Knight Meets Sharpe: Capital Asset Pricing under Ambiguity

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March 26, 2020

Abstract

This paper extends the mean-variance preferences to mean-variance-ambiguity preferences by relaxing the assumption that probabilities are known, and instead assuming that probabilities are uncertain. The optimal portfolio is identified in general equilibrium, demonstrating that the two-fund separation theorem is preserved. Thereby, introducing ambiguity into the capital asset pricing model demonstrates that the ambiguity premium corresponds to systematic ambiguity, which is distinguished from the systematic risk. Using the closed-form measurable beta ambiguity, performance measures are generalized to account for ambiguity alongside risk. Use of this model can be extended to other applications including investment decisions and valuations. The introduced asset pricing model is empirically applicable and provides insight into empirical asset pricing anomalies.

Keywords and Phrases: Ambiguity Index, Ambiguity measurement, Knightian uncertainty, Perceived probabilities, Mean-variance, Optimal portfolio, Investment decisions, General equilibrium.

JEL Classification: D81, G4, G11, G12.
1 Introduction

The capital asset pricing model (CAPM), delivered by modern portfolio theory (MPT), has become the theoretical pillar of modern finance and is widely used in investment decisions (Barber et al., 2016). However, evidence has shown that the theoretical predictions of expected returns delivered by the CAPM are inconsistent with empirical findings (Fama and French, 2004). In particular, the intercept of the empirical security market line (SML) is found to be too high and the slope too flat to be justified by the theoretical CAPM.¹ A possible reason for this discrepancy is that these classical theories focus on risk, assuming away other dimensions of uncertainty in the economy.

MPT (Markowitz, 1952; Tobin, 1958; Treynor, 1961) and the CAPM (Sharpe, 1964; Lintner, 1965; Mossin, 1966) are built upon the concept of mean-variance preferences. The assumption underlying these theories is that the probabilities of future returns are known, establishing a unique mean-variance space upon which such preferences apply. In reality, however, probabilities of future returns are usually not precisely known, and financial decision makers (investors) face uncertainty about these probabilities, referred to as ambiguity or Knightian uncertainty (Knight, 1921; Ellsberg, 1961).² The presence of ambiguity implies that the standard mean-variance preferences cannot portray a realistic picture of pricing decisions, since these standard preferences ignore information regarding the probabilities of future returns, which is invaluable in portfolio and pricing decisions.

This paper extends the standard mean-variance space to a mean-variance-ambiguity space, equipped with the appropriate preferences. In this space, it solves for the general equilibrium with heterogeneous investors in order to identify the set of optimal portfolios. Thereby, it reestablishes the capital market line (CML), and proves that the two-fund separation theorem is maintained also in the presence of ambiguity. This framework is then utilized to introduce ambiguity into the classical CAPM, distinguishing ambiguity from risk, and systematic ambiguity, dominated by economy-wide shocks, from idiosyncratic ambiguity, dominated by firm-specific shocks. This extended model provides a closed-form solution for beta ambiguity, which corresponds to the systematic ambiguity associated with an asset. Analogous to the risk premium, the ambiguity premium is derived from the commonality of asset ambiguity and market ambiguity, rather than from the asset’s own ambiguity. In other words, investors are rewarded for systematic ambiguity and systematic risk, but not for idiosyncratic ambiguity or idiosyncratic risk. The notion of (systematic) ambiguity allows for the introduction of ambiguity into the Treynor and Sharpe ratios, as well as into Jensen’s alpha, thereby delivering ex-

¹See, for example, Black et al. (1972), Merton (1973), and Frazzini and Pedersen (2014).
²Risk refers to conditions in which the event to be realized is a priori unknown, but the odds of all possible events are perfectly known. Ambiguity refers to conditions in which not only the event to be realized is a priori unknown, but the odds of events are also not uniquely assigned.
tended performance measures. The introduced asset pricing model and the performance measures can be estimated from the data and be employed in empirical studies of the cross-sectional implications of ambiguity.

In reality, investors face two tiers of uncertainty: one with respect to future returns and the other with respect to the probabilities associated with these returns. Since investors are assumed to be ambiguity averse, having a prior over probability distributions (priors), they do not compound the probability distributions of returns (beliefs) linearly with this prior when assessing expected utility. Instead, they overweight the probabilities of unfavorable returns and underweight the probabilities of favorable returns (Tversky and Kahneman, 1992; Wakker, 2010). Expected utility is, therefore, negatively affected by both risk and ambiguity, for a given expected rate of return.

To represent preferences for ambiguity, the standard mean-variance space is extended by adding ambiguity—the uncertainty of probabilities—as a third dimension. Suppose that rates of return are normally distributed with a density function $\phi(r \mid \mu, \sigma)$, where the mean, $\mu$, and the standard deviation, $\sigma$, are uncertain. Then, in this three-dimensional mean-variance-ambiguity space, ambiguity is measured by the expected volatility of probabilities (Izhakian, 2020). That is,

$$\bar{\Omega}^2[r] = \int E[\phi(r \mid \mu, \sigma)] \text{Var}[\phi(r \mid \mu, \sigma)] \, dr,$$

where the expectation, variance and covariance of probabilities are taken using a second-order probability distribution (a distribution over a set of possible distributions). Risk, $\text{Var}[r]$, in this space, is measured by the volatility of returns, taken using expected probabilities.

Preferences for risk and ambiguity are naturally reflected in the mean-variance-ambiguity space, as investors maximize their expected return, for a given level of risk and ambiguity. In general equilibrium, these preferences imply that in any optimal portfolio the relative proportion of any two risky and ambiguous assets is the same for all (heterogeneous) investors, independent of their aversion to ambiguity or to risk. This means that, in general equilibrium, every investor possesses a portfolio consisting of only two funds: the market portfolio and the risk-free asset (which is also ambiguity free). That is to say, the two-fund separation theorem (Tobin, 1958) holds true also in the presence of ambiguity. The proportions of these two funds, as well as the proportions and values (prices) of the assets composing the market portfolio, are determined in general equilibrium by investors’ preferences for risk and ambiguity. Since two-fund separation is maintained, along with the appropriate risk and ambiguity preferences, a construct analogous to the CML can be derived.

In the presence of ambiguity, the proportions of the assets composing the market portfolio might be different from those in an economy with no ambiguity or with ambiguity neutral preferences. The
reason being that, in the presence of ambiguity, these proportions reflect market values which also price ambiguity. In reality, the market values (proportions) of the assets in the market portfolio are unique and observable. However, the classical CAPM faces difficulty in explaining these values. The proposed model aims to improve the theoretical explanation of these observable market values. The improvement of the predictions of this model is a question for future empirical research.

The mean-variance-ambiguity framework allows for the introduction of ambiguity into the classical CAPM. In this extended model, referred to as Capital Asset Pricing Model under Ambiguity (ACAPM), the expected return of asset \( j \) corresponds not only to the covariation of its return, \( r_j \), with the market-portfolio return, \( r_m \), but also to the covariation of the possible probability distributions of \( r_j \) with the possible probability distributions of \( r_m \). Formally, the expected return of asset \( j \) satisfies

\[
\mathbb{E}[r_j] = r_f + \zeta^p_j (\mathbb{E}[r_m] - r_f) + \beta^R_j (1 - \zeta^p_j) (\mathbb{E}[r_m] - r_f) + \beta^A_j (1 - \zeta^p_j) (\mathbb{E}[r_m] - r_f),
\]

where

\[
\zeta^p_j = \sqrt{\frac{\Omega^2 [r_m]}{1 + \Omega^2 [r_m]}} I\{j \neq f\}
\]

is the zeta participation; \( I\{j \neq f\} \) is an indicator function that takes the value one for non risk-free assets and zero otherwise;

\[
\beta^R_j = \frac{\text{Cov}[r_m, r_j]}{\text{Var}[r_m]} \frac{1 + \Omega^2 [r_m]}{1 + \Omega^2 [r_m] + \Lambda [r_m, r_m]}
\]

is the beta risk;

\[
\beta^A_j = \frac{\Lambda [r_m, r_j]}{1 + \Omega^2 [r_m] + \Lambda [r_m, r_m]}
\]

is the beta ambiguity;

\[
\Lambda [r_m, r_j] = \int \mathbb{E}[\phi(r | \mu_m, \sigma_m)] \text{Cov}[\phi(r | \mu_m, \sigma_m), \phi(r | \mu_m, \sigma_m)] \lambda(r | \mu_m, \mu_j, \sigma^2_m, \sigma_{m,j}) dr;
\]

and

\[
\lambda(r | \mu_m, \mu_j, \sigma^2_m, \sigma_{m,j}) = \frac{r - \mu_m}{\sigma^2_m} \left( \frac{\sigma_{m,j}}{\sigma^2_m} (r - \mu_m) + \mu_j \right) - \frac{\sigma_{m,j}}{\sigma^2_m}.
\]

The risk-free rate of return is denoted \( r_f \), and the expectation, variance and covariance of returns are taken using the expected probabilities.

In the ACAPM, beta risk and beta ambiguity are independent of individual attitudes toward risk and ambiguity, reflecting only beliefs (information). Beta risk captures the effect of systematic

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3Several studies derived a consumption-based CAPM (Breeden, 1979; Duffie and Zame, 1989). Other studies extend the CAPM by introducing various risk factors including skewness (Kraus and Litzenberger, 1976; Harvey and Siddique, 2000), stochastic volatility in intratemporal settings (Campbell et al., 2018), probability weights (Barberis and Huang, 2008), liquidity risk (Acharya and Pedersen, 2005; Liu, 2006), long-run risk in aggregate consumption (Ai and Kiku, 2013), extrapolated past prices (Barberis et al., 2015), and index investments (Baruch and Zhang, 2017).
risk, measured by the covariation of asset return with market return; it departs from the standard theory due to the uncertainty about the probabilities used to assess risk. Beta ambiguity captures the effect of systematic ambiguity, measured by the covariation of asset return probabilities and market return probabilities. In contrast to other models, in which the ambiguity premium is attributed to the asset’s own ambiguity and does not consider the relation between asset ambiguity and market ambiguity, in the current model the entire ambiguity of an asset is not the relevant determinant of its expected return, but only its systematic component. In addition, the model introduces a fixed \textit{participation} premium that rewards for bearing the fundamental ambiguity in the economy. Different than risk, where a marginal exposure to the market portfolio implies a marginal exposure to risk, in an ambiguous economy a marginal exposure to the market portfolio exposes the investor to a discrete level of ambiguity, for which the participation premium is rewarding. A special case occurs when probabilities are known (or investors are ambiguity neutral), in which the ACAPM reduces to the classical CAPM.

Existing empirical findings are inconsistent with the predictions of the classical CAPM. Specifically, the slope of the SML is found to be flatter and its intercept higher than predicted by the traditional theory. As Figure 1 depicts, the ACAPM delivers a new structure of a SML that, in addition to risk, accounts for ambiguity. This new structure may help explain some of the empirical inconsistencies and anomalies related to the traditional CAPM, including the fact that expected returns may differ from the risk-free rate even for assets having no systematic risk (the zero-beta anomaly, Black et al., 1972; Merton, 1973); the empirical SML being too flat to be explained by the theoretical prediction of the CAPM (the beta anomaly, Black et al., 1972; Frazzini and Pedersen, 2014); the idiosyncratic volatility being priced in a sharp contrast to the prediction of the CAPM (the idiosyncratic volatility anomaly, Ang et al., 2006; Liu et al., 2018); and the additional positive premia associated with firms with a small market capitalization and a high book-to-market equity ratio (the size and value anomalies, Fama and French, 1992). Since the ACAPM can be estimated using trading data, it can be used in cross-sectional empirical tests of the rates of return, which may address the aforementioned anomalies.

The new formulations of the CML and the SML in the mean-variance-ambiguity space allows for the extension of the classical performance measures. The \textit{Sharpe} (1966) ratio, which measures the reward in terms of excess return per unit of the total (systematic and idiosyncratic) risk borne, can be extended to measure, for any risky and ambiguous asset \( j \), the reward per unit of risk and ambiguity borne:

\[
\frac{\mathbb{E}[r_j] - r_f}{\text{Std}[r_j] \sqrt{1 + \Omega^2[r_j]}}.
\]
The Treynor (1965) ratio, which measures the reward per unit of systematic risk borne, can be extended to measure, for any risky and ambiguous asset $j$, the reward per unit of systematic risk and ambiguity borne:

$$\frac{\mathbb{E}[r_j] - r_f}{\zeta_j^p + \left(1 - \zeta_j^p\right) \left(\beta_j^R + \beta_j^A\right)}.$$ 

In this framework, Jensen’s (1968) alpha, which measures the abnormal return over the theoretical expected return, is written:

$$r_j - r_f - \zeta_j^p \left(\mathbb{E}[r_m] - r_f\right) - \left(\beta_j^R + \beta_j^A\right) \left(1 - \zeta_j^p\right) \left(\mathbb{E}[r_m] - r_f\right).$$

These new performance measures preserve the same key analytical properties of the Sharpe and Treynor ratios and Jensen’s alpha. They are empirically applicable to capital budgeting estimations and to the evaluation of professionally managed portfolios.

The implications of ambiguity have been studied in different aspects of asset pricing and portfolio selection, including the equity premium (Izhakian and Benninga, 2011; Ui, 2011), market participation (Cao et al., 2005; Easley and O’Hara, 2009), zero trade (Dow and Werlang, 1992), portfolio inertia (Simonsen and Werlang, 1991; Illeditsch, 2011), portfolio choice (Pflug and Wozabal, 2007; Garlappi et al., 2007; Gollier, 2011; Boyle et al., 2012), learning (Leippold et al., 2008; Ju and Miao, 2012), the term structure of interest rates (Gagliardini et al., 2009), and credit default swaps spreads (Augustin and Izhakian, 2020). Unlike these studies, which consider the ambiguity of an asset independently of the surrounding ambiguity in the market, the current paper studies the pricing of asset ambiguity.

4The implications of ambiguity for asset pricing is surveyed by Guidolin and Rinaldi (2010).
relative to the surrounding market ambiguity.

In related studies, Chen and Epstein (2002), and Epstein and Ji (2013), introduce ambiguity into the consumption CAPM using dynamic recursive max-min preferences (Gilboa and Schmeidler, 1989), assuming a representative investor; the latter also assumes ambiguity only with respect to the covariance matrix.\(^5\) In these models, an asset’s beta is subject to the investor preferences for ambiguity and risk. Adding to this literature, the current paper considers heterogeneous investors, and accounts for both ambiguous means and ambiguous covariance matrices, through ambiguous probabilities. Contrary to other studies, in the current paper, beta ambiguity is independent of investors’ (heterogenous) preferences for ambiguity and risk. Therefore, the ACAPM can be tested empirically using the methodology of estimating ambiguity from the data, as suggested in recent literature (Izhakian and Yermack, 2017; Brenner and Izhakian, 2018).

The ACAPM provides a theoretical foundation for cross-sectional empirical tests of the rates of return. Prior studies have focused mainly on the implications of ambiguity for the time-series of asset prices (e.g., Izhakian and Benninga, 2011; Ui, 2011). Unlike these studies, the current paper focuses on the implication of cross-sectional asset ambiguity (relative to the surrounding market ambiguity) for assets return. The extended performance measures that the current paper introduces provide a theoretical foundation for better assessment of portfolios performance.

