - (\ell^u_t + x^u_t)(1 - \pi^u_t) + (\ell^g_t + x^g_t) \pi^g_t \right] \\
\quad + m \pi^g_t \left[ (\ell^u_t + x^u_t) - (\ell^g_t + x^g_t) \right]
\]

If \( \ell^g_t + x^g_t = \ell^u_t + x^u_t = 0 \) and \( \ell^b_t + x^b_t = 1 \) then \( \dot{q}_t = m q_t (1 - \pi^u_t - \pi^g_t) > 0 \). On the other hand, by the skimming property, we have that:

1. If \( \ell^g_t + x^g_t = 1 \), then \( \ell^u_t + x^u_t = \ell^b_t + x^b_t = 1 \) so \( \dot{q}_t = 0 \).
2. If \( \ell^b_t + x^b_t = 0 \), then \( \ell^g_t + x^g_t = \ell^u_t + x^u_t = 0 \) and also \( \dot{q}_t = 0 \).
3. If \( \ell^g_t + x^g_t = 0 \) and \( \ell^u_t + x^u_t = 1 \), then \( \ell^b_t + x^b_t = 1 \) so \( \dot{q}_t = m \pi^g_t (1 - q_t) > 0 \).

Hence, in any equilibrium, the trajectory of \( q_t \) must be non-decreasing in time so an equilibrium must be monotone.

\[\square\]

### A.1.4 Proof of Proposition 4

**Proof.** Lemma 2 shows that the constraint \( y \leq c \) must bind at time \( t_b \). However, depending on the parameters of the problem, the constraint might be slack at time \( t_g \). In particular, we can show that the constraint on \([t_b, \infty)\) is monotonic, that is, there exists \( t_c \) such that \( y^b_t = c \) on \([t_b, t_c)\) and \( y^b_t < c \) after \( t_c \). Note that if such a \( t_g \) satisfies \( t_c > t_g \), then the financial constraint always binds until the borrower refines with the market. As in the case in which we ignore the financial constraint, we have that

\[
V^b_{t_g} = V^g_{t_g} + \frac{\phi R (\theta - 1) + m \phi (R - F)(\theta - 1)}{r + \phi + m} \\
= \frac{c + \phi R}{r + \phi} + \frac{\phi R (\theta - 1) + m \phi (R - F)(\theta - 1)}{r + \phi + m}.
\]

Given the boundary, we solve for \( V^b_t \)

\[
(r + \phi) V^b_t = \dot{V}^b_t + c + \phi \theta R \\
V^b_t = \frac{c + \phi R \theta}{r + \phi} + e^{(r + \phi)(t-t_g)} \left[ V^b_{t_g} - \frac{c + \phi R \theta}{r + \phi} \right]
\]
At $t = t_c$, the financial constraint exactly binds so that it must be

$$B^b_t(c) = L + (1 - \beta) (V^b_t - L)$$

$$L + (1 - \beta) (V^b_t - L) - B^b_t(rF) \equiv \frac{c - rF}{r + \phi + m},$$

(37)

where $B^b_t(rF)$ solves

$$(r + \phi + m) B^b_t(rF) = \dot{B}^b_t(rF) + rF + \phi F + m \left[ L + (1 - \beta) (V^b_t - L) \right], \quad t \in (t_c, t_g).$$

Let’s define $Z_t \equiv L + (1 - \beta) (V^b_t - L) - B^b_t(rF)$. Substituting the ODEs for $V^b_t$ and $B^b_t(rF)$, and defining the constant

$$\Gamma_1 \equiv (r + \phi) \beta L + (1 - \beta) (c + \phi \theta R) - (r + \phi \theta) F,$$

we get the following ODE for $Z_t$ on $(t_c, t_g)$

$$(r + \phi + m) Z_t = \dot{Z}_t + \Gamma_1 \quad t \in (t_c, t_g).$$

(38)

At time $t_g$ we have

$$B^b_{t_g}(rF) = \frac{\phi \theta + r + m}{r + \phi + m} F,$$

which means that we can solve $Z_{t_g}$ up to primitives according to

$$Z_{t_g} = L + (1 - \beta) \left( V^b_{t_g} - L \right) - B^b_{t_g}(rF).$$

Solving (38) backward in time and combining with equation (37) we get

$$Z_{t_c} = \frac{\Gamma_1}{r + \phi + m} + e^{(r+\phi+m)(t_c-t_g)} \left[ Z_{t_g} - \frac{\Gamma_1}{r + \phi + m} \right] = \frac{c - rF}{r + \phi + m}.$$  

From here, we can solve for $t_c - t_g$, which is given by

$$t_c - t_g = \frac{1}{r + \phi + m} \log \left( \frac{c - rF - \Gamma_1}{r + \phi + m} \frac{1}{Z_{t_g} - \frac{\Gamma_1}{r + \phi + m}} \right).$$

Next, letting

$$\Gamma_2 \equiv rF + \phi \theta F + m \frac{c - rF}{r + \phi + m}$$

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we can find $B^b_t(rF)$ for $t \in (t_b, t_c)$ solving the following ODE

$$(r + \phi) B^b_t = \dot{B}^b_t + \Gamma_2, \quad t \in (t_b, t_c),$$

with initial condition

$$B^b_{t_b}(rF) + \frac{c - rF}{r + \phi + m} = L.$$  

The solution at time $t_c$ is

$$B^b_{t_c}(rF) = \frac{\Gamma_2}{r + \phi} + e^{(r + \phi)(t_c - t_b)} \left[ L - \frac{c - rF}{r + \phi + m} - \frac{\Gamma_2}{r + \phi} \right]. \quad (39)$$

From equation (37), we also know that at $t_c$

$$B^b_{t_c}(rF) = L + (1 - \beta) \left( V^b_{t_c} - L \right) - \frac{c - rF}{r + \phi + m}, \quad (40)$$

where

$$V^b_{t_c} = \frac{c + \phi R \theta}{r + \phi} + \left[ V^b_{t_g} - \frac{c + \phi R \theta}{r + \phi} \right] \left[ \frac{c - rF - \Gamma_1}{r + \phi + m} \frac{1}{Z_{t_g} - \frac{\Gamma_1}{r + \phi + m}} \right]^{r + \phi}_{r + \phi + m}. \quad (41)$$

