Uncertainty aversion and heterogeneous beliefs in linear models*

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Abstract

This paper proposes a simple perturbation approach for dynamic models with agents who differ in their perception of exogenous shocks. The method characterizes linear dynamics around the steady state, which may differ from any individual agent's long run expectation. It applies when agents agree to disagree, as well as when they differ in aversion to Knightian uncertainty and hence behave *as if* they hold different worst case beliefs. It thus allows us to study uncertainty in a linear setting. Our leading example looks at precautionary savings and gains from insurance in a borrower-lender model with agents who differ in uncertainty aversion.

1 Introduction

In this paper we analyze a general class of dynamic models with agents that have heterogeneous perceptions over the probability distributions of the exogenous shocks. The agents otherwise share structural knowledge of the economy, so that they know the mapping from shocks to endogenous variables. Thus, disagreement about probability distributions of shocks translates into a disagreement about endogenous variables that is determined in equilibrium. We propose a perturbation approach to show how to characterize the linear dynamics of such a heterogeneous beliefs model.

Our approach allows for disagreement that is not only temporary but survives even in the long run. When the disagreement about future endogenous variables matters for steady state, the solution strategy has to jointly determine the dynamics and the steady state. Indeed, while heterogeneity affects the steady state, the latter in turn affects the intensity

*Preliminary and incomplete.

with which these beliefs about endogenous variables matter, through their equilibrium laws of motion. Disagreement in the long run affects the applicability of perturbation methods. The reason is that typically in dynamic models in which future endogenous variables enter the intertemporal conditions, we cannot determine the steady state separately from the policy functions.

To solve jointly for the steady states and for the elasticities, we propose a solution strategy that uses standard linear forward-looking rational expectations solution methods. The strategy consists of the following fixed-point approach. First, guess how the future endogenous variables respond to the exogenous shocks over which there is disagreement. Second, stack all individual optimality conditions, where the intertemporal ones involve agent specific beliefs directly about exogenous shocks and indirectly through the conjectured laws of motion of future endogenous variables. Then find a candidate steady state, which we refer to as the zero-risk steady state, that solves these equilibrium conditions. Third, compute all derivatives of the optimality conditions at the steady state and find elasticities for the policy functions. To solve for this step we can appeal to standard linear rational expectations model solution methods. To do that, we propose adding a vector of agent-specific forward-looking endogenous variables that capture the heterogeneity in beliefs. Finally, we verify the initial guess on the elasticities. If the guess is correct, then agents are forming heterogeneous, but model-consistent, expectations of the future endogenous variables.

Our proposed approach of solving dynamic heterogeneous beliefs models applies to two broad types of economies. One is an expected utility model in which agents agree to disagree. There disagreement is wired in as a-priori assumed differences of perceptions about probability distributions. A second economy is one where the heterogeneity of beliefs arises endogenously because agents differ in their aversion to Knightian uncertainty (ambiguity). Ambiguity aversion is described by the recursive multiple priors preferences that capture agents' lack of confidence in probability assessments.

To describe how linear methods can be applied for models in which agents differ in their perceived uncertainty about the exogenous shocks, we start from a general recursive model with heterogeneity in ambiguity. There optimizing agents act *as if* they use different worst-case beliefs to evaluate future plans. When we focus attention on models with ambiguity about conditional means, the equilibria will look like in models of disagreement. The key property of the ambiguity model is that agents act *as if* the true data generating process (DGP) is given by the worst-case beliefs that support the equilibrium allocation. This means that the degree of disagreement is endogenous, since it will depend on equilibrium laws of motion that map beliefs about exogenous shocks into beliefs about endogenous variables. While at the equilibrium allocations these beliefs will look to an outside observer as dogmatic differences, the ambiguity model is one in which policy interventions that affect equilibrium choices will also alter this disagreement.