2 The equilibrium model

2.1 Ambiguity

Ambiguity, or Knightian uncertainty, provides the basis for a rich literature in decision theory. This literature takes a variety of approaches for modeling decision making under ambiguity (Gilboa and Schmeidler, 1989; Schmeidler, 1989; Klibanoff et al., 2005; Nau, 2006; Chew and Sagi, 2008). One important concept of these models is that, in the presence of ambiguity, ambiguity-averse decision makers act as if they overweight the probabilities of unfavorable outcomes and underweight the probabilities of favorable outcomes, thereby lowering the perceived expected utility. In particular, the higher the degree of ambiguity or the aversion to ambiguity, the lower the perceived expected utility.

Ambiguity may affect asset pricing due to the role it plays in investment decisions. Investment decisions are made based upon perceived expected utility, determined by the perceived expected payoff estimated using subjective perceived probabilities. When an asset’s perceived expected payoff

\(^5\)In an unpublished work, Kogan and Wang (2003) consider a representative investor with max-min preferences, who constructs her subjective set of priors around a reference prior based upon her aversion to ambiguity. Therefore, in their model, the asset’s beta is subject to the investor’s preferences for ambiguity. In addition, their model assumes a known covariance matrix and a known reference prior. Nevertheless, Epstein and Ji (2013) highlight the importance of ambiguous covariance matrix.
is relatively low, investors are reluctant to hold it, reducing its equilibrium price. To illustrate, suppose an asset whose payoff is determined by a flip of an unbalanced coin, for which the investors do not know the odds of heads or tails. The payoff of the asset is $100 for heads and $0 for tails. Suppose now that new information increases the assessed degree of ambiguity about the coin. As investors are ambiguity averse, they lower their perceived probabilities of good payoffs and raise their perceived probabilities of bad payoffs. As a result, the expected payoff falls, so that investors find this asset less attractive, and may prefer to reduce their holding in the asset, decreasing its equilibrium price. Instead, suppose that the good payoff increases to $200. In this case, both risk and expected payoff increase, such that the investors may find this asset more attractive, which may increase its price. However, ambiguity has not changed, as investors have no reason to change either the assessed probabilities or the assessed degree of ambiguity, since there is no new information about likelihoods.

This example illustrates that ambiguity is outcome independent up to a state space partition. That is, if the outcomes associated with events change, while the induced partition of the state space into events (set of events) remains unchanged, then the degree of ambiguity remains unchanged, since all probabilities remain unchanged. This is a critical insight, since outcome dependence enforces risk dependence. Furthermore, since ambiguity is outcome independent the related preferences must also be outcome independent, and apply exclusively to probabilities; otherwise, when outcomes are changed, investors would change their perceived probabilities (beliefs) of events even though no new information about the probabilities of events has been obtained.

2.2 The economy

Following Mossin (1966), consider a single-good frictionless exchange economy with one risk-free asset, indexed $j = 0$, and $n$ risky and ambiguous real assets (firms), indexed $j = 1, \ldots, n$. Thereby, the risk-free rate of return is also endogenously determined.\(^6\) Each investor brings to the market her present holdings of assets, and an exchange takes place. Then, assets’ payoffs are realized and consumption takes place. The consumption good is perishable, and the only way to transfer consumption between individuals is through the capital market. Prices and payoffs of assets are denominated in units of the single consumption good.

The payoff of the risk-free asset, which is also ambiguity free, is one, $y_0 = 1$.\(^7\) The payoff vector of the risk-free, and the risky and ambiguous assets is $y = [y_0, y_1, \ldots, y_n]'$. These payoffs are characterized

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\(^6\)Note that in Sharpe (1964) the risk-free rate of return is exogenously given.

\(^7\)The following notational convention is used. All vectors are column vectors. The transpose operation is denoted by a single quotation mark. Bold lowercase (Greek or upright Roman) letters denote vectors. Bold uppercase (Greek or upright Roman) letters denote matrices. No special notation is used to distinguish random variables from their realizations. The context should make the intention clear. Constants and variables are italicized, operators are in regular font (followed by square parentheses), and sets are in capital calligraphic font.
homogeneous expectations, variances and covariances (Sharpe, 1964; Lintner, 1965; Mossin, 1966). Investors have the same information (symmetric information), they have homogeneous beliefs and thus by Lemma 1 in Appendix A.1, the variance of the portfolio payoff is variance of probabilities taken using the second-order probabilities, \( \xi \). The implication of \( \Sigma_{yy} \) being full ranked (nonsingular) is that there are no redundant assets.\(^8\) Since ambiguity is present, the distribution of \( y \) is not unique. Instead, there is a set \( \mathcal{P} \) of (joint) probability measures (priors), where each \( P \in \mathcal{P} \) is associated with a (joint) probability density function \( \varphi (\cdot) \), according to which \( y \) may be distributed. Since \( y_0 = 1 \) is constant (the risk-free asset), all \( P \in \mathcal{P} \) agree on its probability. A second-order probability distribution, \( \xi \), determines which \( P \in \mathcal{P} \) is realized.\(^9\) As a consequence, \( \mu_y \) and \( \Sigma_{yy} \) are not unique, but jointly distributed according to \( \xi \). All individuals are assumed to have an identical perception of \( \mathcal{P} \) and \( \xi \) (homogenous beliefs, symmetric information). The price vector of the assets is \( p = [p_0, p_1, \ldots, p_n]' \). A portfolio of assets is a vector \( x = [x_0, x_1, \ldots, x_n]' \) of number of shares. The number of shares outstanding of each asset is perfectly divisible.

Let double-struck capital font (\( \mathbb{E} [\cdot] \) and \( \mathbb{V} \text{ar} [\cdot] \)) denote expectation or variance of payoffs taken using the expected probabilities, and regular straight font (\( E [\cdot] \) and \( \text{Var} [\cdot] \)) denote expectation or variance of probabilities taken using the second-order probabilities, \( \xi \).\(^10\) With these notations in place, the expected portfolio payoff, taken using the expected probabilities, is \( x'E [\mu_y] = x'E [y] \) and, by Lemma 1 in Appendix A.1, the variance of the portfolio payoff is \( x'E [\Sigma_{yy}] x + x' \Sigma_{\mu \mu} x \). Since all investors have the same information (symmetric information), they have homogeneous beliefs and thus homogeneous expectations, variances and covariances (Sharpe, 1964; Lintner, 1965; Mossin, 1966).

2.3 The decision theory framework

To develop a general equilibrium model of the relation between ambiguity, risk, and expected return, the proposed model rests on a key requirement that is necessary to differentiate ambiguity from risk: preferences for ambiguity that are outcome independent. To this end, this paper utilizes the theoretical framework of Expected Utility with Uncertain Probabilities (EUUP, Izhakian, 2017), as preferences for ambiguity in this framework are outcome independent. A by-product of the EUUP model is a

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\(^8\) The case of two perfectly correlated payoffs with different means implies that one could short one asset, long the other asset, and create an infinite expected payoff with no risk and no ambiguity. However, such a case is a violation of the law of one price, which is ruled out in equilibrium, as proved later.

\(^9\) Formally, there is a probability space \((\mathcal{S}, \mathcal{E}, P)\), where \( \mathcal{S} \) is an infinite state space; \( \mathcal{E} \) is a \( \sigma \)-algebra of subsets of the state space (a set of events); a \( \lambda \)-system \( \mathcal{H} \subset \mathcal{E} \) contains the events with an unambiguous probability (i.e., events with a known, objective probability); and \( P \in \mathcal{P} \) is an additive probability measure. The set of all probability measures \( \mathcal{P} \) is assumed to be endowed with an algebra \( \Pi \subset 2^\mathcal{P} \) of subsets of \( \mathcal{P} \) that satisfies the structure required by Kopylov (2010). \( \Pi \) is equipped with a unique countably-additive probability measure \( \xi \) that assigns each subset \( A \in \Pi \) with a probability \( \xi (A) \).

\(^10\) Formally, the expectation and the variance of the probability of \( y \) occurring are defined, respectively, by \( E [\varphi (y)] = \int_\mathcal{P} \varphi (y) d\xi \) and \( \text{Var} [\varphi (y)] = \int_\mathcal{P} (\varphi (y) - E [\varphi (y)])^2 d\xi \). The expected payoff is defined by the double expectation (with respect to probabilities and payoffs) \( E [y] = \int_\mathcal{P} \left( \int_\mathcal{S} y d\varphi \right) d\xi = \int \int \mathbb{E} [\varphi (y)] y dy \). The covariance of payoffs \( y \) and \( z \) is defined by \( \text{Cov} [y, z] \equiv \int_\mathcal{P} \left( \int_\mathcal{S} (y - E [y]) (z - E [z]) d\varphi \right) d\xi = \int \int \mathbb{E} [\varphi (y, z)] (y - E [y]) (z - E [z]) dy dz \).
model-derived risk-independent measure of ambiguity that is rooted in an axiomatic decision theory (Izhakian, 2020).\(^{11}\)

The EUUP model assumes two tiers of uncertainty: one with respect to outcomes, and the other with respect to the probabilities of these outcomes. An individual in this framework applies two differentiated phases of the decision process, each reflecting one of these tiers. In the first phase, she forms a representation of her perceived probabilities for all the events relevant to her decision, as the certainty equivalent probabilities of the uncertain probabilities. In the second phase, she assesses the expected utility of each alternative using these perceived probabilities. Since preferences for ambiguity apply exclusively to the probabilities of events (independently of their associated outcomes), aversion to ambiguity takes the form of aversion to mean-preserving spreads in \textit{probabilities}.\(^{12}\)

Investors have heterogeneous distinct preferences for ambiguity and for risk. As is usual, investors are assumed to be risk averse with constant absolute risk aversion (Kraus and Litzenberger, 1973; Brennan, 1979; Acharya and Pedersen, 2005): 

\[
U(c) = \frac{e^{-\gamma k} - e^{-\gamma c}}{\gamma},
\]

where \(c\) is consumption; \(\gamma > 0\) is the coefficient of (absolute) risk aversion; and \(k\) is a \textit{reference point} that satisfies \(U(k) = 0\). Consumption (outcome) lower than \(k\) is unfavorable, consumption higher than \(k\) is favorable, and \(k\) is relatively close to the expected consumption, \(E[c]\).\(^{13}\)

Similarly, investors are assumed to be ambiguity averse with constant absolute ambiguity aversion:  

\[
\Upsilon(P(c)) = -\frac{e^{-\eta P(c)}}{\eta},
\]

where \(0 < \eta \leq \frac{1}{\text{var}[\phi(c)]}\) is the coefficient of (absolute) ambiguity aversion.\(^{14}\) As investors are averse to ambiguity, they do not compound the set of priors \(P\) and the prior \(\xi\) over \(P\) in a linear way (compounded lotteries), but instead they use \(\Upsilon\) to aggregate these probabilities in a non-linear way to form their perceived probabilities. Intuitively, in the EUUP model, ambiguity aversion is exhibited

\(^{11}\)Outcome-independent preferences for ambiguity are necessary for the separation of ambiguity from risk, as well as attitudes from beliefs (Izhakian, 2020). The risk-independent measurement of ambiguity poses a challenge for other frameworks since they do not separate ambiguity from attitudes toward ambiguity (Gilboa and Schmeidler, 1989; Schmeidler, 1989) or preferences for ambiguity are outcome dependent and therefore risk dependent (Klibanoff et al., 2005; Nau, 2006; Chew and Sagi, 2008).

\(^{12}\)In Rothschild and Stiglitz (1970), aversion to risk takes the form of aversion to mean-preserving spreads in outcomes.

\(^{13}\)Close in the sense that the third and higher absolute moments of \(c\) around \(k\) are of a smaller order than the second absolute central moment, and are therefore negligible.

\(^{14}\)This assumption is supported by experimental evidence (Baillon and Placido, 2019).

\(^{15}\)Ambiguity aversion takes the form of a concave \(\Upsilon\), ambiguity loving takes the form of a convex \(\Upsilon\), and ambiguity neutral takes the form of a linear \(\Upsilon\). The condition on \(\Upsilon\) bounds the level of ambiguity aversion (the concavity of \(\Upsilon\)) to ensure that the approximated perceived probabilities are nonnegative and satisfy set monotonicity with respect to set-inclusion.
when an investor prefers an outcome with the expectation of its uncertain probability rather than with the uncertain probability itself.\textsuperscript{16}

The assumptions regarding the investors’ preferences representation are made, without loss of generality, for tractability only, and as shown below, they imply a natural closed-form solution for asset allocations, prices and expected returns. Moreover, as shown below, since the two-fund separation theorem holds true also in the presence of ambiguity, other preference representations are supported.

Within the EUUP model the perceived probabilities are formed by the certainty equivalent probabilities of uncertain probabilities. That is, the perceived probability is the minimum (maximum) unique certain probability value that an individual is willing to accept in exchange for the uncertain probability of a given favorable (unfavorable) event. Therefore, following Izhakian (2020, Theorem 2), the expected utility can be formed by

\[
V(c) = \int_{c \leq k} U(c) \dfrac{E[\varphi(c)](1 + \eta \text{Var}[\varphi(c)])}{\text{Perceived Probability of Unfavorable Outcome}} dc + \int_{c \geq k} U(c) \dfrac{E[\varphi(c)](1 - \eta \text{Var}[\varphi(c)])}{\text{Perceived Probability of Favorable Outcome}} dc + R_2(c),
\]

where the remainder \(R_2(c) = o\left(\int E[|\varphi(c) - E[\varphi(c)]|^3] \, dc\right)\) as \(|\varphi(c) - E[\varphi(c)]| \to 0.\textsuperscript{17}

When investors are ambiguity neutral (\(\Upsilon\) is linear), they compound probabilities linearly and Equation (3) reduces to the conventional expected utility. The same holds true when ambiguity is not present. In contrast, when investors are ambiguity averse (\(\Upsilon\) is concave), they do not aggregate probabilities linearly such that they overweight the probabilities of the unfavorable outcomes and underweight the probabilities of favorable outcomes. In particular, the higher the ambiguity or the aversion to ambiguity, the higher the perceived probabilities of unfavorable outcomes and the lower the perceived probabilities of favorable outcomes. As a result, when ambiguity increases, the expected utility assessed using the perceived probabilities decreases.

The notion of the variance of probabilities in Equation (3) allows for the degree of ambiguity to be measured by the expected volatility of probabilities (Izhakian, 2020):

\[
\bar{\sigma}^2[c] = \int E[\varphi(c)] \text{Var}[\varphi(c)] dc.
\]

\textsuperscript{16}Note that risk aversion is exhibited when an investor prefers the expectation of the uncertain outcome over the uncertain outcome itself.