Combining equations (39) and (40) we get

$$\frac{\Gamma_2}{r + \phi} + e^{(r + \phi)(t_c - t_b)} \left[ L - \frac{c - rF}{r + \phi + m} - \frac{\Gamma_2}{r + \phi} \right] = \Gamma_3$$

$$t_c - t_b = \frac{1}{r + \phi} \log \left( \frac{\Gamma_3 - \frac{\Gamma_2}{r + \phi}}{L - \frac{c - rF}{r + \phi + m} - \frac{\Gamma_2}{r + \phi}} \right), \quad (42)$$

where

$$\Gamma_3 \equiv L + (1 - \beta) \left( \frac{c + \phi R \theta}{r + \phi} + \left[ V^b_{t_g} - \frac{c + \phi R \theta}{r + \phi} \right] \left[ \frac{c - rF - \Gamma_1}{r + \phi + m} \frac{1}{Z_{t_g} - \frac{\Gamma_1}{r + \phi + m}} \right]^{r + \phi}_{r + \phi + m} - L \right) - \frac{c - rF}{r + \phi + m}, \quad (43)$$

Thus, we get that

$$t_b = t_g + \frac{1}{r + \phi + m} \log \left( \frac{c - rF - \Gamma_1}{r + \phi + m} \frac{1}{Z_{t_g} - \frac{\Gamma_1}{r + \phi + m}} \right) - \frac{1}{r + \phi} \log \left( \frac{\Gamma_3 - \frac{\Gamma_2}{r + \phi}}{L - \frac{c - rF}{r + \phi + m} - \frac{\Gamma_2}{r + \phi}} \right). \quad (44)$$

The previous solution can be simplified significantly when, $y_t = c$ for all $t \in [t_b, t_g]$. This happens if

$$\beta L + (1 - \beta)V^b_{t_g} > B^b_{t_g}(rF) + \frac{c - rF}{r + \phi + m}. \quad (45)$$
which reduces to
\[
L - \frac{c + (\phi \theta + m)F}{r + \phi + m} + (1 - \beta) \left[ \frac{c}{r + \phi} + \frac{\phi}{r + \phi} \left( (1 - \theta) \frac{mF}{r + \phi + m} + \theta R \right) - L \right] > 0
\]

In this case, let \( B_{\text{max}}^b(t|t_g) \) be the maximum continuation value that a bank can obtain at time \( t \) given that the loan rate after time \( t \) is \( c \) and the firm switches to market financing after time \( t_g \).

\[
B_{\text{max}}^b(t|t_g) = \frac{c + \phi \theta F}{r + \phi} \left( 1 - e^{-(r+\phi)(t_g-t)} \right) + e^{-(r+\phi)(t_g-t)} \frac{c + (\phi \theta + m)F}{r + \phi + m}
\]

In equilibrium, the thresholds \( \{t_b, t_g\} \) must be such \( B_{\text{max}}^b(t|t_g) \geq L \) for all \( t \in (t_b, t_g) \), and in particular

\[
B_{\text{max}}^b(t_b|t_g) = \frac{c + \phi \theta F}{r + \phi} \left( 1 - e^{-(r+\phi)(t_g-t_b)} \right) + e^{-(r+\phi)(t_g-t_b)} \frac{c + (\phi \theta + m)F}{r + \phi + m} = L
\]

which implies (25).

\[\square\]

A.1.5 Proof of Proposition 5

The following Lemma is very useful for the proof:

Lemma 6. For any \( t \in (t_b, t_g) \),

\[
V_t^u = q_0 V_t^g + (1 - q_0) V_t^b
\]

Proof. On \( t_a \in (t_b, t_g) \), the continuation values \( V_t^i \) follow

\[
(r + \phi) V_t^u = \hat{V}_t^u + c + \phi [q_0 + (1 - q_0) \theta] R \\
(r + \phi) V_t^g = \hat{V}_t^g + c + \phi R \\
(r + \phi) V_t^b = \hat{V}_t^b + c + \phi \theta R,
\]

which implies

\[
V_t^u = \frac{c + \phi [q_0 + (1 - q_0) \theta] R}{r + \phi} \left( 1 - e^{-(r+\phi)(t_g-t)} \right) + e^{-(r+\phi)(t_g-t)} V_{t_g}^u \\
V_t^g = \frac{c + \phi R}{r + \phi} \left( 1 - e^{-(r+\phi)(t_g-t)} \right) + e^{-(r+\phi)(t_g-t)} V_{t_g}^g \\
V_t^b = \frac{c + \phi \theta R}{r + \phi} \left( 1 - e^{-(r+\phi)(t_g-t)} \right) + e^{-(r+\phi)(t_g-t)} V_{t_g}^b.
\]
Therefore,

\[ q_0 V_t^g + (1 - q_0)V_t^b - V_t^u = e^{-(r+\phi)(t_g-t)} \frac{m}{r+\phi+m} \left(q_0 \bar{V}^g + (1 - q_0)\bar{V}^b - \bar{V}^u\right), \]

where \( \bar{V}^i \) is derived from (4) with \( q_m = q^- \):

\[
\bar{V}^u = \frac{c + \phi \left[q + (1 - \bar{q}) \theta\right] F}{\delta + \phi} + \frac{\phi\left[q_0 + (1 - q_0)\theta\right] (R - F)}{r + \phi},
\]

\[
\bar{V}^b = \frac{c + \phi \left[q + (1 - \bar{q}) \theta\right] F}{\delta + \phi} + \frac{\phi (R - F)}{r + \phi},
\]

\[
\bar{V}^g = \frac{c + \phi \left[q + (1 - \bar{q}) \theta\right] F}{\delta + \phi} + \frac{\phi (R - F)}{r + \phi},
\]

\[ q_0 \bar{V}^g + (1 - q_0)\bar{V}^b - \bar{V}^u = 0, \text{ and therefore } q_0 V_t^g + (1 - q_0)V_t^b - V_t^u = 0. \]