The endogeneity of disagreement shows up in our proposed solution approach in two major ways. First, even in models where agents have ex-ante exogenously specified differences of perceptions about the probability distributions of shocks, the equilibrium effects of these perceptions will be in general a function of the other model parameters. A second manifestation is in models where agents face ambiguity but otherwise have ex-ante the same degree of confidence. If in equilibrium agents take trading positions of different signs, the ex-ante homogeneity manifests as ex-post heterogeneity of worst-case beliefs. For example, as we will analyze in one of our application, when credit is in nominal terms, the worst-case beliefs are different even if ex-ante agents shared the same uncertainty about inflation: the lender acts *as if* future inflation is high while the borrower acts *as if* future inflation is low.

The general logic of the proposed strategy to solve for heterogeneous ambiguity in linear models starts by finding the linear equilibrium laws of motion. This entails the following. First, we conjecture the worst case mean of each agent as a linear function of state variables. Conditional on this worst-case belief, the model is always observationally equivalent with an expected-utility (EU) model. If there is heterogeneity in these beliefs, then the observationally equivalent EU model is one with belief disagreement. Second, given these beliefs, we solve for the loglinearized model. This step usually requires jointly finding the steady state and the law of motion. Third, we verify the initial conjecture from a first order expansion of value function. Having solved for the equilibrium policy functions, the last step is to determine the actual dynamics of model by using the true Data Generating Process (DGP) of shocks to simulate and compute moments.[1]

The second part of the paper consists of using our proposed strategy to solve a model of uncertainty sharing in which two types of agents engage in financial trading to smooth consumption. Each agent receives an endowment stream. The assets consist of a one-period non-contingent bond, whose trading is subject to a convex borrowing cost, and a tree, which pays dividends and has collateral value by lowering leverage. We illustrate heterogeneity in uncertainty by assuming that one type, called type A, perceives his endowment to be uncertain while the endowment stream for the second type, called type B, is known to be constant. In the baseline version agent A is also more patient than agent B, a feature than we turn off for some experiments.

We analyze two versions of modeling uncertainty sharing, which we compare to the case

[1]It is useful to note that the discrepancy between true and perceived moments is sometimes important. Indeed, the econometrician that analyzes data produced by the model will measure premia or 'wedges', such as ex-post excess returns on uncertain assets, between equilibrium objects determined under the two different measures.

where uncertainty does not affect decision rules. In the expected utility version, uncertainty refers only to risk, where all agents agree on the same probability distribution of shocks as given by the true DGP. For this 'risky' model, we need to use nonlinear solution for the policy functions to analyze the general equilibrium effects of heterogeneous uncertainty. In the second version, the uncertain endowment stream is also ambiguous. In particular, the type A agent entertains a set of probability distributions about the conditional mean of his endowment, while type B knows the true DGP. For this 'ambiguity' model, we use our proposed solution strategy to show that we can study equilibrium effects of the heterogeneous uncertainty using linearization methods. In particular, we guess and verify what is the worst-case belief of agent A. Under this belief, which differs from the true DGP, the model looks like one with belief disagreement.

We develop several exercises to help understand the economics of the model. First, we confirm that the qualitative equilibrium effects of uncertainty are similar, when modeled either as risk or ambiguity. Moreover, we find that for an appropriately chosen width of the set of beliefs of agent A the steady state of the two uncertainty versions is the same and that policy rules behave very similarly. In particular, when compared to the deterministic case, uncertainty generates in the ergodic steady state gains from trade that come in addition to the difference in patience. This is reflected in the higher equilibrium steady state leverage and debt. The precautionary saving motive of agent A, coming either in the form of risk or worst-case belief, incentivizes agent A to lend more. This not only raises the quantity of debt but also the bond price so that uncertainty aversion leads to a lower interest rate. From the perspective of the borrower, the higher collateral value of the tree means that agent B is willing to pay a higher price for the tree.

Second, we focus on the ambiguity model to study time-variation in uncertainty. An advantage of the linearity behind the solution method is that we can expand easily the state space. We study shocks to the ambiguity, or, in the language of heterogeneous beliefs, shocks to the degree of disagreement between the two agents. We analyze a one-time increase in agent A's ambiguity about his future income. This results in a strong desire to save from agent A which leads to a fall in his consumption, a reduction in the interest rate and an increase in debt. At the same time, the tree price increases as its collateral value is higher. The transitional dynamics back to steady state differ whether the shock leads to an initial negative real rate.