\textsuperscript{17}The remainder of order \(o\left(\int E[|\varphi(c) - E[\varphi(c)]|^3] \, dc\right)\) means that the fourth and higher absolute central moments of the uncertain consumption, \(c\), are of a strictly smaller order than the third absolute central moment of \(c\), and are therefore negligible. This is equivalent to a cubic expansion, i.e., a remainder of order \(o\left(E[|c - E[c]|^3]\right)\), in which the fourth and higher absolute central moments of consumption are of a strictly smaller order than the third absolute central moment as \(|c - E[c]| \to 0\), and are therefore negligible.
The measure \( \mathcal{U}^2 \) (mho\(^2\)) is outcome independent and risk independent, always positive, and attains its minimum value, 0, only when all probabilities are perfectly known.\(^{18}\)

### 2.4 Asset allocation decision

Suppose a large number of investors, labeled \( i = 1, 2, \ldots, m \). Each investor brings to the market her present holdings of assets, \( \bar{x}^i = [\bar{x}_0^i, \bar{x}_1^i, \ldots, \bar{x}_n^i]' \). Thus, the budget set of investor \( i \) is

\[ B^i = \left\{ \mathbf{x}^i \in \mathbb{R}^{n+1} \mid (\bar{x}^i - \mathbf{x}^i)' \mathbf{p} = 0 \right\}. \tag{5} \]

Subject to her budget constraint, each investor chooses the portfolio of assets, \( \mathbf{x}^i \), that maximizes her expected utility \( V(\cdot) \) as defined in Equation (3). Accordingly, investor \( i \) solves the maximization problem

\[ \max_{\mathbf{x}^i \in B^i} V^i \left( \mathbf{x}^i'y \right). \tag{6} \]

Investors are heterogeneous in the sense that each investor \( i \) may have a different aversion to risk, \( \gamma^i \), and a different aversion to ambiguity, \( \eta^i \). Thus, for each investor \( i \), the solution for the maximization problem in Equation (6) depends on her risk and ambiguity aversion. The solution also depends on prices, which determine the budget constraint. However, for brevity, the notation does not show this dependency. A general equilibrium is a vector of prices, \( \mathbf{p} \), such that the market clears for all assets. That is,

\[ \sum_i x^i = \sum_i \bar{x}^i. \tag{7} \]

### 2.5 Mean-variance-ambiguity preferences

To solve the optimization problem in Equation (6), and thereby the equilibrium prices, investors’ preferences can be represented as mean-variance-ambiguity preferences. To this end, the next theorem identifies the certainty equivalent utility in Equation (3).\(^{19}\)

**Theorem 1.** Suppose that \( \mathbb{E}[c] > 0 \). The expected utility then satisfies

\[ V^i(c) = V^i(\mathbb{E}[c] - K), \tag{8} \]

\(^{18}\)Some studies interpret the volatility of the volatility or the volatility of the mean as measures of ambiguity. These measures, however, are outcome dependent, and therefore risk dependent. Moreover, \( \mathcal{U}^2 \) solves some major issues that arise from the use of the volatility of the volatility or the volatility of the mean as measures of ambiguity. For example, comparing two assets with different degrees of ambiguity but each with a constant volatility, or two assets with different degrees of ambiguity but each with a constant mean.

\(^{19}\)Considering constant absolute risk aversion, Ljungqvist and Sargent (2004) show that the mean-variance preferences can be extracted using a Taylor approximation.
where $\mathbb{E}[c] - K$ is the certainty equivalent consumption;

$$K = \frac{\gamma^i}{2} \text{Var}[c] + \frac{\eta^i}{2 \mathbb{E}[c]} \mathbb{E}[c] \bar{\sigma}^2[c] + R_2(c)$$

is the risk and ambiguity premium; and the remainder $R_2(c) = o\left(\mathbb{E}\left[c - \mathbb{E}[c]|^2\right]\right)$ as $|c - \mathbb{E}[c]| \to 0$.

Theorem 1 shows that risk and ambiguity have separate negative effects on expected utility, while the expected consumption has a positive effect. In particular, the higher the risk, $\text{Var}[c]$, or the aversion to risk, $\gamma^i$, the lower the certainty equivalent utility. Similarly, the higher the ambiguity, $\bar{\sigma}^2[c]$, or the aversion to ambiguity, $\eta^i$, the lower the certainty equivalent utility. The remainder of order $o\left(\mathbb{E}\left[|c - \mathbb{E}[c]|^2\right]\right)$ means that the third and higher absolute central moments of the uncertain consumption, $c$, are of a strictly smaller order than the second absolute central moment of $c$, and are therefore negligible. Thus, henceforth, the remainder is ignored (Ljungqvist and Sargent, 2004), and for simplicity the equal sign is used instead of the approximation sign. The use of the approximate expected utility, rather than the expected utility, has no impact on the conclusions of the model.

Theorem 1 implies that, in order to maximize her expected utility, every investor $i$ maximizes

$$F^i(\mathbb{E}[c], \text{Std}[c], \text{Std}[c] \bar{\sigma}[c]) = \mathbb{E}[c] - \frac{\gamma^i}{2} \text{Var}[c] - \frac{\eta^i}{2 \mathbb{E}[c]} \mathbb{E}[c] \bar{\sigma}^2[c],$$

subject to her budget constraint, where $\bar{\sigma}[c] = \sqrt{\int \mathbb{E}[\varphi(c)] \text{Var}[\varphi(c)] dc}$ and $\text{Std}[c] = \sqrt{\text{Var}[c]}$.

With this representation in place, the mean-variance-ambiguity preferences representation is immediately obtained.

**Theorem 2.** The preferences of a risk- and ambiguity-averse investor can be formed in the mean-variance-ambiguity space by

$$F^i(\mathbb{E}[c], \text{Std}[c], \text{Std}[c] \bar{\sigma}[c]),$$

where

$$\frac{\partial F^i}{\partial \mathbb{E}[c]} > 0; \quad \frac{\partial F^i}{\partial \text{Std}[c]} \leq 0; \quad \text{and} \quad \frac{\partial F^i}{\partial \bar{\sigma}[c]} \leq 0.$$

Theorem 2 establishes a representation of preferences for risk and ambiguity as mean-variance-ambiguity preferences in the mean-variance-ambiguity space. The mean-variance-ambiguity space extends the mean-variance space by adding ambiguity as a third dimension.\(^\text{20}\) In this three-dimensional space, to scale ambiguity to the units of consumption, as the other two dimensions, $\mathbb{E}[c]$ and $\text{Std}[c]$, the third dimension is represented by $\text{Std}[c] \bar{\sigma}[c]$. An alternative representation of this third dimension

\(^{20}\)Other extensions of the mean-variance space to $\mathbb{R}^3$ have been proposed. For example, Kraus and Litzenberger (1976) extend it to mean-variance-skewness space.
by $\bar{U}[c]$ would not alter the conclusions of the model.

Since the only source of consumption available to investor $i$ is the payoff of her asset portfolio $x^i$, investor $i$'s maximization problem in Equation (6) can be reframed as

$$\max_{x^i \in B^i} F^i \left( \mathbb{E} \left[ x^i' y \right], \text{Std} \left[ x^i' y \right], \text{Std} \left[ x^i' y \right] \bar{U} \left[ x^i' y \right] \right). \quad (10)$$

The solution to this optimization problem, subject to the market clearing conditions in Equation (7), provides the optimal portfolio holding of each investor and thereby the general equilibrium prices.

Every portfolio with a non-zero cost has a return $r_x = \frac{x'y}{x'p} - 1$, where $x'p$ is the cost (value) of the portfolio and $x'y$ is its payoff. Thus, the equivalent representation of the maximization problem in Equation (10) is

$$\max_{x^i \in B^i} F^i \left( \mathbb{E} \left[ x^i' (1 + r) \right], \text{Std} \left[ x^i' (1 + r) \right], \text{Std} \left[ x^i' (1 + r) \right] \bar{U} \left[ x^i' (1 + r) \right] \right), \quad (11)$$

where

$$B^i = \left\{ x^i \in \mathbb{R}^{n+1} \mid (x^i - x^i)' 1 = 0 \right\}; \quad (12)$$

$x \in \mathbb{R}^{n+1}$ is the values in terms of consumption units allocated to each asset instead of asset units; $r \in \mathbb{R}^{n+1}$ is the vector of returns of the assets in the economy; and $1 \in \mathbb{R}^{n+1}$ is a vector of 1s.

Modern portfolio theory asserts that, in an efficient market, a rational investor holds an asset portfolio that maximizes expected return for a given level of risk. The maximization problem in Equation (11) generalizes this concept to ambiguity: a rational (risk- and ambiguity-averse) investor holds an asset portfolio that maximizes the expected return for a given level of risk and a given level of ambiguity.$^{21}$ With this notion, a portfolio $x$ is efficient if there is no other portfolio with the same risk, the same ambiguity, and a strictly higher expected return.

The preferences for risk and ambiguity define sets of portfolios over which the investor is indifferent. Every such indifference set represents a specific level of expected utility. A rational investor chooses from among all feasible portfolios the one placing her on the indifference set that represents the highest level of expected utility.

### 3 Equilibrium prices and assets allocation

To characterize the mean-variance preferences and to measure risk by the variance of returns, it is commonly assumed that returns are normally distributed (Lintner, 1965; Merton, 1973; Acharya and Pedersen, 2005), so that the probability distributions are completely characterized by the mean and the

$^{21}$In particular, given two portfolios with identical risk and ambiguity, a rational (risk- and ambiguity-averse) investor prefers the portfolio with the higher expected return; given two portfolios with identical expected return and risk, she prefers the portfolio with the lower ambiguity; given two portfolios with identical expected return and ambiguity, she prefers the portfolio with the lower risk.
variance of returns. To maintain our settings as closely as possible to those of the standard CAPM, it
is assumed that returns are normally distributed; however, the parameters governing the distributions,
mean and variance, are uncertain. This assumption allows for a closed-form formalization of the effect
a change in a portfolio composition has on its degree of ambiguity. Since the EUUP model is not
restricted to a special class of probability distributions, the model that the current paper introduces
can be generalized to elliptically distributed returns.\textsuperscript{22} In addition, all other assumptions of the CAPM
are maintained.\textsuperscript{23}

Formally, $r \sim N(\mu_r, \Sigma_{rr})$, where $\Sigma_{rr}$ is a symmetric, positive definite matrix of rank $n + 1$. Since ambiguity is present, $\mu_r$ and $\Sigma_{rr}$ are uncertain, and determined by a (joint) prior $P \in \mathcal{P}$, with probability $\xi$. Each $P \in \mathcal{P}$ is associated with a (joint) normal probability density function $\phi(\cdot)$ according to which $r$ may be distributed. The second-order probability distribution $\xi$, determining which $P \in \mathcal{P}$ is realized, is assumed to induce a symmetric distribution over $\mu_r$. Note that the assumption that the set $\mathcal{P}$ of probability distributions consists of parametric (normal) probability distributions implies that a change in the parameters of the distributions may cause a change in the risk, $\text{Var}[r]$, as well as in the ambiguity, $\bar{\Omega}^2[r]$. Risk may be changed directly by the change in the parameters. In the particular case of normal distribution, an increase in the parameter $\Sigma_{rr}$ is an increase in risk. Ambiguity may be changed because the change in the parameters alters the priors within the set of priors. By Lemma 3 in Appendix A.1, in the particular case of normal distribution, an increase in the parameter $\Sigma_{rr}$ decreases the degree of ambiguity.

### 3.1 Optimal portfolio selection

To solve for investor $i$’s optimal portfolio, the maximization problem in Equation (11) can be written
explicitly as

$$\max_{x^i \in \mathcal{B}^i} \mathbb{E} \left[ x^i' (1 + r) \right] - \gamma^i \frac{1}{2} \text{Var} \left[ x^i' r \right] - \eta^i \frac{1}{2} \mathbb{E} \left[ x^i' \left( \frac{1 + 1}{(1 + r)} \right) \right] \text{Var} \left[ x^i' r \right] \bar{\Omega}^2 \left[ x^i' r \right].$$

(13)

The Lagrangian of this maximization problem can then be written as

$$\mathcal{L} (x^i, \theta) = \mathbb{E} \left[ x^i' (1 + r) \right] - \gamma^i \frac{1}{2} \text{Var} \left[ x^i' r \right] - \eta^i \frac{1}{2} \mathbb{E} \left[ x^i' \left( \frac{1 + 1}{(1 + r)} \right) \right] \text{Var} \left[ x^i' r \right] \bar{\Omega}^2 \left[ x^i' r \right] - \theta (\bar{x}^i - x^i)' 1.$$  

(14)

Using the Lagrangian, the next theorem identifies the optimal portfolio. To this end, the next theorem defines $r \in \mathbb{R}^n$ as the return vector of the risky and ambiguous assets, and $r_f \in \mathbb{R}$ as the return of

\textsuperscript{22}The normal probability distribution is a subclass of elliptical distributions, which are fully characterized by the first
two moments, mean and variance (Owen and Rabinovitch, 1983; Zhou, 1993).

\textsuperscript{23}That is, markets are efficient in the sense that all information is available to all investors, who behave competitively.
All of them have equal access to all assets in a market with no taxes and no commissions, and they can short any asset and hold any fraction of any asset.
the risk-free asset. Accordingly, henceforth, \( x^{si} \in \mathbb{R}^n \) is investor \( i \)'s optimal portfolio of risky and ambiguous assets.

**Theorem 3.** Investor \( i \)'s optimal portfolio of risky and ambiguous assets, \( x^{si} \in \mathbb{R}^n \), satisfies

\[
x^{si} = \left( \left( \frac{\Theta}{\gamma^i + \eta^i} \frac{\Theta^2 \left[ x^{si}' \mathbf{r} \right]}{\mathbb{E} \left[ x^{si}' (1 + r_f) + x^{si}' (1 + \mathbf{r}) \right]} \right) (E \left[ \Sigma \mathbf{r} \mathbf{r} \right] + \Sigma \mu_r \mu_r + \eta^i \mathbb{E} \left[ x^{si}' (1 + r_f) + x^{si}' (1 + \mathbf{r}) \right])^{-1} \right) \\
\times \left( \left( \frac{\Theta}{\gamma^i + \eta^i} \frac{\Theta^2 \left[ x^{si}' \mathbf{r} \right]}{\mathbb{E} \left[ x^{si}' (1 + r_f) + x^{si}' (1 + \mathbf{r}) \right]} \right) \left( \frac{\mathbb{E} [\mathbf{r}] - 1 r_f}{1 + \eta^i \frac{1}{2} \mathbb{E} \left[ x^{si}' (1 + r_f) + x^{si}' (1 + \mathbf{r}) \right]} \right) \right),
\]

where \( \mathbf{r} \in \mathbb{R}^n \) is the vector of returns on the risky and ambiguous assets; \( \mathbf{1} \in \mathbb{R}^n \) is a vector of 1s; \( x^{si}_j \in \mathbb{R} \) is the investor’s optimal allocation to the risk-free asset; \( r_f \) is the risk-free rate of return; and \( \Theta \) satisfies \( \frac{\partial \Theta^2 \left[ x^{si}' \mathbf{r} \right]}{\partial x^{si}} = \Theta x^{si}, \) as detailed in Lemma 6 in Appendix A.1.

In Theorem 3, \( x^{si} \) describes the optimal values that investor \( i \) allocates to each of the risky and ambiguous assets. The difference between the value of her total initial endowment and the total value allocated to the risky and ambiguous assets defines the optimal allocation to the risk-free asset, which can be positive (lending) or negative (borrowing). The total value allocated to the risky and ambiguous assets can also be positive (long position) or negative (short position). However, in equilibrium, the investor either has positive holdings in every risky and ambiguous asset, or negative holdings in every risky and ambiguous asset, as shown in Section 3.3.

Investor \( i \)'s optimal allocation is a function of her aversion to risk, \( \gamma^i \), and aversion to ambiguity, \( \eta^i \). When she is ambiguity neutral (\( \eta^i = 0 \)), her optimal allocation is

\[
x^{si} = \frac{1}{\gamma^i} (E \left[ \Sigma \mathbf{r} \mathbf{r} \right] + \Sigma \mu_r \mu_r)^{-1} (\mathbb{E} [\mathbf{r}] - 1 r_f),
\]

and, in the absence of ambiguity, her optimal allocation is

\[
x^{si} = \frac{1}{\gamma^i} \Sigma^{-1} \mathbf{r} - 1 r_f, \]

similar to standard asset pricing models (Sharpe, 1964; Treynor, 1965).