Let us first show that in any equilibrium characterized by \{t_b, t_g\}, banks never learn after \( t_b \). To see this, suppose that instead the bank learns during \([t_b, t_g] \), where \( t_a > t_b \), the HJB equations are as follows for \( t \in [t_a, t_g] \)

\[
(r + \phi + m) B_t^u = \dot{B}_t^u + y_t F + \phi[q_0 + (1 - q_0)\theta] F + m[L + (1 - \beta)(V_t^u - L)]
\]

\[
(r + \phi + m) B_t^g = \dot{B}_t^g + y_t F + \phi F + m[L + (1 - \beta)(V_t^g - L)]
\]

\[
(r + \phi + m) B_t^b = \dot{B}_t^b + y_t F + \phi \theta F + m[L + (1 - \beta)(V_t^b - L)].
\]

From here, we get

\[
(r + \phi + m) \Gamma_t = \dot{\Gamma}_t + m(1 - \beta)q_0 V_t^g + (1 - q_0)V_t^b - V_t^u = \dot{\Gamma}_t,
\]

where \( \Gamma_t \equiv q_0 B_t^g + (1 - q_0) B_t^b - B_t^u \), and the second equality follows from Lemma 6. Since \( \Gamma_{t_g} = 0 \), it implies that \( \Gamma_t = 0 \) for \( \forall t \in [t_a, t_g] \). In this case, the net benefit of learning \( \Omega_t = \Gamma_t - \frac{\psi}{\lambda} \) is always negative at \( t_a \). Therefore, if learning happens in an interval, this interval must be a sub-interval of \([0, t_b] \).

Next, we prove that it if learning happens at all, it must be that the bank learns during \([0, t_a] \), where \( t_a < t_b \). Note that the threshold \( t_b \) is given by the condition \( q_{t_b} = \bar{q} \). Using the fact that \( \Gamma_{t_b} = 0 \), we can solve \( \Gamma_t \) backward in time and solve for \( t_a \) such that \( \Gamma_{t_a} = \psi/\lambda \). Once we have solved for \{t_a, t_b, t_g\}, the only step left is to verify that \( \Gamma_t \) single-crosses \( \psi/\lambda \).
from above at time \( t_a \). Consider the regions \( t < t_a \), in this region, we have

\[
(r + \phi + m) B^u_t = \dot{B}^u_t + y_tF + \phi[q_0 + (1 - q_0)\theta]F - \psi + m[L + (1 - \beta)(V^u_t - L)] + \lambda\Gamma_t
\]

\[
(r + \phi + m) B^q_t = \dot{B}^q_t + y_tF + \phi F + m[L + (1 - \beta)(V^q_t - L)]
\]

\[
(r + \phi + m) B^b_t = \dot{B}^b_t + y_tF + \phi\theta F + mL,
\]

so

\[
(r + \phi + m + \lambda) \Gamma_t = \dot{\Gamma}_t + m(1 - \beta)((1 - q_0)L + q_0 V^q_t - V^u_t) + \psi
\]

Let \( H_t \equiv (1 - q_0)L + q_0 V^q_t - V^u_t \), and combine the previous ODE with the ODE for \( \Gamma_t \) on \( (t_a, t_b) \) we get

\[
(r + \phi + m + \lambda) \Gamma_t = \dot{\Gamma}_t + m(1 - \beta)H_t + \psi, \ t \in [0, t_a]
\]

\[
(r + \phi + m) \Gamma_t = \dot{\Gamma}_t + m(1 - \beta)H_t, \ t \in [t_a, t_b]
\]

Take left and right limit at \( t_a \) and \( \dot{\Gamma}_{t_a-} = \dot{\Gamma}_{t_a+} \). Let \( \Omega_t \equiv \Gamma_t - \psi/\lambda \), and \( \eta \equiv (r + \phi + m) \frac{\psi}{\lambda} \), so

\[
(r + \phi + m + \lambda) \Omega_t = \ddot{\Omega}_t + m(1 - \beta)H_t - \eta, \ t \in (0, t_a)
\]

\[
(r + \phi + m) \Omega_t = \ddot{\Omega}_t + m(1 - \beta)H_t - \eta, \ t \in (t_a, t_b)
\]

Differentiating \( \Omega_t \), we get

\[
(r + \phi + m + \lambda) \dot{\Omega}_t = \dddot{\Omega}_t + m(1 - \beta)\dot{H}_t, \ t \in (0, t_a)
\]

\[
(r + \phi + m) \dot{\Omega}_t = \dddot{\Omega}_t + m(1 - \beta)\dot{H}_t, \ t \in (t_a, t_b)
\]

If we can prove \( \dot{H}_t \leq 0 \) on \( (0, t_b) \), then once \( \dot{\Omega}_t = 0 \), it immediately implies \( \dddot{\Omega}_t \geq 0 \). Hence, \( \dddot{\Omega}_t \) single crosses 0 from negative to positive. This implies that if \( \dot{H}_t \leq 0 \), then \( \Omega_t \) is quasi-convex on \( (0, t_b) \), which means that if \( \Omega_{t_a} = 0 \) and \( \Omega_{t_b} < 0 \) (which necessarily holds since \( \Gamma_{t_b} = 0 \) it must be the case that \( \Omega_t \geq 0 \) for \( t < t_a \) and \( \Omega_t \leq 0 \) on \( (t_a, t_b) \). Therefore, it only remains to show to show that \( \dot{H}_t \leq 0 \). We have

\[
(r + \phi + \lambda) H_t = \dot{H}_t + (r + \phi)(1 - q_0)L - (1 - q_0)(c + \phi\theta R) + \psi - \lambda(1 - q_0)(V^b_t - L), \ t \in [0, t_a]
\]

\[
(r + \phi) H_t = \dot{H}_t + (r + \phi)(1 - q_0)L - (1 - q_0)(c + \phi\theta R), \ t \in [t_a, t_b],
\]

where \( H_{t_b} = (1 - q_0)V^b_{t_b} + q_0 V^q_{t_b} - V^u_{t_b} = 0 \). Under Assumption 1, this implies that \( \dot{H}_{t_b} < 0 \).
Differentiating the previous equation we get

\begin{align*}
(r + \phi + \lambda) \dot{H}_t &= \ddot{H}_t - \lambda (1 - q_0) \dot{V}_t^b, \quad t \in (0, t_a) \\
(r + \phi) \dot{H}_t &= \dot{H}_t, \quad t \in (t_a, t_b).
\end{align*}