Third, we modify our baseline model so that there is no heterogeneity in patience. In this case, in the deterministic steady state there are no gains from trade. This means that there is zero debt and, importantly, that the asset and consumption allocations are indeterminate. Standard perturbation methods that rely on approximations around the deterministic steady state cannot handle such a case and need to impose some prior differences between agents to make these allocations determinate. The purpose of this exercise is to highlight that our strategy does not use the deterministic steady state and instead involves solving jointly for the equilibrium elasticities and the zero-risk steady state. In the latter there will be differences between agents because of the heterogeneity in uncertainty.

The key property that emerges is that ambiguity generates gains from trade. The less confident agent, of type A, is concerned about future income and wants to save. Indeed, we show analytically how heterogeneity in ambiguity works like differences between subjective discount factors. Importantly, the resulting difference is endogenous. Changes in the environment will lead to agent A behaving *as if* his subjective discount factor has changed.

Part of the a-priori assumed heterogeneity in beliefs is the selection of the identity of the agent whose income is uncertain. However, the proposed approach can be extended to economies in which agents self-select in equilibrium into types whose beliefs will ex-post differ. To illustrate this, the fourth experiment we consider is a version of the baseline model in which we introduce nominal credit and ambiguous inflation. Ambiguity is modeled as a set of conditional means for inflation. In equilibrium, the lender is concerned that the mean for inflation is high, because this lowers the future continuation utility by eroding the real value of the nominal bond. Thus, agent A, the lender, acts as if the probability distribution for future inflation is the one with the highest mean. At the same time, agent B, who in equilibrium becomes the borrower, is concerned that future inflation is low since this raises the real value of the repayments the agent has to make, which lowers his continuation value.

We find that the endogenous disagreement generated by inflation uncertainty lowers significantly the gains from trade, since both agents now perceive lower real returns to their trading strategies. The uncertainty premium lowers the price of nominal bonds and it leads to less debt. The price of the tree also decreases due to the lower value of collateral.

The paper is organized as follows. Section 2 proposes a solution approach for a general class of recursive models with heterogeneous perceptions of exogenous shocks. There we first characterize models with heterogeneous ambiguity and then describe a representation of a recursive model with heterogeneous beliefs. Section 3 introduces an application consisting of a model of uncertainty sharing, whose economic intuition is discussed in section 4 based on a numerical example.

2 Solving heterogeneous beliefs models by linearization

In this section we first introduce a general class of recursive models with heterogeneity in ambiguity. We then move towards a representation of a recursive model with heterogeneous beliefs by noting the observational equivalence of the ambiguity model with an expected utility (EU) model, given the worst-case beliefs. Once we focus on ambiguity about conditional means, we develop an approach to solve for this EU model with disagreement using log-linearization. The approach usually requires finding jointly the steady state and the equilibrium law of motion. We show that there is an easy implementation that uses standard solution methods, implementable in packages such as Dynare, for an *as if* linear forward-looking rational expectations representation. In the heterogeneous ambiguity model we need to verify the conjectured worst-case belief by analyzing the equilibrium value functions. This step is not required for EU models, which can be solved using our solution approach, but which impose dogmatic disagreement. Once the linear policy functions are determined, the equilibrium dynamics of the model are found by simulating under the true DGP.

2.1 Generalized recursive model

Let Z_t denote a $k \times 1$ vector of exogenous state variables, X_t an $m \times 1$ vector of endogenous state variables and Y_t a $n \times 1$ vector of other endogenous variables.

There are I agents, each agent i with a vector of actions, denoted by D^i. These actions are part of X or Y and include the consumption choice C^i. Let $B^i(X, Y, Z)$ denote the agent i's constraint set.

Given the evolution of Z, the evolution of the endogenous state variables is given by

$$X' = T(X, Y, Z)$$

An example of such law of motion is the capital accumulation equation, where X will include capital. An important component is the law of motion of the vector of ambiguities, $a^i \in X$, given by some transition

$$a^i = A^i(X, Z)$$

As detailed below, the vector of ambiguities controls the size of the belief sets for each agent, and in turn for each shock. In particular, each agent is characterized by belief sets such that one-step ahead conditional mean of innovation to the jth shock Z_j belongs to the interval $\left[-a_j^i, a_j^i\right]$. In principle, the ambiguity process can be a function of the endogenous state-variables X, as it would happen for example in models of learning.