An important insight delivered by Theorem 3 is that the optimal allocation (optimal asset portfolio) in an ambiguous economy is different from the optimal allocation in a non-ambiguous economy. Since the allocation to each asset determines its equilibrium price, the prices of assets in an ambiguous economy would also be different from those in a non-ambiguous economy. Theorem 3 implies that

\footnote{In Sharpe (1964), investors can borrow or lend unlimited quantities at a constant risk-free rate of return which is exogenous. Here, following Mossin’s (1966) approach, the quantity of the risk-free assets is limited and the risk-free rate is endogenously determined in general equilibrium.}
accounting for ambiguity has the potential to provide an explanation for observable asset allocations and prices, thereby helps resolve some asset pricing anomalies (Brennan and Xia, 2001; Fama and French, 2008).25 Extant literature highlights the discrepancy between the observed asset allocations and the predicted optimal ones (Canner et al., 1997). The asset allocations, suggested in Theorem 3, may be empirically compelling to the extent that they are consistent with asset allocation puzzles.

3.2 Two-fund separation

The identification of the optimal portfolio allocation in Theorem 3 delivers an important property: a description of the optimal relative proportion of the value allocated to each risky and ambiguous asset.

**Theorem 4.** In general equilibrium, the proportional allocation of any two risky and ambiguous assets \( j \) and \( h \) satisfies

\[
\frac{x^i_j}{x^i_h} = \frac{\mathbb{E}[r_j] - r_f}{\mathbb{E}[r_h] - r_f},
\]

for each investor \( i \).

Theorem 4 suggests an ambiguity-adjusted version of the optimal allocation in a risk-only economy (Sharpe, 1964; Treynor, 1965). In an ambiguous economy, the relative proportions of risky and ambiguous assets might be different than those in a non-ambiguous economy due to the ambiguity premium that affects asset expected returns (Theorem 1). The description of the optimal allocation in Theorem 4 delivers an important insight: the two-fund separation theorem holds true also in the presence of ambiguity.

**Theorem 5.** Suppose there exist \( n \) risky and ambiguous assets, whose returns are normally distributed with uncertain means and uncertain variances, and a risk-free (and ambiguity-free) asset.

(i) There exists a unique pair of efficient portfolios (mutual funds): one containing only the risk-free asset and the other containing only the risky and ambiguous assets, such that independent of preferences (attitudes toward risk and ambiguity) or wealth, all investors are indifferent between choosing portfolios from the original \( n + 1 \) assets or from these two funds.

(ii) The return of the risky and ambiguous fund is normally distributed with uncertain mean and uncertain variance.

(iii) The relative proportion of an investor’s initial wealth invested in the \( j \)th risky and ambiguous asset

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25 A few studies use different approaches to analyze the effect of ambiguity on optimal asset allocations. Pflug and Wozabal (2007) consider an optimal portfolio problem with a confidence set of probability distributions. Garlappi et al. (2007) and Boyle et al. (2012) consider a similar problem with a confidence interval for the estimated mean returns, and Maenhout (2004) explores the effect of model uncertainty, which may be interpreted as ambiguity.
is the same for any investor $i$, independent of her preferences for risk or ambiguity.

Theorem 5 is an ambiguity-adjusted version of the Markowitz-Tobin separation theorem (Merton, 1973, Theorem 1). Tobin’s (1958) separation theorem asserts that, in equilibrium, any investor holds the risk-free asset and a unique optimal portfolio of risky assets, called the market portfolio. Theorem 5 shows that Tobin’s insight holds true also for an ambiguous economy. This implies that investment decisions can be broken into two separate phases: the first phase considers the choice of a unique optimal risky and ambiguous asset portfolio (the market portfolio); the second phase considers the allocation of funds between the risk-free asset and the risky and ambiguous portfolio (the market portfolio).

Investors may have different intensities of aversion to risk and to ambiguity. Nevertheless, in their investment decisions, they are different only in their decisions regarding the proportions of funds allocated to the risk-free asset and to the risky and ambiguous portfolio. Thus, in equilibrium, every investor holds risky and ambiguous assets in the same relative proportions as the assets in the market portfolio, which means the same relative proportions as represented by the market value of assets (Theorem 3). The nature of the market portfolio in an ambiguous economy, however, is different from Tobin’s market portfolio.²⁶ Whereas Tobin’s market portfolio demonstrates minimum risk for a given expected return, in an ambiguous economy, the market portfolio has minimum consolidated risk and ambiguity for a given expected return, but not necessarily minimum risk. The reason being the tradeoff between risk and ambiguity.²⁷

As investors are different in their intensities of aversion to risk and to ambiguity, the proportions of the risk-free asset and the market portfolio they choose may be different. More conservative investors, for example, choose to allocate a larger proportion of their initial wealth to the risk-free asset. More aggressive investors may decide to borrow money, i.e., to make a negative allocation to the risk-free asset, in order to invest more than their initial wealth in the market portfolio.

### 3.3 Equilibrium

To extract the optimal allocation, Theorem 3 constitutes $n$ equations for each investor, describing her demand for the $n$ risky and ambiguous assets in the economy. These equations can also be used to identify the demand for the risk-free asset. To determine a general equilibrium, an equality between demand and supply must be satisfied for each asset. That is, the market clearing condition

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²⁶In a model of investors with neutral preferences for parameter uncertainty, Klein and Bawa (1976) and Brown (1979) show that the market portfolio in the presence of parameter uncertainty is different from the market portfolio in the absence of parameter uncertainty, due to the Bayesian approach used.

²⁷A few recent studies investigate the negative relation between risk and ambiguity (e.g., Brenner and Izhakian, 2018; Augustin and Izhakian, 2020).
in Equation (7) must be satisfied. Since the budget constraint in Equation (12) holds true for every investor \(i\) at their optimum, summing the budget equations over all investors delivers the required market clearing condition in Equation (7), which completes the conditions describing the general equilibrium.

As in Mossin (1966), the equilibrium allocation represents a Pareto optimum. That is, due to the property of a competitive equilibrium, in which preferences are concave, it would be impossible to increase one investor’s expected utility by a new allocation without reducing the expected utility of at least one other investor. It is important to note that, in equilibrium, the problem of negative asset holdings is ruled out. By Theorems 4 and 5, in equilibrium, the relative proportion invested in any risky and ambiguous asset is the same for every investor. Thus, if one investor has a negative relative allocation to a given asset, then all other investors have a negative relative allocation in that asset, implying a violation of the market clearing condition. Therefore, the equilibrium allocation is positive for all risky and ambiguous assets.

**Corollary 1.** In equilibrium, every risky and ambiguous asset \(j\) has a strictly positive proportion in the market portfolio and, therefore, a strictly positive value \(x_j\).

Furthermore, the market portfolio is unique and, therefore, so is the equilibrium.

**Corollary 2.** In equilibrium, the proportions of assets in the market portfolio are unique. Therefore, the market portfolio and the equilibrium are unique.

To recognize the uniqueness of the market portfolio, note that since the market is in equilibrium, which is governed by supply and demand, the proportion of each asset in the market portfolio is determined by its capital market value divided by the capital value of the whole market. The capital market value of an asset (total worth of its shares) is unique, which implies that the proportion of each asset in the market portfolio is unique. Therefore, the market portfolio is unique.

### 3.4 Fund allocation decision

Suppose that the asset market is in equilibrium. The total resources available to investor \(i\) are \(w^i = \bar{x}^i'1\), where \(\bar{x}^i\) is the value in terms of consumption units of all the assets that investor \(i\) brings to the market (including the risk-free asset). Based on the equilibrium prices, each investor allocates these resources into an optimal portfolio with a particular expected return, a particular level of risk, and a particular level of ambiguity. By Theorem 5, this portfolio consists of two funds: one containing only the risk-free asset and the other containing all the risky and ambiguous assets—the market portfolio, denoted \(m\).
To maximize her consumption, \( c^i \), conditional on her preferences for risk and ambiguity, investor \( i \) chooses a proportion \( \alpha \) of her resources \( w^i \) to invest in the market portfolio and a proportion \( 1 - \alpha \) to invest in the risk-free asset. Thus, investor \( i \)'s maximization problem can be simplified to

\[
\max_{\alpha} \ E[c^i] - \gamma \frac{1}{2} \text{Var}[c^i] - \eta \frac{1}{2} \frac{1}{E[c^i]} \text{Var}[c^i] \right) \Omega^2[c^i], \tag{17}
\]

where

\[
c^i = w^i (1 - \alpha) (1 + r_f) + \alpha (1 + r_m),
\]

and \( E[c^i] > 0 \). That is, \( \alpha \) is the investor’s decision in equilibrium. This means that all investors solve the same optimization problem to maximize expected return, conditional on the degrees of risk and ambiguity. Since all investors have the same investment opportunities to choose from, the same information and the same decision procedure, every portfolio selected by a rational investor is in the set of efficient portfolios, i.e., the set of portfolios that maximize the expected return for a given level of risk and a given level of ambiguity.

4 The capital market line

The set of feasible efficient portfolios defines the capital market line (CML). In the mean-variance-ambiguity space, a rational investor maximizes the expected return for given degrees of risk and ambiguity. Therefore, all portfolios lying on the CML are efficient in the sense that they attain the maximum expected return for a given degree of consolidated risk and ambiguity. The CML identifies the reward (in terms of expected return) of efficient portfolios per unit of consolidated risk and ambiguity borne; i.e., the price of risk and ambiguity. Therefore, in equilibrium, subject to the investor’s of aversion to risk or to ambiguity, every investment decision is made on the CML. The CML is described as follows.

**Definition 1.** The capital market line is defined by

\[
E[r] = r_f + \frac{E[r_m] - r_f}{\text{Std}[r_m]} \frac{1}{\sqrt{1 + \Omega^2[r_m]}}, \tag{18}
\]

Definition 1 is obtained geometrically by the slope of the piecewise line segment originating from the risk-free rate, \( r_f \), and passing through the market portfolio’s expected return, \( E[r_m] \), in the three-dimensional mean-variance-ambiguity space. The ratio between the expected excess return, \( E[r] - r_f \), and the consolidated risk and ambiguity borne, \( \text{Std}[r] \sqrt{1 + \Omega^2[r]} \), is the same for every investor, regardless of her intensity of aversion to risk or to ambiguity. This ratio is also the same for every asset, including the market portfolio.

The expected return of an optimal portfolio rewards for three elements: the time value of money,
the risk borne, and the ambiguity borne. By Definition 1, the reward for the time value of money is equal to the risk-free rate, \( r_f \). The reward for risk is a premium, proportional to the amount of risk borne, which is proportional to the holdings in the market portfolio. The reward for ambiguity consists of two elements: a fixed participation premium and a premium proportional to the amount of ambiguity borne, which is proportional to the holdings in the market portfolio. Once an investor holds any portion of the market portfolio (or of any other risky and ambiguous asset), she bears a fixed amount of normalized ambiguity, \( \text{Std} [r_0] \overline{U} [r_0] = \text{Std} [r_m] \overline{U} [r_m] \), for which she is rewarded by a positive participation premium of magnitude \( \mathbb{E} [r_0] - r_f \). The next corollary elicits this premium.

**Corollary 3.** The expected rate of return of a portfolio with infinitely small proportional holdings in the market portfolio is

\[
\mathbb{E} [r_0] = r_f + (\mathbb{E} [r_m] - r_f) \sqrt{\frac{\overline{U}^2 [r_m]}{1 + \overline{U}^2 [r_m]}}.
\]

(19)

Corollary 3 implies that any non-zero holdings of risky and ambiguous assets exposes the investor to the discrete inherited market ambiguity. Thus, in the presence of ambiguity, all the risky and ambiguous portfolios with \( \text{Std} [r] \overline{U} [r] \in (0, \text{Std} [r_m] \overline{U} [r_m]) \) are unfeasible, which implies that the CML has a segment of unfeasible portfolios. In this respect, Bossaerts et al. (2010) find that investors who are sufficiently ambiguity averse have open sets of prices for which they refuse to hold ambiguous portfolios. Figure 4 depicts the two-dimensional section of the mean-variance-ambiguity space that contains the CML.

![Figure 2: The capital market line](image)

The portfolio with expected return \( \mathbb{E} [r_0] \) is referred to as the zero-beta portfolio (Merton, 1973). In
the absence of ambiguity, the reward per unit of risk is equal to \( \frac{\mathbb{E}[r_m] - r_f}{\text{Std}[r_m]} \). In the absence of ambiguity, Definition 1 reduces to the standard CML (Sharpe, 1964; Lintner, 1965; Mossin, 1966) in which the CML implies that the rate of substitution between a unit of expected excess return and a unit of risk is constant. Analogously, in the presence of ambiguity, the CML is a straight line, which means that the rate of substitution between a unit of expected excess return and a unit of uncertainty (consolidated risk and ambiguity) is constant. However, the consolidated risk and ambiguity, \( \text{Std}[r] \sqrt{1 + \bar{U}^2[r]} \), is not linear in the proportion allocated to the market portfolio. In particular, by Lemma 3 in Appendix A.1, \( \text{Std}[\alpha r_m] \sqrt{1 + \bar{U}^2[\alpha r_m]} = \text{Std}[r_m] \sqrt{\alpha^2 + \bar{U}^2[r_m]} \).

In equilibrium, by Corollary 3, the expected return of the market portfolio is at least as high as the expected return of the zero-beta portfolio, which is at least as high as the risk-free rate. The risk-free rate is lower than the expected return of the portfolio with the minimum possible risk and ambiguity, i.e., the global minimum risk and ambiguity portfolio; otherwise, all investors with mean-variance-ambiguity preferences would attempt to short this portfolio, which cannot represent an equilibrium.

By Definition 1 and Corollary 3, regardless of their aversion to risk and ambiguity, in the mean-variance-ambiguity space \((\mathbb{E}[r], \text{Std}[r], \text{Std}[r] \cup [r])\) all investors share the same goal: to maximize the expected excess return for a given level of consolidated risk and ambiguity (uncertainty). Therefore, in equilibrium, each investor can be thought of as solving the following optimization problem:

\[
\max_{x \in B} \frac{\mathbb{E}(x'r) - \mathbb{E}[r_0]}{\text{Std}[x'r] \sqrt{1 + \bar{U}^2[x'r] - \text{Std}[r_0] \bar{U}[r_0]}},
\]

where

\[
B = \{ x \in \mathbb{R}^n \mid x'1 = 1 \};
\]

and \( x \) is the proportional capital value of her assets relative to the capital value of her total assets. Therefore, \( x \) can also be viewed as the proportion of each asset in the market portfolio, which is determined by its capital market value divided by the capital value of the market portfolio.

## 5 Capital asset pricing

The simplified optimization problem in Equation (20) can be utilized to extract the expected return of assets. To this end, a calculus of variations-type argument can be applied to extend the classical CAPM to account for ambiguity. In the obtained closed-form pricing model, labeled the Capital Asset Pricing Model under Ambiguity (ACAPM), the expected return of an asset corresponds to its ambiguity and risk, relative to the market ambiguity and risk, rather than to its own ambiguity and

\[28\] For the same reason, in an economy with no ambiguity, the risk-free rate is lower than the expected return of the portfolio with the minimum possible risk (Cochrane, 2005).
Theorem 6. Suppose that the rates of return are normally distributed with uncertain mean, $\mu$, and uncertain variance, $\sigma^2$. The expected return of asset $j$ is then

$$E[r_j] = r_f + \frac{\zeta^P_j}{\mathcal{L}} (E[r_m] - r_f) + \beta^R_j (1 - \zeta^P_j) (E[r_m] - r_f) + \beta^A_j (1 - \zeta^P_j) (E[r_m] - r_f). \tag{21}$$

Zeta participation is

$$\zeta^P_j = \sqrt{\frac{\mathcal{L}^2[r_m]}{1 + \mathcal{L}^2[r_m]}} I_{\{j \neq f\}}, \tag{22}$$

where the indicator function $I_{\{j \neq f\}}$ takes the value one for non-risk-free assets, and zero otherwise.