Since \( \dot{V}_t^b \geq 0 \), we immediately get the result that \( \ddot{H}_t \geq 0 \) if \( \dot{H}_t = 0 \). Hence, \( \dot{H}_t \) single crosses 0 from negative to positive, so \( \dot{H}_{t_b} < 0 \Rightarrow \dot{H}_t < 0, \forall t \in (0, t_b) \).

Finally, we provide conditions to characterize the equilibrium and therefore the parametric assumptions needed to validate it. Note that in the equilibrium characterized by \( \{t_a, t_b, t_g\} \), beliefs evolve on \( t \in (t_a, t_b) \) according to

\begin{align*}
\dot{\pi}_t^u &= m \pi_t^u \pi_t^b, \\
\dot{\pi}_t^g &= m \pi_t^g \pi_t^b, \\
\dot{\pi}_t^b &= -m \pi_t^b (1 - \pi_t^b).
\end{align*}

In particular,

\( \dot{q}_t = mq_t \pi_t^b \),

so

\( q_t = q_{t_a} e^{m \int_{t_a}^t \pi_s^b ds} \).

Solving for \( \pi_t^b \) we get

\( \pi_t^b = \frac{\pi_{t_a}^b}{\pi_{t_a}^b + (1 - \pi_{t_a}^b)e^{m(t-t_a)}}. \)

We have that

\begin{align*}
m \int_{t_a}^t \pi_s^b ds &= \int_{t_a}^t \frac{-\pi_s^b}{1 - \pi_s^b} ds \\
&= \log(1 - \pi_s^b) \bigg|_{t_a}^t \\
&= \log \left( \frac{1 - \pi_t^b}{1 - \pi_{t_a}^b} \right)
\end{align*}

so

\( e^{m \int_{t_a}^t \pi_s^b ds} = \frac{1 - \pi_t^b}{1 - \pi_{t_a}^b} e^{m(t-t_a)} = \frac{\pi_{t_a}^b}{\pi_{t_a}^b + (1 - \pi_{t_a}^b)e^{m(t-t_a)}} \)

so

\( q_t = \frac{1}{1 - \pi_{t_a}^b + \pi_{t_a}^b e^{-m(t-t_a)} q_{t_a}}. \)
To find $\pi_{t_a}^b$, we use equations (32), (34), and (35) in region $[0, t_a]$:

$$\pi_{t_a}^b = 1 - \frac{q_{t_a}}{q_0} (q_0 + (1 - q_0)e^{-\lambda t_a}).$$

Notice that as $t \to \infty$ we have that

$$\frac{1}{1 - \pi_{t_a}^b + \pi_{t_a}^b e^{-m(t-t_a)}} \to \frac{q_{t_a}}{1 - \pi_{t_a}^b}.$$

This limit is greater than $\bar{q}$ if and only if

$$q_{t_a} \geq (1 - \pi_{t_a}^b)\bar{q} = \frac{q_{t_a}}{q_0} (q_0 + (1 - q_0)e^{-\lambda t_a}) \bar{q} \implies t_a \geq t_a \equiv \frac{1}{\lambda} \log \left( \frac{\bar{q}}{1 - \bar{q}} \frac{1 - q_0}{q_0} \right).$$

Next, we derive a system of equations for $t_a, t_b, t_g$. For $t \in (t_a, t_b)$ we have the following ODE for $B_t^i$

$$(r + \phi + m) B_t^u = \dot{B}_t^u + y_t F + \phi[q_0 + (1 - q_0)\theta]F + m[L + (1 - \beta)(V_t^u - L)]$$
$$(r + \phi + m) B_t^b = \dot{B}_t^b + y_t F + \phi F + m[L + (1 - \beta)(V_t^g - L)]$$
$$(r + \phi + m) B_t^1 = \dot{B}_t^1 + y_t F + \phi \theta F + mL.$$

Thus, for $\Gamma_t \equiv q_0 B_t^g + (1 - q_0) B_t^b - B_t^u$,

$$(r + \phi + m) \Gamma_t = \dot{\Gamma}_t + m(1 - \beta)((1 - q_0)L + q_0 V_t^g - V_t^u)$$

with boundary $\Gamma_{t_b} = 0$. Thus,

$$\Gamma_t = \int_t^{t_b} e^{-(r+\phi+m)(s-t)} m(1 - \beta) [(1 - q_0)L + q_0 V_s^g - V_s^u] \, ds.$$ 

Next,

$$V_t^u = \frac{c + \phi [q_0 + (1 - q_0)\theta] R}{r + \phi}(1 - e^{-(r+\phi)(t_g - t)}) + e^{-(r+\phi)(t_g - t)} V_{t_g}^u$$

$$q_0 V_t^g = \frac{q_0 c + q_0 \phi R}{r + \phi}(1 - e^{-(r+\phi)(t_g - t)}) + e^{-(r+\phi)(t_g - t)} q_0 V_{t_g}^g$$
so

\[(1 - q_0)L + q_0 V_t^g - V_t^u = (1 - q_0) \left[ L - \frac{c + \phi \theta R}{r + \phi} (1 - e^{-(r + \phi)(t_0 - t)}) \right] + e^{-(r + \phi)(t_0 - t)} (q_0 V_t^g - V_t^u) \]

Thus, we get

\[ \Gamma_t = \frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left( L - \frac{c + \phi \theta R}{r + \phi} \right) (1 - e^{-(r + \phi + m)(t_b - t)}) + \]

\[ (1 - \beta)(1 - q_0) \left( \frac{c + \phi \theta R}{r + \phi} - V_t^b \right) e^{-(r + \phi)(t_0 - t)} (1 - e^{-m(t_b - t)}) \].