Finally, the model economy consists of additional restrictions that determine the other endogenous variables, Y, through some relationship

$$\Gamma\left(X, Y, Z\right) = 0$$

An example would be market clearing conditions in the goods or asset markets.

The recursive equilibrium consists of functions for the endogenous variables $X'\left(X, Z\right)$, $Y\left(X, Z\right)$, value functions $V^i\left(X.Z\right)$ that satisfy three conditions.

First, agents optimize. In particular, we describe agents' preferences by the recursive multiple priors according to which

$$V^i\left(X, Z\right) = \max_{D^i \in B^i(Y,X,Z)} \left\{ u^i(C^i) + \beta \min_{|\mu_j^i| \leq A_j^i(X,Z)} E^{\mu}\left[V^i(X', Z')\right] \right\} \tag{1}$$

$$s.t.\ X' = T(X, Y, Z) \tag{2}$$

Agents' valuation of state variables X and Z is given by $V^i\left(X, Z\right)$, which is obtained by maximizing per period utility $u^i(C^i)$ and the continuation value over the available actions D^i from the budget set $B^i\left(Y, X, Z\right)$.

A key element of the setup is that each agent has a set of beliefs over the one-step ahead conditional mean, denoted by μ_j^i, of the innovation to each shock Z_j. This set is given by the restriction that $|\mu_j^i| \leq A_j^i\left(X, Z\right)$, where the ambiguity vector A^i is potentially specific to each agent.

In the recursive representation in (1), agents react to the ambiguity (Knightian uncertainty) about the conditional probability distributions of the shocks by taking optimal actions *as if* the true data generating is characterized by each shock Z_j having the probability distribution with the lowest mean μ_j^i out of the set of beliefs. The fact that agents have different sets thus gives rise to heterogeneity in beliefs. Finally, in order the forecast the next period state variables, agents use the law of motion in (2).

Second, the endogenous variables are determined according to the restrictions summarized by $\Gamma\left(Y, X, Z\right) = 0$.

Third, the dynamics of this equilibrium need to be characterized under some law of motion for the exogenous variables Z. In the two steps above we have found the decision rules and laws of motion that arise from agents' aversion to uncertainty, as expressed by the optimization of the recursive multiple priors preference. Given these functions X' and Y, the realized equilibrium outcomes are obtained feeding in the true exogenous Markov state process for Z.

2.2 Recursive equilibrium with general beliefs

The general system consists of three types of equations. There are n equations that determine the other endogenous variables as an implicit function of the state variables:

$$f\left(X_t, X_{t-1}, Y_t, Z_t\right) = 0 \qquad (n \text{ equations})$$

There are m equations that capture intertemporal behavior by agents. In a model with heterogeneous beliefs, we need to distinguish between what belief is used for form expectations. We say there m_i equations involving beliefs of agent i, where $\sum_i m_i = m$.

$$E_t^i\left[g^i\left(X_t, X_{t-1}, Y_t, Y_{t+1}, Z_t\right)\right] = 0 \qquad (m_i \text{ equations})$$

Notice that the expectation operator uses the conditional distribution that emerges as the worst-case distribution from the minimization in the utility represented in (1). Once the worst-case belief is determined, or guessed, the model becomes observationally equivalent with an expected utility model with heterogeneity in beliefs.

There are k equations that describe the law of motion of the exogenous variables. We introduce the risk directly on innovations to the log state variables:

$$\log Z_t = (I - P)\log \bar{Z} + P \log Z_{t-1} + \varepsilon_t \tag{3}$$

We assume that $I - P$ is invertible so the mean of a stationary solution to this difference equation is $\log \bar{Z}$.

We assume that agents may disagree about the exogenous variables, but they share structural knowledge of the economy, that is, they agree on the mapping from exogenous to endogenous variables.