Beta risk is

$$\beta^R_j = \frac{\text{Cov}(r_m, r_j)}{\text{Var}(r_m)} \frac{1 + \mathcal{L}^2[r_m]}{1 + \mathcal{L}^2[r_m] + \Lambda [r_m, r_m]}.$$

Beta ambiguity is

$$\beta^A_j = \frac{\Lambda [r_m, r_j]}{1 + \mathcal{L}^2[r_m] + \Lambda [r_m, r_m]}, \tag{24}$$

where

$$\Lambda [r_m, r_j] = \int E[\phi(r | \mu_m, \sigma_m)] \text{Cov} \left[ \phi(r | \mu_m, \sigma_m), \phi(r | \mu_m, \sigma_m) \lambda (r | \mu_m, \mu_j, \sigma^2_m, \sigma_{m,j}) \right] dr \tag{25}$$

and

$$\lambda (r | \mu_m, \mu_j, \sigma^2_m, \sigma_{m,j}) = \frac{r - \mu_m}{\sigma^2_m} \left( \frac{\sigma_{m,j}}{\sigma^2_m} (r - \mu_m) + \mu_j \right) - \frac{\sigma_{m,j}}{\sigma^2_m}. \tag{26}$$

Theorem 6 decomposes the price of an asset in terms of expected return into four components: the price of time, the price of risk, the price of ambiguity, and the price of market participation. The price of time, $r_f$, is the pure risk-free rate of return, rewarding for the time value of money. The price of risk, $\beta^R_j (1 - \zeta^P_j) (E[r_m] - r_f)$, is an additional expected return, rewarding for the systematic risk borne, referred to as the risk premium. The price of ambiguity, $\beta^A_j (1 - \zeta^P_j) (E[r_m] - r_f)$, is a second additional expected return, rewarding for the systematic ambiguity borne, referred to as the ambiguity premium. The participation price, $\zeta^P_j (E[r_m] - r_f)$, is a third discrete additional fixed expected return, rewarding for the exposure to the fundamental ambiguity in the market, referred to as the participation premium.

Different than risk, where a marginal exposure to the market portfolio implies a proportional marginal exposure to risk, a marginal exposure to the market portfolio exposes the investor to a discrete fixed level of ambiguity, for which the participation premium is rewarding. In other words, when moving away from pure risk-free and ambiguity-free holdings, there is a discrete change in ambiguity, which exposes the investor to the entire ambiguity of the market portfolio. The sum of risk. The next theorem, which is the central result of the current paper, introduces the ACAPM.
these premia—the uncertainty premium—on the market portfolio, \( E[r_m] - r_f \), is the aggregate excess return, rewarding for risk and ambiguity borne by the market portfolio, \( m \). The participation, risk and ambiguity premia on asset \( j \) are proportional to the uncertainty premium on \( m \), as determined by the coefficients \( \zeta^p_j \), \( \beta^R_j \) and \( \beta^A_j \), respectively.

In the absence of ambiguity, \( \mu \) and \( \sigma \) are certain for all assets, and so are \( \mu_m \) and \( \sigma_m \). In this case, Theorem 6 reduces to the classical CAPM, in which only the systematic risk is rewarded. This also holds true when all investors are ambiguity neutral, since they compound probabilities linearly.

**Corollary 4.** In a non-ambiguous economy or in an economy with ambiguity-neutral investors, for every asset \( j \),

\[ \zeta^p_j = 0, \quad \beta^R_j = \frac{\text{Cov}[r_m, r_j]}{\text{Var}[r_m]}, \quad \text{and} \quad \beta^A_j = 0. \]

In Theorem 6, beta risk, \( \beta^R \), measures the sensitivity of asset return to market return. It corresponds to the covariation of asset return and market return, which is assessed using expected probabilities (linearly compounded first- and second-order probabilities). Since there is uncertainty about probabilities, when investors are sensitive to ambiguity, the risk premium is adjusted for this uncertainty through \( \zeta^p \). In Theorem 6, beta ambiguity, \( \beta^A \), measures the sensitivity of asset return probabilities to the market return probabilities. It corresponds to the covariation of asset return probabilities and market return probabilities, which is assessed using the second-order probabilities (the joint probability distribution of the uncertain parameters \( \mu \) and \( \sigma \)). This correspondence is formulated by \( \lambda \) in the expression of beta ambiguity in Equation (24). By Equation (26), in \( \lambda \), the component \( \mu_j - \frac{\sigma_{m,j}}{\sigma_m} \mu_m \) reflects a shock to the location of asset \( j \)'s return distribution.\(^{29}\) The additional component \( \frac{\sigma_{m,j}}{\sigma_m} \) reflects a shock to the precision of asset \( j \)'s return distribution (a shock to \( \sigma^2_j \)). Since return distributions are fully characterized by their mean and variance (normally distributed), the uncertainty of the location and precision jointly generate the uncertainty of the return distribution.

The greater the absolute value of \( \Lambda \), the greater the sensitivity of asset \( j \)'s return distribution to the market return distribution and, accordingly, the greater the absolute value of \( \beta^A_j \). A positive \( \Lambda \) implies a positive relation between asset \( j \)'s ambiguity and the market ambiguity, resulting in a positive \( \beta^A_j \) and a positive ambiguity premium. A negative \( \Lambda \) implies a negative relation between asset \( j \)'s ambiguity and the market ambiguity, resulting in a negative \( \beta^A_j \) and a negative ambiguity premium. The intuition for negative \( \beta^A_j \) is that investors are willing to pay (in terms of a negative premium) for holding the asset in order to hedge against the ambiguity in the market portfolio. For an asset with

\(^{29}\)The difference \( \mu_j - \frac{\sigma_{m,j}}{\sigma_m} \mu_m \) can also be interpreted as the unexpected mean return. In this respect, Merton (1980) argues that the mean return (the location of the distribution) is difficult to estimate precisely.
a positive $\beta^A$ (a positive covariation of asset return probabilities and market return probabilities),
investors ask for an additional positive premium as a reward for bearing ambiguity.

The risk-free asset bears no risk and no ambiguity; accordingly, all three premia are identically zero.

**Corollary 5.** For the risk-free asset,

$\zeta_f^p = 0, \quad \beta_f^R = 0, \quad \text{and} \quad \beta_f^A = 0.$

It is possible for asset $j$ to be characterized by $\beta_j^R \neq 0$ and $\beta_j^A = 0$. This may happen when the market return probabilities and asset $j$’s return probabilities are perfectly known.\(^{30}\) Notice that, when the market return probabilities are uncertain, an asset with no ambiguity (with perfectly known probabilities) may still have a non-zero beta ambiguity, due to the uncertainty about the correlation of its return with the return of the market portfolio. In addition, in this case, zeta participation would be positive. It is also possible for an asset to be characterized by $\beta_j^R = 0$ and $\beta_j^A \neq 0$. This may happen when the covariance between $r_j$ and $r_m$, assessed using the expected probabilities, is zero; e.g., when $\mathbb{E}[\sigma_{r_m, j}] = 0$ and $\text{Cov}[\mu_m, \mu_j] = 0$ (Lemma 1 in Appendix A.1). A special case is the zero-beta portfolio (or asset), for which $\beta_0^R = 0$ and $\beta_0^A = 0$.

**Corollary 6.** For the zero-beta portfolio,

$\zeta_0^p = \sqrt{\frac{\mathbb{U}^2 [r_m]}{1 + \mathbb{U}^2 [r_m]}}, \quad \beta_0^R = 0, \quad \text{and} \quad \beta_0^A = 0.$

Corollary 6 suggests that, in the presence of ambiguity, the excess return of the zero-beta portfolio is not identically zero. Merton (1973) shows that the expected return of a risky asset may differ from the risk-free rate, even for an asset with no systematic risk. He attributes this difference to shifts in the investment opportunity set that are correlated with a zero-beta portfolio. In the ACAPM, this difference is attributed to zeta participation, which is non-zero even when beta risk (systematic risk) and beta ambiguity (systematic ambiguity) are identically zero. Moreover, in the ACAPM, the additional hedging portfolio, implied by Merton’s three-fund theorem, is not required.\(^{31}\)

A special case considers the market portfolio, as defined in the next corollary.

**Corollary 7.** For the market portfolio,

$\zeta_m^p = \sqrt{\frac{\mathbb{U}^2 [r_m]}{1 + \mathbb{U}^2 [r_m]}},$
\[
\beta_m^R = \frac{1 + \Omega^2 [r_m]}{1 + \Omega^2 [r_m] + \Lambda [r_m, r_m]} \quad \text{and} \quad \beta_m^A = \frac{\Lambda [r_m, r_m]}{1 + \Omega^2 [r_m] + \Lambda [r_m, r_m]}.
\]

Corollary 7 implies the following.

**Corollary 8.** For the market portfolio,

\[
\zeta_m^P + \beta_m^R (1 - \zeta_m^P) + \beta_m^A (1 - \zeta_m^P) = 1.
\]

Thus,

\[
\beta_m^R + \beta_m^A = 1.
\]

Corollary 8 implies that \(\beta_m^R + \beta_m^A = 1\), while in the classical CAPM \(\beta_m^R = 1\). In this respect, the ACAPM decomposes the observable beta of the market portfolio into three components: the first is derived from the systematic risk, the second from the systematic ambiguity, and the third from the participation in the ambiguous market.

An important property of beta risk and beta ambiguity is stressed in the next proposition.

**Proposition 1.** Beta risk and beta ambiguity are both additive. That is,

\[
\beta_x^R = \mathbf{x}'\beta^R \quad \text{and} \quad \beta_x^A = \mathbf{x}'\beta^A,
\]

where \(\beta\) is a vector of the assets’ betas, and \(\mathbf{x}\) is a vector of the proportions of the assets in the portfolio.

Proposition 1 suggests that the beta ambiguity (risk) of an asset portfolio is the value weighted average of the betas ambiguity (risk) of the individual assets composing the portfolio. It implies that, similarly to the classical CAPM, the ACAPM is a linear beta pricing model. Consider a portfolio consisting of \(n\) risky and ambiguous assets with proportions \(\mathbf{x} = (x_1, \ldots, x_n)\). The expected excess return of portfolio \(\mathbf{x}\) can then be expressed as

\[
\mathbb{E} [r_x] - r_f = \mathbf{x}'\mathbb{E} [\mathbf{r}] - r_f = \left(\mathbf{x}'\zeta^P + \mathbf{x}'\beta^R \left(1 - \zeta^P\right) + \mathbf{x}'\beta^A \left(1 - \zeta^P\right)\right) (\mathbb{E} [r_m] - r_f).
\]

In the ACAPM, an optimal portfolio has the maximal expected return for a given level of consolidated risk and ambiguity. Since risk and ambiguity diversification do not necessarily coincide, systematic risk and systematic ambiguity are optimal but not necessarily minimal with which to remain. Similar to the classical CAPM, investors are rewarded via a higher rate of return for the systematic risk and ambiguity borne, while the idiosyncratic risk and ambiguity are not rewarded. However, idiosyncratic risk in the ACAPM is different than in the CAPM, since in the latter optimal portfolios have minimal risk. This difference in measuring idiosyncratic risk may help explain the
idiosyncratic volatility anomaly (Ang et al., 2006; Liu et al., 2018).

The ACAPM suggests that there may be a tradeoff between risk diversification and ambiguity diversification. Theoretically, Uppal and Wang (2003) and Boyle et al. (2012) show that the presence of ambiguity leads to a strong bias in portfolio holdings (under diversification), such that full risk diversification is not optimal. Empirically, risk and ambiguity can be inversely related, such that risk reduction incurs higher ambiguity (Izhakian and Yermack, 2017; Brenner and Izhakian, 2018; Augustin and Izhakian, 2020). This inverse relation implies that, in the presence of ambiguity, a full risk diversification may not be optimal. Recall that, in the presence of ambiguity, the equilibrium asset proportions composing the market portfolio may be different from those in a non-ambiguous economy. By identifying the equilibrium prices, Theorem 6 characterizes the optimal amounts of risk and ambiguity, accounting for the tradeoff between the two.

Theorem 6 generalizes the classical CAPM and shows that the ambiguity premium on an asset is proportional to the part of its ambiguity that is derived from the market ambiguity. In earlier models, the ambiguity premium is attributed to the entire ambiguity of the market (e.g., Izhakian and Benninga, 2011; Ui, 2011) or of the asset (e.g., Chen and Epstein, 2002; Epstein and Ji, 2013). Theorem 6 adds to this literature the insight that the ambiguity premium corresponds to the covariation of asset ambiguity with market ambiguity.

6 The Security market line

In the ACAPM, the security market line (SML) characterizes the linear relation between systematic risk and ambiguity (captured by beta risk and ambiguity) and expected return. By Theorem 6, the SML of the classical CAPM can be generalized to accommodate ambiguity and be defined as

$$
E [r_j] = r_f + \zeta_j^P (E [r_m] - r_f) + (\beta_j^R + \beta_j^A) (1 - \zeta_j^P) (E [r_m] - r_f).
$$

The intercept, $r_0 = r_f + \zeta_j^P (E [r_m] - r_f)$, captures the time value of money and the participation premium. The slope, $(1 - \zeta_j^P) (E [r_m] - r_f)$, is the adjusted risk and ambiguity premium on the market portfolio. The coefficient $\beta_j^R + \beta_j^A$ corresponds to the level of systematic risk and ambiguity of asset $j$.

Figure 1 provides a graphical representation of the SML. The x-axis depicts the magnitude of $\beta_j^R + \beta_j^A$, and the y-axis depicts the expected rate of return. The sloped dashed line describes the SML in a non-ambiguous economy, and the sloped solid line describes it in an ambiguous economy. The SML slope in an ambiguous economy is flatter than in a non-ambiguous economy. Specifically, in an ambiguous economy, the slope is $(1 - \zeta_j^P) (E [r_m] - r_f)$, while in a non-ambiguous economy, it is $E [r_m] - r_f$. This implies that, in the presence of ambiguity, assets with low standard beta risk have greater excess return (equity premium) than in the classical CAPM, and assets with high standard
beta risk have smaller excess return than in the classical CAPM.

All possible portfolios, efficient and inefficient, lie on the SML, where the risk-free asset is a point of discontinuity on the SML. Every non-zero (even very small) holding of a non-risk-free asset bears an exposure to the market fundamental ambiguity, which is rewarded by a fixed discrete participation premium of size $E[r_0] - r_f$. In other words, the SML reflects a fixed premium, attributed to the ambiguity borne by stock market participation. Since market values (prices) also reflect both ambiguity and the participation premia, in an ambiguous economy, the equilibrium asset proportions composing the market portfolio may be different from those in a non-ambiguous economy.

The theoretical SML in Equation (27) offers a possible explanation for the inconsistency between the empirical SML and the SML predicted by the classical CAPM. The SML delivered by the ACAPM might be more consistent with the empirical findings than the SML delivered by the classical CAPM, and may explain some well-known related anomalies, including the zero-beta anomaly—the expected return being higher than the risk-free rate, even for assets having no systematic risk (Black et al., 1972; Merton, 1973); the beta anomaly—the empirical SML being too flat to be explained by the theoretical prediction of the classical CAPM (Black et al., 1972; Frazzini and Pedersen, 2014); the idiosyncratic-volatility anomaly—the idiosyncratic volatility being priced in contrast to the prediction of the classical CAPM (Ang et al., 2006; Liu et al., 2018)\(^{32}\); and the size and value anomalies—the additional positive premia associated with firms with small market capitalization and high book-to-market ratio (Fama and French, 1992).