Substituting \( V_t^b \) we get the following equation for \( t_a \):

\[ \frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left( L - \frac{c + \phi \theta R}{r + \phi} \right) (1 - e^{-(r + \phi + m)(t_b - t_a)}) + \]

\[ (1 - \beta)(1 - q_0) \left( \frac{c + \phi \theta R}{r + \phi} - \frac{c + \phi \theta R + mV_t^b}{r + \phi + m} \right) e^{-(r + \phi)(t_0 - t_a)} (1 - e^{-m(t_b - t_a)}) = \frac{\psi}{\lambda}. \] (41)

Therefore, we have three equations to characterize the thresholds: \( \{ t_a, t_b, t_g \} \)

\[ \bar{q} = \frac{1}{1 - \pi_{t_a}^b + \pi_{t_a}^b e^{-m(t_b - t_a)} q_a} \]

\[ \psi = \frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left( L - \frac{c + \phi \theta R}{r + \phi} \right) (1 - e^{-(r + \phi + m)(t_b - t_a)}) + \]

\[ (1 - \beta)(1 - q_0) \left( \frac{c + \phi \theta R}{r + \phi} - \frac{c + \phi \theta R + mV_t^b}{r + \phi + m} \right) e^{-(r + \phi)(t_0 - t_a)} (1 - e^{-m(t_b - t_a)}) \].

\( t_g - t_b \) is unchanged from Proposition 2, which can be simplified to

\[ t_g = t_b + \frac{1}{r + \phi} \log \left( \frac{m}{r + \phi + m (r + \phi) L - (c + \phi \theta R)} \right) \].

The final step for the equilibrium to hold is to verify conditions for bank’s learning policy. Let \( \bar{t}_a \) be the threshold the first time \( q_t = \bar{q} \) in the benchmark model in which \( \psi = 0 \), which is the same as if \( t_a = t_b \). On the other hand, recall that \( t_a \equiv \frac{1}{\lambda} \log \left( \frac{\bar{q}}{1 - \bar{q} q_0} \right) \) in which case the we have that \( \inf \{ t > t_a : q_t = \bar{q} \} = \infty \). In particular, we need to verify that \( t_a \in [t_a, \bar{t}_a] \).

We have already shown that if \( t_a = \bar{t}_a \), we have \( \Gamma_{t_a} = 0 < \psi/\lambda \). Hence, we only need to show that if \( t_a = \bar{t}_a \), then \( \Gamma_{t_a} > \psi/\lambda \). In this case both \( t_g \to \infty \) and \( t_b \to \infty \), which means
that

\[
\frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left( L - \frac{c + \phi \theta R}{r + \phi} \right) (1 - e^{-(r+\phi+m)(t_g-t_a)}) \\
+ (1 - \beta)(1 - q_0) \left( \frac{c + \phi \theta R}{r + \phi} - \frac{c + \phi \theta R + m\bar{V}}{r + \phi + m} \right) e^{-(r+\phi)(t_g-t_a)} \left( 1 - e^{-m(t_g-t_a)} \right) \rightarrow \\
\frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left( L - \frac{c + \phi \theta R}{r + \phi} \right)
\]

Hence, a solution exists if and only if

\[
\frac{\psi}{\lambda} < \frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left( L - \frac{c + \phi \theta R}{r + \phi} \right)
\]

Next, we show that if this condition is not satisfied, then there is no learning in equilibrium. Suppose that the firm never learns and never goes to the market. In this case, we have the value of the project being

\[
V^u = NPV^u = \frac{c + \phi (q_0 + (1 - q_0)\theta) R}{r + \phi},
\]

so that the value of bank at loan rate \( y \) is

\[
B^u = \frac{yF + m(L + (1 - \beta)(V^u - L)) + \phi(q_0 + (1 - q_0)\theta)R}{r + \phi + m}
\]

Suppose that the bank is informed (which only occurs off the equilibrium path). In this case, for any loan rate \( y \),

\[
B^b = \frac{y + mL + \phi \theta F}{r + \phi + m}
\]

and

\[
B^g = \frac{y + m(L + (1 - \beta)(V^g - L)) + \phi F}{r + \phi + m}
\]

where

\[
V^g = \frac{c + \phi R}{r + \phi}.
\]

Combining the previous expressions, we get that

\[
q_0B^g + (1 - q_0)B^b - B^u = \frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left( L - \frac{c + \phi \theta R}{r + \phi} \right),
\]
which means that not learning is optimal if

\[
\frac{\psi}{\lambda} \geq \frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left( L - \frac{c + \phi \theta R}{r + \phi} \right)
\]

A.1.6 Proof of Proposition 6

Proof. Next, we consider the case in which the entrepreneur is financially constrained. We consider the case in which the constraint \( y_t F = c \) binds for all types. By similar arguments as the ones in the unconstrained case, the bank never monitors after time \( t_b \). Hence, we can restrict attention to the case in which \( t_a < t_b \). If \( y_t = c \), the value function of the bank satisfies the following HJB equation on \((t_a, t_b)\)

\[
(r + \phi) B^a_t = \dot{B}^a_t + c + \phi [q_0 + (1 - q_0)\theta F]
\]

\[
(r + \phi) B^b_t = \dot{B}^b_t + c + \phi F
\]

\[
(r + \phi + m) B^b_t = \dot{B}^b_t + c + \phi \theta F + mL.
\]