For a general definition of equilibrium, suppose that each agent has in mind a probability distribution over Z_t, possibly different from (3). A recursive equilibrium is given by functions $X'(X, Z)$ and $Y(X, Z)$ such that the equations above are satisfied for all (X, Z). The resulting system of functional equations is

$$
\begin{aligned}
f\left(X'\left(X,Z\right),X,Y\left(X,Z\right),Z\right) &= 0, \\
E^i\left[g^i\left(X'\left(X,Z\right),X,Y\left(X,Z\right),Y\left(X'\left(X,Z\right),Z'\right),Z\right)|Z\right] &= 0,
\end{aligned}
$$

where in the second equation Z' is uncertain and expectations in equation i are computed using agent i's conditional distribution over it. These are $m+n$ functional equations in $m+n$ functions.

2.2.1 Recursive equilibrium: zero risk steady state

If agents' beliefs differ in conditional means, then they will disagree in steady state. Denote steady state value by bars. Suppose also there are no shocks, but agents differ in conditional beliefs at the steady state $E^i\left[Z'|\bar{Z}\right]$. The steady state is then characterized by

$$
\begin{aligned}
f\left(\bar{X},\bar{X},\bar{Y},\bar{Z}\right) &= 0, \\
g^i\left(\bar{X},\bar{X},\bar{Y},Y\left(\bar{X},E^i\left[Z^i|\bar{Z}\right]\right),\bar{Z}\right) &= 0,
\end{aligned}
$$

In general, the beliefs $E^i\left[Z'|\bar{Z}\right]$ matter both directly, in the second equation, and through their effects on future endogenous variables.

2.3 Recursive equilibrium: log-linear approximation

Disagreement at the steady state affects the applicability of perturbation methods. Indeed, if future endogenous variables Y enter into the intertemporal conditions, we cannot determine the steady state separately from the policy function $Y\left(X,Z\right)$. This problem is relevant for example in models with an intertemporal Euler equation of a long lived agent, where Y typically includes consumption, and it is not possible to find consumption in closed form. It does not occur for example in models with 2 period lived agents, where we have a closed form solution for consumption as a function of X and Z.

2.3.1 Loglinear approximation

We shoot for approximate solutions that are loglinear expansions around the steady state. Denote log deviation by lower case letters with hats, that is, $X = \bar{X}e^{\hat{x}}$ and so on. We conjecture solutions

$$X_t = \bar{X} \exp\left(\varepsilon_{xx}\hat{x} + \varepsilon_{xz}\hat{z}\right)$$
$$Y_t = \bar{Y} \exp\left(\varepsilon_{yx}\hat{x} + \varepsilon_{yz}\hat{z}\right)$$

Here $\hat{x}$ corresponds to the lagged deviation of X from steady state and $\hat{z}$ corresponds to the current deviation of Z from steady state. The undetermined coefficients are therefore $m+n$ constants, and $(m+n)(m+k)$ elasticities.

If all we want is loglinear approximate solutions, we do not have to model explicitly all features of beliefs: all that matters is differences in conditional means. We assume

$$E_t^i \log Z_{t+1} = (I-P)\log\bar{Z} + P\log Z_t + A^i \log Z_t$$

This assumes that conditional means can be written as a linear function of exogenous state variables. The matrix A^i determines the belief adjustment relative to the true law of motion (3). In the exogenous ambiguity case, an ambiguity state variable would be part of Z_t and we can then use different A^i matrices to pick different worst cases.

In a rational expectations setup, we can solve the constants separately from the elasticities. This is not possible here since agents disagree in steady state about what happens to the state variables next period. However, we can still loglinearize to obtain as many equations as coefficients we need to determine.

Loglinearizing the first set of equations, we have (writing the jth partial derivative as f_j and dropping arguments):

$$0 = f\left(\bar{X},\bar{X},\bar{Y},\bar{Z}\right) + \bar{X}f_1\left(\varepsilon_{xx}\hat{x}+\varepsilon_{xz}\hat{z}\right) + \bar{X}f_2\hat{x} + \bar{Y}f_3\left(\varepsilon_{yx}\hat{x}+\varepsilon_{yz}\hat{z}\right) + \bar{Z}f_4\hat{z}.$$

This delivers n equations for constants, and $n\times(m+k)$ on elasticities in the usual way.