Other extensions to the SML have been proposed in the literature. For example, Merton (1973) extends the CAPM to hedging portfolios using three funds; Kraus and Litzenberger (1976) extend the CAPM to accommodate return skewness, also using three different funds; and Acharya and Pedersen (2005) extend the CAPM to accommodate liquidity risk, consisting of a constant liquidation cost premium. In contrast to other models, to draw the SML in the ACAPM, only two funds are required: the risk-free asset and the market portfolio.

7 Performance measures

A natural application of the mean-variance-ambiguity preferences would be to measure portfolios’ performance relative to their associated risk and ambiguity. A broadly used performance measure is the Sharpe ratio (Sharpe, 1966), which measures the reward in terms of excess return per unit of the entire (systematic and idiosyncratic) risk borne. This ratio can be extended to account for

\(^{32}\)Notice that under the ACAPM the idiosyncratic volatility is reformulated due to the new structure of the SML, which redefine the optimal portfolios.
ambiguity using the CML in the mean-variance-ambiguity space.\footnote{Modigliani and Modigliani (1997) adjust the Sharpe ratio for portfolio leverage.} Definition 1 implies that the risk and ambiguity premium per unit of the entire consolidated risk and ambiguity borne can be measured by

\[
\frac{\mathbb{E}[r_j] - r_f}{\text{Std}[r_j] \sqrt{1 + \Omega^2[r_j]}}.
\]

A second broadly used performance measure is the Treynor ratio (Treynor, 1965), which measures the reward in terms of excess return per unit of systematic risk borne. This ratio can be extended to account for systematic ambiguity using the SML in Equation (27).\footnote{Hüblner (2005) extends the Treynor ratio to multiple indices.} Equation (27) implies that the risk and ambiguity premium per unit of the systematic risk and ambiguity borne can be measured by

\[
\frac{\mathbb{E}[r_j] - r_f}{\zeta_P^j + \left(1 - \zeta_P^j\right) \left(\beta_R^j + \beta_A^j\right)}.
\]

A third broadly used performance measure is the Jensen alpha (Jensen, 1968), which measures the abnormal return over the theoretical expected return. The Jensen alpha can be extended to account for ambiguity, using the theoretical expected return defined by the SML in Equation (27), as follows:\footnote{Connor and Korajczyk (1986) develop multi-factor counterparts of the Jensen alpha.}

\[
r_j - r_f - \zeta_P^j \left(\mathbb{E}[r_m] - r_f\right) - \left(\beta_R^j + \beta_A^j\right) \left(1 - \zeta_P^j\right) \left(\mathbb{E}[r_m] - r_f\right).
\]

These three new performance measures preserve the same key geometric and analytical properties of the original Sharpe and Treynor ratios, and of the Jensen alpha.

\section{Empirical applications}

The main contribution of ACAPM is portraying a more realistic picture of uncertainty and its effects on pricing. A second main contribution of ACAPM is providing a theoretical foundation of cross-sectional empirical tests. ACAPM and the new structure of the SML it introduces in Equation (27) can be tested empirically, paving the way for further understanding of the effect of ambiguity on pricing. In this regard, it is important to note that the slope coefficient of a linear regression of assets’ excess return on the market’s excess return does not capture the effect of ambiguity, since linear regressions assume a known unique covariance matrix. Therefore, beta risk, beta ambiguity and zeta participation must be computed directly as formulated in Theorem 6. This can be done using the methodology to compute ambiguity presented in recent literature (e.g., Izhakian and Yermack, 2017; Brenner and Izhakian, 2018; Augustin and Izhakian, 2020). These estimates can than be used as the first-stage estimates in Fama and MacBeth’s (1973) cross-sectional regression tests.

The new structure of the theoretical SML delivered by the ACAPM may address the inconsistency...
between the empirical evidence about the SML and the theoretical SML predicted by the classical CAPM (Fama and French, 2004). Since the SML delivered by the ACAPM can be estimated from the data, it may help explain some of the well-known asset pricing anomalies. For example, the theoretical intercept of the SML in the ACAPM is higher than that in the classical CAPM (due to the participation premium), which may explain the zero-beta anomaly (Black et al., 1972; Merton, 1973). In particular, the intercept of the SML in ACAPM is $E[r_0] = r_f + \zeta_j (E[r_m] - r_f)$, while in the classical CAPM it is $r_f$. The theoretical slope of the SML in the ACAPM is flatter than that in the classical CAPM, which may explain the beta anomaly (Black et al., 1972; Frazzini and Pedersen, 2014). In particular, the slope of the SML in ACAPM is $(1 - \zeta_j) (E[r_m] - r_f)$, while in the classical CAPM it is $(E[r_m] - r_f)$.

The SML in ACAPM defines the set of optimal portfolios in an ambiguous economy, and is different from the SML in the classical CAPM. Therefore, it redefines the idiosyncratic risk and the idiosyncratic uncertainty (the consolidated idiosyncratic risk and ambiguity). This difference in defining and measuring idiosyncratic risk may help explain the idiosyncratic volatility anomaly (Ang et al., 2006; Liu et al., 2018). The size and value anomalies (Fama and French, 1992) may also be explained by ACAPM, since firms with high book-to-market ratios are typified by highly ambiguous investment opportunities (Herron and Izhakian, 2018, 2019).

9 Conclusion

This paper introduces a new capital asset pricing model that accounts for ambiguity—the uncertainty of probabilities—a real world situation in which probabilities of outcomes are not uniquely assigned. It relaxes the main assumption of modern portfolio theory, according to which the probabilities of returns are known, and instead assumes that probabilities are uncertain. In this view, the mean-variance paradigm is generalized to mean-variance-ambiguity, in which preferences are characterized. The three-way tradeoff between risk, ambiguity and expected return sheds new light on asset pricing and optimal portfolio selection.

In general equilibrium, the mean-variance-ambiguity preferences deliver an important fundamental result: the Tobin two-fund separation theorem holds true in the presence of ambiguity. That is, optimally, every investor holds only two funds: the risk-free asset and the market portfolio (a unique optimal portfolio of risky and ambiguous assets). The proportions allocated to these two funds are determined by the investor’s aversions to risk and to ambiguity. Asset proportions, composing the market portfolio, may be different from those in the absence of ambiguity, since in the presence of ambiguity market values (prices) also reflect ambiguity and participation premia.
The mean-variance-ambiguity preferences provide the theoretical underpinning for the extension of the classical capital asset pricing model (CAPM) to the Capital Asset Pricing Model under Ambiguity (ACAPM). In this extended model, a closed-form representation of beta ambiguity, in addition to the ambiguity-adjusted beta risk, is obtained. Asset prices in this model correspond to their systematic risk and systematic ambiguity borne. In addition, asset prices consist of an added fixed participation premium, generated by the market fundamental ambiguity. A natural application of the proposed model is the generalization of the Treynor (1965) and Sharpe (1966) ratios to account for ambiguity, allowing for the measurement of portfolios’ performance relative to their consolidated risk and ambiguity borne. A generalization of the Jensen (1968) alpha is demonstrated as well. These measures are applicable in capital-budgeting estimations and in evaluating professionally managed portfolios.

The predictions of the classical CAPM are inconsistent with existing empirical findings, suggesting that the slope of the empirical SML is flatter and the intercept is higher than predicted by the traditional theory; inconsistencies which generate multiple anomalies. The model that the current paper introduces may help explain these empirical inconsistencies and the related anomalies, including the zero-beta anomaly (Black et al., 1972; Merton, 1973); the beta-anomaly (Black et al., 1972; Frazzini and Pedersen, 2014); the idiosyncratic volatility anomaly (Ang et al., 2006; Liu et al., 2018); and the size and value anomalies (Fama and French, 1992).

The novel theoretical model, the ACAPM, introduced in this paper, provides important insights that pave the way for further research into the three-way risk-ambiguity-return relation. This model provides a theoretical foundation for empirical cross-sectional tests of the three-way tradeoff between risk, ambiguity and expected return. A notable merit of the model is that it is trackable, applicable and can be utilized in empirical studies, improving our understanding of capital asset pricing in the financial markets. The model can also be used in other applications, including portfolio selection and value at risk.

For more than half a century the CAPM has been criticized for not portraying a realistic picture reflecting the empirical evidence regarding the risk-return relation. While advancing the literature toward addressing this puzzle, the model introduced in this paper may be further developed to support broader settings. The concepts, introduced in this paper, may also stimulate further thinking that will advance the literature toward a better understanding of the implication of ambiguity.
References


Appendix

A.1 Lemmata

Lemma 1. Suppose that \( y \) and \( z \) are distributed with uncertain means, \( \mu_y \) and \( \mu_z \), and uncertain variance, \( \sigma^2_y \) and \( \sigma^2_z \). Their covariance, computed using the expected probabilities, is then

\[
\text{Cov}(y, z) = E[\sigma_y z] + \text{Cov}(\mu_y, \mu_z).
\]

Lemma 2. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be some function. The volatility of the probabilities of \( x \), \( \text{Var}[\varphi(x)] \), is then uncorrelated with \( f(x) \). That is, \( \text{Cov}(f(x), \text{Var}[\varphi(x)]) = 0 \), implying that

\[
\mathcal{E}[f(x)\text{Var}[\varphi(x)]] = \mathcal{E}[f(x)] \mathcal{E}[\text{Var}[\varphi(x)]],
\]

where \( \mathcal{E} \) is the expectation taken using either any \( P \in \mathcal{P} \) or the expected probabilities; and \( \text{Cov} \) is the covariance taken using either any \( P \in \mathcal{P} \) or the expected probabilities.

Lemma 3. Let \( \delta \) and \( \alpha \) be constants. When \( r \) is normally distributed with uncertain mean and uncertain variance, \( \mathcal{V}^2[\delta + \alpha r] = \frac{1}{\alpha^2} \mathcal{U}^2[r] \) for any \( \alpha \neq 0 \), and \( \mathcal{V}^2[\delta + \alpha r] = 0 \) for \( \alpha = 0 \).

Lemma 4. The expression

\[
I = \int \mathcal{E} \left[ \phi (r | \mu_r, x'\Sigma_{rr}x) \left( \left( \frac{(r - x'\mu_r)^2}{x'\Sigma_{rr}x} - 1 \right) \frac{\Sigma_{rr}x}{x'\Sigma_{rr}x} + \frac{(r - x'\mu_r) \mu_r}{x'\Sigma_{rr}x} \right) \right] \text{Var}[\phi (r | \mu_r, x'\Sigma_{rr}x)] \, dr
\]

is identically zero.

Lemma 5. The expression

\[
I = \int \mathcal{E} \left[ \phi (r | \mu_m, \sigma_m) \lambda (r | \mu_m, \sigma^2_m, \sigma_{j,m}) \right] \text{Var}[\phi (r | \mu_m, \sigma_m)] \, dr
\]

is identically zero.

Lemma 6. Suppose that \( \mathcal{P} \) consists of only normal probability distributions. The ambiguity \( \mathcal{Q}^2[x'r] \) of the return \( x'r \) of asset portfolio \( x \) then satisfies

\[
\frac{\partial \mathcal{Q}^2[x'r]}{\partial r} = \Theta x^2,
\]

where

\[
\Theta = \int \mathcal{E} [\phi (r | \mu_r, x'\Sigma_{rr}x)] \text{Cov} \left[ \phi (r | \mu_r, x'\Sigma_{rr}x), \phi (r | \mu_r, x'\Sigma_{rr}x) \right] \left( \frac{(r - x'\mu_r)^2}{x'\Sigma_{rr}x} - 1 \right) \frac{\Sigma_{rr}x}{x'\Sigma_{rr}x} \, dr.
\]
A.2 Proofs

Proof of Lemma 1. The covariance can be written explicitly
\[
\text{Cov}[y, z] = \int \int E[\varphi(y, z)] (y - E[y]) (z - E[z]) dydz = \int \int E[\varphi(y, z)] (y - E[\mu_y]) (z - E[\mu_z]) dydz,
\]
where \( \varphi(y, z) \) stands for the joint distribution of \( y \) and \( z \); and \( E[\mu_y] \) is the expectation taken using the second-order probabilities. The expectation over probabilities can be taken out to obtain
\[
\text{Cov}[y, z] = E\left[\int \int \varphi(y, z) ((y - \mu_y) + (\mu_y - E[\mu_y])) (z - \mu_z) + (\mu_z - E[\mu_z]) dydz\right].
\]
Organizing terms provides
\[
\text{Cov}[y, z] = E\left[\int \int \varphi(y, z) (y - \mu_y) (z - \mu_z) dydz\right] + E\left[\int \int \varphi(y, z) (y - \mu_y) (\mu_z - E[\mu_z]) dydz\right] +
E\left[\int \int \varphi(y, z) (\mu_y - E[\mu_y]) (z - \mu_z) dydz\right] + E\left[\int \int \varphi(y, z) (\mu_y - E[\mu_y]) (\mu_z - E[\mu_z]) dydz\right],
\]
which simplifies to
\[
\text{Cov}[y, z] = E[\sigma_{yz}] + \text{Cov}[\mu_y, \mu_z].
\]

Proof of Lemma 2. Let \( y = \varphi(x) \), then \( \text{Var}[\varphi(x)] \) can be written \( \text{Var}[y|x] = E[y^2|x] - E^2[y|x] \).
In turn, \( \text{Cov}[\text{Var}[\varphi(x)], f(x)] \) can be written explicitly
\[
\text{Cov}[\text{Var}[\varphi(x)], f(x)] = E\left[(E[y^2|x] - E^2[y|x] - E[E[y^2|x] - E^2[y|x]]) (f(x) - E[f(x)])\right]
= E\left[f(x)(E[y^2|x] - E^2[y|x])\right] - E\left[f(x)E(E[y^2|x] - E^2[y|x])\right].
\]
Applying the tower property to the first term and the law of iterated expectations to the second term (e.g., Goldberger, 1991, page 47, T8), provides
\[
\text{Cov}[\text{Var}[\varphi(x)], f(x)] = E\left[f(x)(E[y^2]) - E^2[y]\right] - E\left[f(x)E[E[y^2] - E^2[y] | x]\right].
\]
By Karlin and Taylor (2012, page 8), \( E[f(x)E[g(y)|x]] = E[f(x)g(y)] \). Therefore,
\[
E\left[f(x)E[E[y^2] - E^2[y] | x]\right] = E\left[f(x)(E[y^2] - E^2[y])\right],
\]
which completes the proof.

Proof of Lemma 3. Since \( \text{Var}[\delta] = 0 \), \( \text{Var}[\delta + \alpha r | \sigma^2] = \alpha^2 \sigma^2 \). Thus, the ambiguity of \( \delta + \alpha r \)
can be written explicitly
\[
\mathcal{U}^2[\delta + \alpha r] = \int E\left[\frac{1}{\sqrt{2\pi\alpha\sigma}} e^{-\frac{(r-\delta-\alpha\mu)^2}{2\alpha^2\sigma^2}}\right] \text{Var}\left[\frac{1}{\sqrt{2\pi\alpha\sigma}} e^{-\frac{(r-\delta-\alpha\mu)^2}{2\alpha^2\sigma^2}}\right] dr.
\]

36
When \( \alpha \neq 0 \), changing the integration variable to \( r + \delta \) and then to \( \alpha r \), provides
\[
\mathcal{U}^2 [\delta + \alpha r] = \int \mathbb{E} \left[ \frac{1}{\sqrt{2\pi\alpha\sigma}} e^{-\frac{(r - \alpha r)^2}{2\alpha^2 \sigma^2}} \right] \text{Var} \left[ \frac{1}{\sqrt{2\pi\alpha\sigma}} e^{-\frac{(r - \alpha r)^2}{2\alpha^2 \sigma^2}} \right] dr = \frac{1}{\alpha^2} \mathcal{U}^2 [r].
\]
When \( \alpha = 0 \), the probability of \( \delta \) is constant and thus \( \mathcal{U}^2 [\delta] = 0 \).