It follows that \( \Gamma_t \) satisfies the following ODE

\[
(r + \phi) \Gamma_t = \dot{\Gamma}_t + m(1 - q_0)(L - B^b_t)
\]

\[
= \dot{\Gamma}_t + m(1 - q_0) \left( L - \frac{c + \phi \theta F + mL}{r + \phi + m} \left( 1 - e^{-(r+\phi+m)(t_b-t)} \right) \right) - e^{-(r+\phi+m)(t_b-t)} L
\]

\[
= \dot{\Gamma}_t - m(1 - q_0) \left( \frac{c + \phi \theta F - (r + \phi)L}{r + \phi + m} \right) (1 - e^{-(r+\phi+m)(t_b-t)})
\]

From here, we get that

\[
\Gamma_t = \int_{t_a}^{t_b} e^{-(r+\phi)(s-t)} m(1 - q_0) \left( \frac{(r + \phi)L - c - \phi \theta F}{r + \phi + m} \right) \left( 1 - e^{-(r+\phi+m)(t_b-s)} \right) ds
\]

\[
= \frac{m(1 - q_0) (r + \phi)L - c - \phi \theta F}{r + \phi + m} \left( 1 - e^{-(r+\phi)(t_b-t)} \right)
\]

\[
- (1 - q_0) \left( \frac{(r + \phi)L - c - \phi \theta F}{r + \phi + m} \right) e^{-(r+\phi)(t_b-t)} \left( 1 - e^{-m(t_b-t)} \right)
\]

It follows that

\[
\frac{\psi}{\lambda} = (1 - q_0) \left( \frac{(r + \phi)L - c - \phi \theta F}{r + \phi + m} \right) \left( \frac{m}{r + \phi} \left( 1 - e^{-(r+\phi)(t_b-t_a)} \right) \right)
\]

\[
- e^{-(r+\phi)(t_b-t_a)} \left( 1 - e^{-m(t_b-t_a)} \right) \right) \quad (42)
\]

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As before, when $t_a$ converges to $\frac{1}{λ} \log \left( \frac{q}{1-q} \frac{1-qa}{q0} \right)$ the threshold $t_b$ converges to infinity. Thus, as in the proof of Proposition 5, a solution exists if

$$\frac{ψ}{λ} < \frac{m(1-q0)}{r+ϕ+μ} \left( L - \frac{c+ϕθF}{r+ϕ} \right)$$

(43)

Finally, we need to verify that $Γ_t$ single crosse $ψ/λ$. Because $B^b_t$ is increasing on $(0,t_b)$, it follows that $Γ_t$ is quasi-convex on $(t_a,t_b)$. On $(0,t_a), Γ_t$ satisfies

$$(r+ϕ+λ)Γ_t = \hat{Γ}_t - m(1-q0)(B^b_t - L) + ψ.$$  

Again, we can verify that $Γ_t$ is quasi-convex on $(0,t_a)$. Moreover, as $Γ_{ta-} = \hat{Γ}_{ta+}$, we can conclude that $Γ_t$ is quasi-convex on $[0,t_a]$. From here, we can conclude that if $Γ_{ta} = \psi/λ > 0$ and $Γ_{t_b} = 0$ , then it must be the case that $Γ_t < \psi/λ$ on $(t_a,t_b)$. On the other hand, if there is $t_a < t_a$ such that $Γ_{t_a} < \psi/λ$, then it must be that case that $Γ_t$ has local maximum on $[0,t_b)$. However, this cannot be the case as $Γ_t$ is quasi-convex. We can conclude that $Γ_t ≥ ψ/λ$ for all $t < t_a$.

The rest of the equilibrium is determined as in the case with exogenous learning. The threshold $t_a$ must be such $q_{t_a} = q$, where $q_t$ is given by the same expression as in the proof of Proposition 5, while $t_g - t_b$ is the same as in Proposition 4.

The final step is to find conditions such that the constraint $yt ≤ c$ is binding. Letting $z^i_t = βL + (1-β)V^i_t - B^i_t$, we get that the constraint is binding if $z^i_t > 0$. Using the equations for $V^i_t$ and $B^i_t$ we can derive an ODE for $z^i_t$. In particular,

\[ z^g_t = (r+ϕ)z^g_t - (r+ϕ)βL + βc - (1-β)R - F \]

\[ z^b_t = (r+ϕ + m1_{t<t_a})z^b_t - (r+ϕ)βL + βc - (1-β)R - F \]

\[ z^u_t = (r+ϕ + λ1_{t<t_a})z^u_t - (r+ϕ)βL + βc - (1-1_{t<t_a})ψ - (1-q0)θ[(1-β)R - F] \]

Solving the previous expressions we get

\[ z^g_t = \frac{1}{r+ϕ} \left( (r+ϕ)βL + (1-β)R - F - βc \right) (1 - e^{-(r+ϕ)(t_g-t)}) + e^{-(r+ϕ)(t_g-t)}z^g_{t_g} \]

\[ z^b_t = \int_t^{t_g} e^{-(r+ϕ)(s-t) - m(s\wedge t_b - t)} (r+ϕ)βL + φθ[(1-β)R - F] - βc ds + e^{-(r+ϕ)(t_g-t) - m(t_b-t)}z^b_{t_g} \]

\[ z^u_t = \int_t^{t_g} e^{-(r+ϕ)(s-t) - λ(s\wedge t_a - t)} (r+ϕ)βL + φ(q_0 + (1-q0)θ)[(1-β)R - F] \]

\[ - β(s - 1_{s<t_a}ψ) + λ1_{s<t_a} [q_0 z^g_t + (1-q0)z^b_t] ds + e^{-(r+ϕ)(t_g-t) - λ(t_a-t)}z^u_{t_g} \]
When $\beta$ goes to zero we get that the previous expressions converge to

\[ z_t^g = \frac{1}{r + \phi} \phi (R - F) \left( 1 - e^{-(r+\phi)(t_g-t)} \right) + e^{-(r+\phi)(t_g-t)} z_{t_g}^g \]

\[ z_t^b = \int_t^{t_g} e^{-(r+\phi)(s-t)} \phi \theta (R - F) ds + e^{-(r+\phi)(t_g-t)} z_{t_g}^b \]

\[ z_t^u = \int_t^{t_g} e^{-(r+\phi)(s-t)} \left( \phi (q_0 + (1 - q_0)\theta) (R - F) + \lambda \mathbb{1}_{s < t_a} [q_0 z_s^g + (1 - q_0) z_s^b] \right) ds + e^{-(r+\phi)(t_g-t) - \lambda (t_a-t)} z_{t_g}^u \]