Loglinearizing the second set of equations, for each i, we have

$$\begin{aligned}0 &= g^i\left(\bar{X},\bar{X},\bar{Y},\bar{Y}\exp(\varepsilon_{yz}A^i\log\bar{Z}),\bar{Z}\right) + \bar{X}g_1\left(\varepsilon_{xx}\hat{x}+\varepsilon_{xz}\hat{z}\right) + \bar{X}g_2\hat{x}\\ &+ \bar{Y}g_3\left(\varepsilon_{yx}\hat{x}+\varepsilon_{yz}\hat{z}\right) + \bar{Y}g_4\left[\varepsilon_{yx}(\varepsilon_{xx}\hat{x}+\varepsilon_{xz}\hat{z}) + \varepsilon_{yz}\left(P+A^i\right)\hat{z}\right] + \bar{Z}g_5\hat{z}\end{aligned}$$

This delivers m equations from the constant terms, and $m\times(m+k)$ equations from matching coefficients on the state variables. Note that all the derivatives f_j and g_j also depend on the constants.

2.3.2 Proposed solution strategy: implementation

To solve jointly for the constants and the elasticities, we propose a solution strategy that uses standard linear forward-looking rational expectations solution methods. The strategy consists of the following steps:

1. Guess the elasticities of the other endogenous variables,Y, with respect to the endogenous and exogenous state variables, ε_{yx} and ε_{yz}, respectively. This step is useful so that the agents have a guess on the law of motion of future endogenous variables, which they need to forecast.

2. Solve for candidate steady state from constant terms

$$f\left(\bar{X},\bar{X},\bar{Y},\bar{Z}\right)=0$$
$$g^i(\bar{X},\bar{X},\bar{Y},\bar{Y}\exp(\varepsilon_{yz}A^i\log\bar{Z}),\bar{Z})=0$$

   Given the guessed elasticities, this is a nonlinear equation system of dimension $m+n$.

3. Compute all derivatives at steady state and find elasticities

   This becomes finding a linear system of dimension$(m+n)\,(m+k)$.

   To solve for this step we can appeal to standard linear rational expectations model solution methods. To do that, we propose adding a vector of forward-looking endogenous variables $\tilde{Y}^i_{t+1}$, with a size equal to the number of Y variables that enter as leads in the expectation equations $g^i(.)$.

   In particular, we modify the model to read

$$0=f\left(X_t,X_{t-1},Y_t,Z_t\right) \tag{4}$$
$$0=E_t\left[g^i(X_t,X_{t-1},Y_t,\tilde{Y}^i_{t+1},\bar{Z})\right] \tag{5}$$
$$\log Z_t=(I-P)\log\bar{Z}+P\log Z_{t-1}+\varepsilon_t \tag{6}$$

   where the new defined variable follows

$$\tilde{Y}^i_t=\bar{Y}\exp(\varepsilon_{yx}\left(\log X_{t-1}-\log\bar{X}\right)+\varepsilon_{yz}\left(\log Z_t-\log\bar{Z}+A^i\log Z_{t-1}\right)) \tag{7}$$

   The variable $\tilde{Y}^i_t$ thus controls for the forward looking variable Y_t. but from the perspective of each agent i. The heterogeneity of beliefs shows up in introducing agent-specific $\tilde{Y}^i_t$ variables. The law of motion for this variable follows from the guess in step 1: it responds to the endogenous and exogenous state variables according to elasticities ε_{yx} and ε_{yz}. Importantly, the agent specific matrix A^i, which determines the belief adjustment about the exogenous state Z, relative to the true law of motion, affects the belief about the endogenous variable through the elasticity ε_{yz}.

4. Finally, for the solution of the economy represented by the equations in (4)-(7) to be the equilibrium in which we have imposed structural knowledge of the economy, we need to check the consistency of these beliefs. This amounts to check if the guess of the elasticities ε_{yx} and ε_{yz} is correct. If the guess is correct, then agents are forming model-consistent expectations of the future endogenous variables.