**Proof of Lemma 4.** Changing the order of integration provides
\[
I = \mathbb{E} \left[ \int \phi (r | \mu_r, x' \Sigma_{rr} x) \left( \left( \frac{(r - x' \mu_r)^2}{x' \Sigma_{rr} x} - 1 \right) \frac{\Sigma_{rr} x}{x' \Sigma_{rr} x} + (r - \mu_r') \frac{\mu_r}{x' \Sigma_{rr} x} \right) \text{Var} \left[ \phi (r | \mu_r, x' \Sigma_{rr} x) \right] dr \right].
\]
By Lemma 2,
\[
I = \mathbb{E} \left[ \int \phi (r | \mu_r, x' \Sigma_{rr} x) \left( \left( \frac{(r - x' \mu_r)^2}{x' \Sigma_{rr} x} - 1 \right) \frac{\Sigma_{rr} x}{x' \Sigma_{rr} x} + (r - \mu_r') \frac{\mu_r}{x' \Sigma_{rr} x} \right) dr \right] 
N \mathbb{E} \left[ \int \phi (r | \mu_r, x' \Sigma_{rr} x) \text{Var} \left[ \phi (r | \mu_r, x' \Sigma_{rr} x) \right] dr \right]
\]
which implies
\[
I = \mathbb{E} \left[ \int \phi (r | \mu_r, x' \Sigma_{rr} x) \left( \left( \frac{(r - x' \mu_r)^2}{x' \Sigma_{rr} x} - 1 \right) \frac{\Sigma_{rr} x}{x' \Sigma_{rr} x} + (r - \mu_r') \frac{\mu_r}{x' \Sigma_{rr} x} \right) dr \right] = 0.
\]

**Proof of Lemma 5.** Writing the integral explicitly provides
\[
I = \int \mathbb{E} \left[ \phi (r | \mu_m, \sigma_m) \left( \frac{r - \mu_m}{\sigma_m^2} \left( \frac{\sigma_m}{\sigma_m} (r - \mu_m) + \mu_j \right) - \frac{\sigma_m}{\sigma_m^2} \right) \right] \text{Var} \left[ \phi (r | \mu_m, \sigma_m) \right] dr.
\]
Changing the order of integration provides
\[
I = \mathbb{E} \left[ \int \phi (r | \mu_m, \sigma_m) \left( \frac{r - \mu_m}{\sigma_m^2} \left( \frac{\sigma_m}{\sigma_m} (r - \mu_m) + \mu_j \right) - \frac{\sigma_m}{\sigma_m^2} \right) \text{Var} \left[ \phi (r | \mu_m, \sigma_m) \right] dr \right].
\]
By Lemma 2,
\[
I = \mathbb{E} \left[ \int \phi (r | \mu_m, \sigma_m) \left( \frac{r - \mu_m}{\sigma_m^2} \left( \frac{\sigma_m}{\sigma_m} (r - \mu_m) + \mu_j \right) - \frac{\sigma_m}{\sigma_m^2} \right) dr \right] 
\times \mathbb{E} \left[ \int \phi (r | \mu_m, \sigma_m) \text{Var} \left[ \phi (r | \mu_m, \sigma_m) \right] dr \right]
\]
\[
= \mathbb{E} \left[ \frac{\mu_j}{\sigma_m^2} \int \phi (r | \mu_m, \sigma_m) (r - \mu_m) dr \int \phi (r | \mu_m, \sigma_m) \text{Var} \left[ \phi (r | \mu_m, \sigma_m) \right] dr \right] = 0.
\]

**Proof of Lemma 6.** Writing the ambiguity measure explicitly provides
\[
\mathcal{U}^2 [x'r] = \int \mathbb{E} \left[ \frac{1}{\sqrt{2\pi\Sigma_{rr} x}} e^{-\frac{(r - x' \mu_r)^2}{2\Sigma_{rr} x}} \right] \text{Var} \left[ \frac{1}{\sqrt{2\pi\Sigma_{rr} x}} e^{-\frac{(r - x' \mu_r)^2}{2\Sigma_{rr} x}} \right] dr.
\]
Differentiating the expected probability with respect to $\mathbf{x}$ provides

$$I = \frac{\partial \mathbb{E}^2 [\mathbf{x'} \mathbf{y}] \partial \mathbb{E} \cdot \cdot}{\partial \mathbb{E} \cdot \cdot \mathbf{x}}$$

$$= \int \mathbb{E} \left[ e^{-\frac{(r - \mathbf{x'} \mu_r)^2}{2\sigma^2}} \left( \left( \frac{(r - \mathbf{x'} \mu_r)^2}{\mathbf{x'} \Sigma_{rr} \mathbf{x}} - 1 \right) \Sigma_{rr} \mathbf{x} \mathbf{x'} \Sigma_{rr} \mathbf{x} + (r - \mu_r \mu_x) \mu_r \mathbf{x'} \Sigma_{rr} \mathbf{x} \right) \right] \mathbb{E} \left[ e^{-\frac{(r - \mathbf{x'} \mu_r)^2}{2\sigma^2}} \left( \left( \frac{(r - \mathbf{x'} \mu_r)^2}{\mathbf{x'} \Sigma_{rr} \mathbf{x}} - 1 \right) \Sigma_{rr} \mathbf{x} \mathbf{x'} \Sigma_{rr} \mathbf{x} + (r - \mu_r \mu_x) \mu_r \mathbf{x'} \Sigma_{rr} \mathbf{x} \right) \right] dr.$$

By Lemma 4, $I = 0$. Differentiating the variance of probabilities with respect to $\mathbf{x}$ provides

$$II = \frac{\partial \mathbb{E}^2 [\mathbf{x'} \mathbf{y}] \partial \mathbb{E} \cdot \cdot}{\partial \mathbb{E} \cdot \cdot \mathbf{x}}$$

$$= 2 \int \mathbb{E} \left[ e^{-\frac{(r - \mathbf{x'} \mu_r)^2}{2\sigma^2}} \mathbb{Cov} \left[ e^{-\frac{(r - \mathbf{x'} \mu_r)^2}{2\sigma^2}} \left( \left( \frac{(r - \mathbf{x'} \mu_r)^2}{\mathbf{x'} \Sigma_{rr} \mathbf{x}} - 1 \right) \Sigma_{rr} \mathbf{x} \mathbf{x'} \Sigma_{rr} \mathbf{x} + (r - \mu_r \mu_x) \mu_r \mathbf{x'} \Sigma_{rr} \mathbf{x} \right) \right] \right] dr.$$

Since $r$ is symmetrically distributed around $\mathbf{x'} \mu_r$, and $\mu_r$ is symmetrically distributed,

$$II = 2 \int \mathbb{E} \left[ e^{-\frac{(r - \mathbf{x'} \mu_r)^2}{2\sigma^2}} \mathbb{Cov} \left[ e^{-\frac{(r - \mathbf{x'} \mu_r)^2}{2\sigma^2}} \left( \left( \frac{(r - \mathbf{x'} \mu_r)^2}{\mathbf{x'} \Sigma_{rr} \mathbf{x}} - 1 \right) \Sigma_{rr} \mathbf{x} \mathbf{x'} \Sigma_{rr} \mathbf{x} + (r - \mu_r \mu_x) \mu_r \mathbf{x'} \Sigma_{rr} \mathbf{x} \right) \right] \right] dr.$$

Thus,

$$\frac{\partial \mathbb{E}^2 [\mathbf{x'} \mathbf{y}] \partial \mathbb{E} \cdot \cdot}{\partial \mathbb{E} \cdot \cdot \mathbf{x}} = I + II = \Theta \times 2,$$

where

$$\Theta = \int \mathbb{E} [\phi (r | \mu_r, \mathbf{x'} \Sigma_{rr} \mathbf{x})] \mathbb{Cov} \left[ \phi (r | \mu_r, \mathbf{x'} \Sigma_{rr} \mathbf{x}), \phi (r | \mu_r, \mathbf{x'} \Sigma_{rr} \mathbf{x}) \left( \left( \frac{(r - \mathbf{x'} \mu_r)^2}{\mathbf{x'} \Sigma_{rr} \mathbf{x}} - 1 \right) \Sigma_{rr} \mathbf{x} \mathbf{x'} \Sigma_{rr} \mathbf{x} + (r - \mu_r \mu_x) \mu_r \mathbf{x'} \Sigma_{rr} \mathbf{x} \right) \right] dr.$$

**Proof of Theorem 1.** The first-order Taylor expansion of the right hand side (RHS) of Equation (8) with respect to $\mathbf{K}$, around 0, is

$$RHS = U(\mathbb{E} [c] - \mathbf{K}) = U(\mathbb{E} [c]) - \mathbf{K} U'(\mathbb{E} [c]) + o(|c|).$$

By Equations (3), the left hand side (LHS) of Equation (8) can be written

$$LHS = \int \mathbb{E} [\varphi (c)] U(c) \, dc + \eta \int_{c \leq k} U(c) \mathbb{E} [\varphi (c)] \mathbb{Var} [\varphi (c)] \, dc - \eta \int_{c \geq k} U(c) \mathbb{E} [\varphi (c)] \mathbb{Var} [\varphi (c)] \, dc + R_2(c).$$

The second-order Taylor expansion of the first component with respect to $c$, around $\mathbb{E} [c]$, is

$$I = \int \mathbb{E} [\varphi (c)] \left( U(\mathbb{E} [c]) + U'(\mathbb{E} [c]) (c - \mathbb{E} [c]) + \frac{1}{2} U''(\mathbb{E} [c]) (c - \mathbb{E} [c])^2 + o\left(|c - \mathbb{E} [c]|^2\right) \right) \, dc$$

$$= U(\mathbb{E} [c]) + \frac{1}{2} U''(\mathbb{E} [c]) \mathbb{Var} [c] + o\left(\mathbb{E} \left[|c - \mathbb{E} [c]|^2\right]\right).$$
By Judd (2003), the first-order Taylor expansion of $U \left( \sqrt{c^2} \right)$ with respect to $c^2$, around $E^2 [c]$, can be written

$$U (c) = \begin{cases} U (E [c]) - U' (E [c]) \frac{1}{2E [c]} (c^2 - E^2 [c]) + o \left( \left| c - E [c] \right|^2 \right), & c < k \\ U (E [c]) + U' (E [c]) \frac{1}{2E [c]} (c^2 - E^2 [c]) + o \left( \left| c - E [c] \right|^2 \right), & c \geq k. \end{cases}$$

Since $E [c]$ is relatively close to $k$ and $U (k) = 0$, then $U (E [c]) \approx 0$. Therefore, accounting for the sign switch of $E [\varphi (x)]$ when moving from a negative to a positive utility across $k$ (Wakker and Tversky, 1993), the last two terms in Equation (28) can be written

$$II = -\eta \int E [\varphi (c)] \operatorname{Var} [\varphi (c)] \left( U' (E [c]) \frac{1}{2E [c]} (c^2 - E^2 [c]) \right) dc + R_{II,2} (c).$$

Since, by Lemma 2, $\operatorname{Var} [\varphi (c)]$ and $(c^2 - E^2 [c])$ are uncorrelated,

$$II = -\eta U' (E [c]) \frac{1}{2E [c]} \int E [\varphi (c)] (c^2 - E^2 [c]) dc \int E [\varphi (c)] \operatorname{Var} [\varphi (c)] dc + R_{II,2} (c).$$

Combining the LHS and the RHS ($I$ and $II$), and substituting Equation (1) for $U (E [c])$ provides

$$K = \gamma \frac{1}{2} \operatorname{Var} [c] + \eta \frac{1}{2} \frac{1}{E [c]} \operatorname{Var} [c] \bar{U}^2 [c] + R_2 (c).$$

By $I$ and $II$, $R_2 (c) = o \left( \operatorname{Var} \left[ \left| c - E [c] \right|^2 \right] \right) + o \left( \int E \left[ \left| \varphi (c) - E [\varphi (c)] \right|^3 \right] dc \right)$. Since, $o \left( \int E \left[ \left| \varphi (c) - E [\varphi (c)] \right|^3 \right] dc \right)$ is equivalent to $o \left( \operatorname{Var} \left[ \left| c - E [c] \right|^2 \right] \right)$, then $R_2 (c) = o \left( \operatorname{Var} \left[ \left| c - E [c] \right|^2 \right] \right)$ as $\left| c - E [c] \right| \to 0$.

**Proof of Theorem 2.** To simplify notations, the superscript $i$ denoting an investor is omitted. By Equation (9),

$$\frac{\partial F}{\partial E [c]} = 1 + \eta \frac{1}{2} \frac{1}{E^2 [c]} \bar{U}^2 [c] > 0, \quad \frac{\partial F}{\partial \operatorname{Std} [c]} = -\gamma \operatorname{Std} [c] - \eta \frac{\operatorname{Std} [c]}{E [c]} \bar{U}^2 [c] \leq 0,$$

and

$$\frac{\partial F}{\partial \bar{U} [c]} = -\frac{\operatorname{Var} [c]}{E [c]} \bar{U} [c] \leq 0.$$

**Proof of Theorem 3.** To simplify notations, the superscript $i$ denoting an investor is omitted. Let $x \in \mathbb{R}^n$ be the portfolio consisting of the risky and ambiguous assets, and $r \in \mathbb{R}^n$ be the vector of returns on these assets. The Lagrangian in Equation (14) can then be written explicitly

$$L (x_f, x, \theta) = \mathbb{E} \left[ x_f (1 + r_f) + x' (1 + r) \right] - \gamma \frac{1}{2} \operatorname{Var} [x' r] \quad \text{(29)}$$

$$- \eta \frac{1}{2} \mathbb{E} \left[ x_f (1 + r_f) + x' (1 + r) \right] \bar{U}^2 [x' r] - \theta \left( (\bar{x}_f - x_f) + (x - x) ' 1 \right).$$

---

36. Judd (2003) shows that the Taylor expansion of $f(x)$ can be improved by the change of variable $x = h(y)$, i.e., writing $x$ as a non-linear transformation of $y$, to obtain $h$-linearization, and expanding $f(h(y))$ with respect to $y$. Here, the linearization is applied by $c^2$.

37. By Wakker and Tversky (1993), the sign switch is determined by a linear shift, which ensures that capacities (perceived probabilities) are nonnegative. This can also be viewed through the Choquet integration over negative functions, which takes the form $\int f dQ = \int \left( f + c \right) dQ - c$, where $c > 0$ such that $f + c > 0$.

38. Note that $II$ is of the order of cubic probabilities. Thus, it is smaller by two orders of magnitude than the probabilities, and therefore smaller by two orders of magnitude than $I$. 