Noting that $z_{t_g}^u = q_0 z_{t_g}^g + (1 - q_0) z_{t_g}^b$ we find if $z_{t_g}^g$ and $z_{t_g}^b$ are positive then the previous limit is positive for all $t \in [0, t_g]$. As verified in the proof of Proposition 4, $z_{t_g}^b$ is positive if

\[ L - \frac{c + (\phi \theta + m)F}{r + \phi + m} + (1 - \beta) \left[ \frac{c}{r + \phi} + \frac{\phi}{r + \phi} \left( (1 - \theta) \frac{mF}{r + \phi + m} + \theta R \right) - L \right] > 0 \]

On the other hand, $z_{t_g}^g$ is positive if

\[ \beta L + (1 - \beta) \frac{c + \phi R}{r + \phi} > \frac{c + (\phi + m)F}{r + \phi + m} \]

When $\beta$ goes to zero, the previous inequalities converge to

\[ \frac{c + \phi ((1 - \theta) F + \theta R)}{r + \phi} > \frac{c + (\phi + m)F}{r + \phi + m} \]

\[ \frac{c + \phi R}{r + \phi} > \frac{c + (\phi + m)F}{r + \phi + m} \]

Which means that it is enough to verify that

\[ \frac{c + \phi ((1 - \theta) F + \theta R)}{r + \phi} > \frac{c + (\phi + m)F}{r + \phi + m} \]

\[ \square \]

A.1.7 Proof of Proposition 10

Proof. With some abuse of notation, let $V^n_i$ be the continuation value at the $n$-th rollover time (i.e. at time $n/m$. In that case, we have

\[ V_{n_g}^g = \int_0^{1/m} e^{-(r+\phi)s} (c + \phi R) ds + e^{-\frac{r+\phi}{m}} V_{n_g}^g. \]
To simplify notation, let
\[ \nu^g \equiv \int_0^1 e^{-(r + \phi)s} (c + \phi R) \, ds \]
be the flow payoff between two rollover events, which is time independent. We can rewrite
\[ V_{n_g-1}^g = \nu^g + e^{-\frac{\nu^g}{m}} V_{n_g}^g \]
which has to be greater or equal than \( \bar{V} \). On the other hand, the No Deals condition also requires \( V_{n_g-1}^g \geq \bar{V} \). Combining these two conditions we find that
\[ \nu^g + e^{-\frac{\nu^g}{m}} V_{n_g}^g = V_{n_g-1}^g \geq \bar{V} \geq \nu^g + e^{-\frac{\nu^g}{m}} \bar{V}. \]
This means that \( q_{n_g/m} = \qbar \). Because \( q_t \) is constant after the \( n_b \)-th rollover date, it has to be the case that \( q_{n_b/m} = \qbar \). Let
\[ \hat{q}(n) = \frac{q_0}{q_0 + (1-q_0)e^{-\lambda_n/m}} \]
be the beliefs after \( n \) rolled over period given the market conjecture that a bad loan is not rolled over, and let \( \hat{n} \equiv \min \{ n : \hat{q}(n) > \qbar \} \). If \( \hat{n}/m \) is an integer, then \( n_b = \hat{n} \) and \( \alpha_b = 0 \). However, if \( \hat{n}/m \) is not an integer then \( n_b = \hat{n} - 1 \), and fraction \( \alpha_b \) of the bad projects is liquidated so that the belief conditional on a rollover at \( n_b \) is \( q_b = \qbar \). In this case, \( \alpha_b \) satisfies
\[ \qbar = \frac{(1-\alpha_b) \left( 1 - q_{n_b-1} \right)}{q_{n_b-1} + (1-\alpha_b) \left( 1 - q_{n_b-1} \right)} \tag{44} \]
The definition of \( \hat{n} \) together with equation (44) uniquely determine \( n_b \) and \( \alpha_b \). It is only left to determine \( n_g \) and \( \alpha_g \). We do this by turning our attention to the bad type incentive compatibility constraint. The bad type has to be indifferent between liquidating and continue rolling over after \( n_b \). Let \( \hat{V}(n', n_b) \) be the payoff if the bad type rolls over until period \( n' \) and receives a payoff \( \hat{V}^b \), which is given by:
\[ \hat{V}^b(n', n_b) \equiv \int_0^{n'-n_b} e^{-(r + \phi)s} (c + \phi R) \, ds + e^{-(r + \phi)\frac{n'-n_b}{m}} \hat{V}^b \]
For fixed \( n_b \), the function \( \hat{V}(n', n_b) \) is increasing in \( n' \). Let’s define \( \bar{n} \equiv \max \{ n' : \hat{V}^b(n', n_b) \geq L \} \). If \( \hat{V}^b(\bar{n}, n_b) = L \), then we can set \( n_g = \bar{n} \) and \( \alpha_g = 1 \). Otherwise, we have that \( \hat{V}^b(\bar{n}, n_b) > L \) and \( \hat{V}^b(\bar{n} + 1, n_b) < L \) so a mixed strategy is required. In particular, if we set \( n_g = \bar{n} \) and choose \( \alpha_g \) such that
\[ \alpha_g \hat{V}^b(n_g, n_b) + (1-\alpha_g) \hat{V}^b(n_g + 1, n_b) = L, \]
we get that \( V_{n_b}^b = L \) so the low type is indifferent between liquidating and rolling over. Finally, because the market investors make zero profit, they are willing to mix between the two debt prices at the rollover period \( n_g \). Moreover, we have the following corollary.

\[ \square \]

### A.2 Bank and Entrepreneur Value Function

In this subsection, we supplement the details in subsection 3.2.2. Below, we will describe the value function of the entrepreneur and the bank respectively in three different regions.

In the Market Financing region \((t_g, \infty)\), the value of the equity depends on the loan rate determined at the last rollover date before \( t_g \), which we denote by \( y \).

\[
E_i^u = \frac{\phi [q_0 + (1 - q_0) \theta] (R - F) + \lambda [q_0 E_i^q + (1 - q_0) E_i^b]}{r + \phi + \lambda + m} + \frac{(c - y) + m (\bar{D} - F) + m \phi [q_0 + (1 - q_0) \theta] (R - F)}{r + \phi + m} + \frac{\phi [q_0 + (1 - q_0) \theta] (R - F) + \lambda [q_0 E_i^q + (1 - q_0) E_i^b]}{r + \phi + \lambda + m}
\]

\[
E_i^3 = \frac{\phi (R - F) + (c - y) + m (\bar{D} - F) + m \phi (R - F)}{r + \phi + m} (45a)
\]

\[
E_i^b = \frac{\phi \theta (R - F) + (c - y) + m (\bar{D} - F) + m \phi \theta (R - F)}{r + \phi + m} (45b)
\]

where \( \bar{D} = D_{\tau_m} \) in (5) evaluated at \( q_{\tau_m} = \bar{q} \). The value function of bank satisfy

\[
B_i^u = \frac{rF + \phi [q_0 + (1 - q_0) \theta] F + \lambda [q_0 B_i^q + (1 - q_0) B_i^b] + mF}{r + \phi + \lambda + m} (46a)
\]

\[
B_i^q = F (46b)
\]

\[
B_i^b = \frac{rF + \phi \theta F + mF}{r + \phi + m} (46c)
\]

Next, we study the bank’s and the entrepreneur’s value in the other two regions. In general, these values will depend on the loan rate that they have agreed on so that we will use \( E_i^u(y) \) and \( B_i^u(y) \) to denote the values at loan rate is \( y \). Note this interim payment will continued to be made until either the project matures or the loan matures, that is, until \( \tau = \min \{\tau_m, \tau_\phi\} \). Equivalently, we can write the present value of this interim payment as \( T(y) = \frac{y - rF}{r + m + \phi} \) so that \( B_i^u(y) = B_i^u(rF) + T(y) \) and \( E_i^u(y) = E_i^u(rF) - T(y) \).

For type \( u \) and
\( g, \) when \( t \in (0, t_g) \),

\[
(r + \phi + m) B_t^w = \dot{B}_t^w + rF + \phi [g_0 + (1 - q_0) \theta] F + \lambda \left[ g_0 B_t^d + (1 - q_0) B_t^b - B_t^w \right]
\]
\[
+ m \left[ L + (1 - \beta) (V_t^a - L) \right] \quad (47a)
\]

\[
(r + \phi + m) E_t^w = \dot{E}_t^w + (c - rF) + \phi [g_0 + (1 - q_0) \theta] (R - F) + m \beta (V_t^a - L)
\]
\[
+ \lambda \left[ g_0 E_t^d + (1 - q_0) E_t^b - E_t^w \right] \quad (47b)
\]

\[
(r + \phi + m) B_t^g = \dot{B}_t^g + rF + \phi \theta F + mL \quad (48b)
\]

\[
(r + \phi + m) E_t^g = \dot{E}_t^g + (c - rF) + \phi \theta (R - F) + m \beta (V_t^b - L) \quad \forall t \in (t_b, t_g) \quad (48c)
\]

\[
(r + \phi + m) B_t^b = \dot{B}_t^b + rF + \phi \theta F + m \left[ L + (1 - \beta) (V_t^b - L) \right] \quad (48d)
\]

In contrast, the value functions for a bad-type entrepreneur differ across the two regions.

\[
(r + \phi + m) E_t^b = \dot{E}_t^b + (c - rF) + \phi \theta (R - F) \quad \forall t \in (0, t_b) \quad (48a)
\]

\[
(r + \phi + m) E_t^b = \dot{E}_t^b + (c - rF) + \phi \theta (R - F) + m \beta (V_t^b - L) \quad \forall t \in (t_b, t_g) \quad (48c)
\]

Intuitively, in the efficient liquidation region, a bad project gets liquidated when the loan matures, whereas in the zombie lending region, the same loan will get rolled over.

Finally, given that we have shown the value function \( E_t^b \) can be non-monotonic in \( [t_b, t_g) \), Lemma 7 proves that in the region of \( (t_b, t_g) \), \( \dot{E}_t^b \) will change sign at most once. Therefore, the value of \( E_t^b \) is either monotonically increasing, or first increases and then decreases.

**Lemma 7.** \( \dot{E}_t^b > 0 \) for \( t \in (t_b, t_g) \).

**Proof.** Take derivative to both sides of equation (48c), we can get

\[
\ddot{E}_t^b = (r + \phi + m) \dot{E}_t^b - m \beta \dot{V}_t^b.
\]

This implies any local extrema of \( E_t^b \) (which satisfies \( \dot{E}_t^b = 0 \)) is a local maximum. if \( \dot{V}_t^b > 0 \). Therefore, if \( \dot{V}_t^b > 0 \) for any \( t \in (t_b, t_g) \), \( E_t^b \) cannot change sign more than once over \( t \in (t_b, t_g) \).

To show this, let us take derivative to both sides of equation (17c) in region \( t \in [t_b, t_g) \)

\[
\dot{V}_t^b = (r + \phi) \dot{V}_t^b.
\]

At \( t = t_b \), \( \dot{V}_t^b = (r + \phi) L - c - \phi \theta R > 0 \) following Assumption 1. Therefore, since \( \text{sign} \left( \dot{V}_t^b \right) = \text{sign} \left( \dot{V}_t^b \right) \) for any \( t \in (t_b, t_g) \), that implies \( \dot{V}_t^b > 0 \) in this region as well.

\( \square \)