39.
By Lemma 1, the variance of returns can be written $\text{Var}[x'] = x' E[\Sigma_{rr}] x + x' \Sigma_{\mu+\mu_e} x$. The first order condition of the Lagrangian is, therefore,

$$
\frac{\partial L}{\partial x} = E[1 + r] - (E[\Sigma_{rr}] + E[\Sigma'_{rr}] + \Sigma_{\mu+\mu_e} + \Sigma'_{\mu+\mu_e}) x \gamma \frac{1}{2} \frac{\partial^2 [x']}{\partial (x)} + (1 + \gamma E[1 + r] + x' (1 + r)) \frac{\partial^2 [x']}{\partial (x)} 
- (E[\Sigma_{rr}] + E[\Sigma'_{rr}] + \Sigma_{\mu+\mu_e} + \Sigma'_{\mu+\mu_e}) x \eta \frac{1}{2} \frac{\text{Var}[x']}{\text{Var}[x']} - \theta = 0.
$$

Since all covariance matrices are symmetric, and by Lemma 6,

$$
\frac{\partial L}{\partial x} = E[1 + r] \left( 1 + \gamma \frac{1}{2} \frac{\text{Var}[x']}{\text{Var}[x']} \frac{\partial^2 [x']}{\partial (x)} \right) - (E[\Sigma_{rr}] + E[\Sigma'_{rr}] + \Sigma_{\mu+\mu_e} + \Sigma'_{\mu+\mu_e}) x \left( \gamma + \eta \frac{\text{Var}[x']}{\text{Var}[x']} \frac{\partial^2 [x']}{\partial (x)} \right) - \Theta x \eta \frac{\text{Var}[x']}{\text{Var}[x']} + 1 \theta = 0. \tag{31}
$$

The additional conditions are

$$
\frac{\partial L}{\partial x_f} = 1 + r_f + (1 + r_f) \eta \frac{\text{Var}[x']}{\text{Var}[x']} \frac{\partial^2 [x']}{\partial (x_f)} + \theta = 0 \tag{32}
$$

and

$$
\frac{\partial L}{\partial \theta} = (\bar{x}_f - x_f) + (\bar{x} - x)' 1 = 0. \tag{33}
$$

By Equation (32),

$$
\theta = -(1 + r_f) \left( 1 + \gamma \frac{1}{2} \frac{\text{Var}[x']}{\text{Var}[x']} \frac{\partial^2 [x']}{\partial (x_f)} \right). \tag{34}
$$

Substituting for $\theta$ into Equation (31), provides

$$
0 = (E[r] - 1 r_f) \left( 1 + \gamma \frac{1}{2} \frac{\text{Var}[x']}{\text{Var}[x']} \frac{\partial^2 [x']}{\partial (x_f)} \right) - (E[\Sigma_{rr}] + E[\Sigma'_{rr}] + \Sigma_{\mu+\mu_e} + \Sigma'_{\mu+\mu_e}) x \left( \gamma + \eta \frac{\text{Var}[x']}{\text{Var}[x']} \frac{\partial^2 [x']}{\partial (x_f)} \right) - \Theta x \eta \frac{\text{Var}[x']}{\text{Var}[x']} \tag{35}
$$

Organizing terms completes the proof.

\[\square\]

**Proof of Theorem 4.** Immediately obtained from Equation (15).

\[\square\]

**Proof of Theorem 5.**

(i) By Theorem 4, the relative proportion of any two risky and ambiguous assets is the same for any investor, independent of their preferences. Therefore, for any investor, the holding of $n$ risky and ambiguous assets is equivalent to holding a portion of the fund containing the risky and ambiguous assets in the same relative proportions.

(ii) Since all risky and ambiguous assets composing the fund are normally distributed, the returns of
the funds are normally distributed (e.g. Papoulis and Pillai, 2002), with uncertain mean and variance.

(iii) Immediately by Theorem 4.

**Proof of Theorem 6.** Suppose that the investor decides to allocate $1 - \alpha - \varepsilon$ of his wealth to the risk-free asset, $\alpha > 0$ to the market portfolio, and $\varepsilon$ to some asset $j$. The maximization problem in Equation (20) can then be written

$$\max_{\alpha, \varepsilon} \frac{\mathbb{E}[(1 - \alpha - \varepsilon) r_0 + \alpha r_m + \varepsilon r_j] - r_0}{\text{Std}[\alpha r_m + \varepsilon r_j] \sqrt{1 + \bar{\sigma}^2[\alpha r_m + \varepsilon r_j] - \text{Std}[r_0] \bar{\sigma}[r_0]}}.$$ 

Since $\text{Std}[]$ and $\bar{\sigma}^2[\cdot]$ are invariant to linear constant shifts in returns, the maximization problem reduces to

$$\max_{\alpha, \varepsilon} \frac{\mathbb{E}[(1 - \alpha - \varepsilon) r_0 + \alpha r_m + \varepsilon r_j] - r_0}{\text{Std}[\alpha r_m + \varepsilon r_j] \sqrt{1 + \bar{\sigma}^2[\alpha r_m + \varepsilon r_j] - \text{Std}[r_0] \bar{\sigma}[r_0]}}.$$ 

The first order condition with respect to $\alpha$, evaluated at $\alpha = 1$ and $\varepsilon = 0$, is

$$0 = \frac{\mathbb{E}[r_m - r_0]}{\text{Std}[r_m] \sqrt{1 + \bar{\sigma}^2[r_m]} - \text{Std}[r_0] \bar{\sigma}[r_0]} - \frac{1}{2} \frac{\mathbb{E}[r_m - r_0]}{\text{Std}[r_m] \sqrt{1 + \bar{\sigma}^2[r_m]} - \text{Std}[r_0] \bar{\sigma}[r_0]} \frac{\partial^2 \mathbb{E}[\alpha r_m + \varepsilon r_j]}{\partial \alpha} \bigg|_{\alpha=1, \varepsilon=0},$$

where

$$\Lambda[r_m, r_m] = \frac{1}{2} \frac{\partial^2 \mathbb{E}[\alpha r_m + \varepsilon r_j]}{\partial \alpha} \bigg|_{\alpha=1, \varepsilon=0}$$

$$= \frac{1}{2} \int \mathbb{E} \left[ e^{-\frac{(r - \mu_m)^2}{2\sigma_m^2}} \right] \lambda \left( r \mid \mu_m, \sigma_m^2 \right) \var \left[ e^{-\frac{(r - \mu_m)^2}{2\sigma_m^2}} \right] dr + \int \mathbb{E} \left[ e^{-\frac{(r - \mu_m)^2}{2\sigma_m^2}} \right] \text{Cov} \left[ e^{-\frac{(r - \mu_m)^2}{2\sigma_m^2}}, e^{-\frac{(r - \mu_m)^2}{2\sigma_m^2}} \right] \lambda \left( r \mid \mu_m, \sigma_m^2 \right) dr,$$

and

$$\lambda \left( r \mid \mu_m, \sigma_m^2 \right) = \frac{r \left( r - \mu_m \right)}{\sigma_m^2} - 1.$$

By Lemma 5,

$$\Lambda[r_m, r_m] = \int \mathbb{E} \left[ e^{-\frac{(r - \mu_m)^2}{2\sigma_m^2}} \right] \text{Cov} \left[ e^{-\frac{(r - \mu_m)^2}{2\sigma_m^2}}, e^{-\frac{(r - \mu_m)^2}{2\sigma_m^2}} \right] \lambda \left( r \mid \mu_m, \sigma_m^2 \right) dr. \quad (37)$$

Thus, by Equation (36),

$$\mathbb{E}[r_m - r_0] = \mathbb{E}[r_m - r_0] \left( 1 + \frac{\Lambda[r_m, r_m]}{1 + \bar{\sigma}^2[r_m]} \right) \frac{\text{Std}[r_m] \sqrt{1 + \bar{\sigma}^2[r_m]}}{\text{Std}[r_m] \sqrt{1 + \bar{\sigma}^2[r_m]} - \text{Std}[r_0] \bar{\sigma}[r_0]}. \quad (38)$$
The first order condition with respect to $\varepsilon$, evaluated at $\alpha = 1$ and $\varepsilon = 0$, is

$$0 = \frac{E[r_j - r_0]}{\text{Std}[r_m] \sqrt{1 + \hat{\Omega}^2[r_m]}} - \frac{E[r_m - r_0]}{\text{Std}[r_m] \sqrt{1 + \hat{\Omega}^2[r_m]}} \frac{\text{Cov}[r_j, r_m]}{\sqrt{1 + \hat{\Omega}^2[r_m]}} - \frac{1}{2} \left( \frac{\text{Std}[r_m] \sqrt{1 + \hat{\Omega}^2[r_m]}}{\text{Std}[r_m] \sqrt{1 + \hat{\Omega}^2[r_m]}} \right) \frac{\partial \hat{\Omega}^2[\alpha r_m + \varepsilon r_j]}{\partial \varepsilon} \bigg|_{\alpha = 1, \varepsilon = 0},$$

where

$$\Lambda[r_m, r_j] = \frac{1}{2} \left. \frac{\partial \hat{\Omega}^2[\alpha r_m + \varepsilon r_j]}{\partial \varepsilon} \right|_{\alpha = 1, \varepsilon = 0}$$

$$= \frac{1}{2} \int E \left[ \frac{e^{-\frac{(r - \mu)^2}{2\sigma^2_m}}}{\sqrt{2\pi\sigma^2_m}} \lambda(r | \mu_m, \sigma_m, \sigma_{m,j}) \right] \text{Var} \left[ \frac{e^{-\frac{(r - \mu)^2}{2\sigma^2_m}}}{\sqrt{2\pi\sigma^2_m}} \right] \text{dr} + \int E \left[ \frac{e^{-\frac{(r - \mu)^2}{2\sigma^2_m}}}{\sqrt{2\pi\sigma^2_m}} \text{Cov} \left[ \frac{e^{-\frac{(r - \mu)^2}{2\sigma^2_m}}}{\sqrt{2\pi\sigma^2_m}}, \frac{e^{-\frac{(r - \mu)^2}{2\sigma^2_m}}}{\sqrt{2\pi\sigma^2_m}} \lambda(r | \mu_m, \sigma_m, \sigma_{m,j}) \right] \right] \text{dr}$$

and

$$\lambda(r | \mu_m, \mu_j, \sigma^2_m, \sigma_{m,j}) = \frac{r - \mu_m}{\sigma^2_m} \left( \frac{\sigma_{m,j}}{\sigma^2_m} (r - \mu_m - \mu_j) - \frac{\sigma_{m,j}}{\sigma^2_m} \right).$$

By Lemma 5,

$$\Lambda[r_m, r_j] = \int E \left[ \frac{e^{-\frac{(r - \mu)^2}{2\sigma^2_m}}}{\sqrt{2\pi\sigma^2_m}} \text{Cov} \left[ \frac{e^{-\frac{(r - \mu)^2}{2\sigma^2_m}}}{\sqrt{2\pi\sigma^2_m}}, \frac{e^{-\frac{(r - \mu)^2}{2\sigma^2_m}}}{\sqrt{2\pi\sigma^2_m}} \lambda(r | \mu_m, \sigma_m, \sigma_{m,j}) \right] \right] \text{dr}. \quad (40)$$

Thus, by Equation (39),

$$E[r_j - r_0] = E[r_m - r_0] \left( \frac{\text{Cov}[r_m, r_j]}{\text{Var}[r_m]} + \Lambda[r_m, r_j] \right) \frac{\text{Std}[r_m] \sqrt{1 + \hat{\Omega}^2[r_m]}}{1 + \hat{\Omega}^2[r_m] - \text{Std}[r_0] \hat{\Omega}[r_0]}. \quad (41)$$

The ratio of Equations (41) and (38) is

$$\frac{E[r_j - r_0]}{E[r_m - r_0]} = \frac{\text{Cov}[r_m, r_j]}{\text{Var}[r_m]} + \frac{\Lambda[r_m, r_j]}{1 + \hat{\Omega}^2[r_m]},$$

which implies

$$\frac{E[r_j - r_0]}{E[r_m - r_0]} = \frac{\text{Cov}[r_m, r_j]}{\text{Var}[r_m]} + \frac{1 + \hat{\Omega}^2[r_m]}{1 + \hat{\Omega}^2[r_m] + \Lambda[r_m, r_j]} + \frac{\Lambda[r_m, r_j]}{1 + \hat{\Omega}^2[r_m] + \Lambda[r_m, r_m]}.$$

By Corollary 3,

$$E[r_0] = r_f + (E[r_m] - r_f) \sqrt{\hat{\Omega}^2[r_m]} \frac{1 + \hat{\Omega}^2[r_m]}{1 + \hat{\Omega}^2[r_m]},$$

Substituting for $r_0$ and organizing terms completes the proof.

**Proof of Proposition 1.** The portfolio return is the weighted average returns of the assets in the
portfolio. The covariance of the portfolio return and the market return is the weighted average of the covariances of assets return and market return. Thus, by Theorem 6,
\[
\beta^R = \frac{\text{Cov} [r_m, x'R]}{\text{Var} [r_m]} \frac{1 + \bar{\sigma}^2 [r_m]}{1 + \bar{\sigma}^2 [r_m] + \Lambda [r_m, r_m]} = x' \frac{\text{Cov} [r_m, r]}{\text{Var} [r_m]} \frac{1 + \bar{\sigma}^2 [r_m]}{1 + \bar{\sigma}^2 [r_m] + \Lambda [r_m, r_m]} = x' \beta^R.
\]
By the same consideration, \( \lambda \) of a portfolio is the weighted average of \( \lambda \)s of the assets in the portfolio. Thereby, the same holds true for the \( \Lambda \) and \( \beta^A \) of the portfolio: \( \Lambda [r_m, x'r] = x' \Lambda [r_m, r] \) and \( \beta^R = x' \beta^A \).

**Proof of Corollary 1.** In equilibrium, the market clearing condition in Equation (7) holds true for every asset. Thus, \( x_j > 0 \) and the relative value of the asset is \( \frac{x_j}{x'1} > 0 \).

**Proof of Corollary 2.** In equilibrium, the market value of an asset is unique, which implies that the proportion of each asset in the portfolio is unique. Therefore, the market portfolio is unique.

**Proof of Corollary 3.** The rate of return on a portfolio, whose risk tends to zero, can be written
\[
E[r_0] = \lim_{\alpha \to 0} (1 - \alpha) r_f + \alpha E[r_m].
\]
Thus, by Equation (18) and Lemma 3,
\[
E[r_0] = r_f + \lim_{\alpha \to 0} \alpha \text{Std}[r_m] \sqrt{1 + \frac{1}{\alpha^2} \bar{\sigma}^2 [r_m]} \frac{E[r_m] - r_f}{\text{Std}[r_m] \sqrt{1 + \bar{\sigma}^2 [r_m]}} = r_f + (E[r_m] - r_f) \frac{\bar{\sigma}^2 [r_m]}{1 + \bar{\sigma}^2 [r_m]}.
\]

**Proof of Corollary 4.** In the absence of ambiguity, \( \mu_m, \sigma_m, \mu_j, \sigma_j \) and \( \sigma_{m,j} \) are constants. Thus, \( \bar{\sigma}^2 [r_m] = 0 \) and \( \Lambda [r_m, r_j] = 0 \). Equations (22), (23), and (24) then complete the proof.

**Proof of Corollary 5.** Substituting \( r_f \) for \( r_j \) in Equations (22), (23) and (24) completes the proof.

**Proof of Corollary 6.** Immediately by Equations (22), (23) and (24).

**Proof of Corollary 7.** Immediately by Equations (22), (23) and (24).

**Proof of Corollary 8.** By Equation (21), \( \zeta^p_m + \beta^R_m (1 - \zeta^p_m) + \beta^A_m (1 - \zeta^p_m) = 1 \), which implies that \( \beta^R_m + \beta^A_m = 1 \).