Having reached consistency of beliefs about the structure of the economy, the dynamics and the steady state of the rational expectations model described by the equilibrium conditions in (4)-(7) produces the dynamics and the zero-risk steady state of our economy with heterogeneous beliefs. Indeed, this system contains: (i) the original static equations $f(.)$ in (4), (ii) the forward looking equations $E_t g^i(.)$ in (5), where expectations about future endogenous variables are agent specific. These functions contain the agent specific views about future variables, reflected in the guessed laws of motion (7); (iii) dynamics are determined by the true law of motion for the exogenous state, as in (6). At the fixed point for the elasticities $(\varepsilon_{yx}, \varepsilon_{yz})$ agent i's expectation of next period's endogenous variables, as described by equation (7), is consistent with the linearization of the overall system at the zero risk steady state.

3 A model of partial insurance

We present here a model of uncertainty sharing. Agents face uncertain endowment streams and engage in financial trading to optimally smooth consumption.

3.1 The model

Agents and beliefs

There are two types, denoted by A and B, of infinite lived agents. Each agent maximizes

$$E_0^j \left[\sum_{t=0}^{\infty} \beta_j^t \frac{c_t^{1-\gamma}}{1-\gamma} \right] \tag{8}$$

where γ is the CRRA coefficient and β_j is the potentially agent specific subjective discount factor. Each agent will receive a stream of endowment of goods. The beliefs of each agent about these streams are as follows. Type B agents always get $\bar{y}^B$ and all agents know this. At date t, type A agents believe they get $y_{t+1} = \bar{y}^A \exp(-a_t)$. The confidence of agent A is potentially time varying, in the sense that a_t is stochastic. However, this belief simply reflects the concern of agent A, who is not confident about his own income. In fact, according to the true data generating process type A agents get $y_{t+1} = \bar{y}^A$. Type B agents are confident: they know the true DGP and thus know not only that their income $y_t^B = \bar{y}^B$ but also that agent A endowment is constant at $y_t^A = \bar{y}^A$.

The model is casted as one of heterogeneous beliefs, in which for simplicity we have assumed that agent B knows the true DGP and agent A is more pessimistic about his income. A way to interpret agent $A's$ belief is an *as if* pessimism that results from the ambiguity he perceived about his income. In that case, the set of beliefs about the one-step ahead conditional mean of his log-income is given by $[-a_t, a_t]$ and the worst-case belief is simply the lower bound, given by $-a_t$. This leads to agent A maximizing the present discounted value in (8) under the worst-case belief that $y_{t+1} = \bar{y}^A \exp(-a_t)$.

Assets

There are two assets in this economy. One is a one period noncontingent debt, traded at price q_t. The second is a tree, which pays each period a constant dividend d,is sold at price p_t and there are no short sales allowed.

There is a financial friction in this economy: in order for a borrower that owns θ_t of the tree to be able to support a leverage ℓ_t, defined as $-q_t b_t / p_t \theta_t$, there is an additional cost to the borrower given by $k(\ell_t) q_t b_t$. The property of the borrowing cost $k(\ell)$ are that: $k(\ell) = 0$ for $\ell \leq 0$; $k(\ell), k'(\ell), k''(\ell) > 0$ for $\ell > 0$.

The date t budget constraint of each agent is given by

$$c_t + p_t \theta_t + q_t b_t (1 + k(\ell_t)) = y_t + (p_t + d) \theta_{t-1} + b_{t-1} \tag{9}$$

The left hand side reflects expenditures on consumption, tree shares and debt. If leverage is positive, this entails the additional cost described above. The right hand side gives the available resources: the realized endowment of goods plus the dividend and market value of the owned shares, as well as the default-free bonds that mature this period.

Optimality conditions

The optimality of bond holdings leads to the following Euler equations for bond pricing

$$q_t = \beta_A E_t^A \left[\left(\frac{c_{t+1}^A}{c_t^A} \right)^{-\gamma} \right] = \beta_B E_t^B \left[\left(\frac{c_{t+1}^B}{c_t^B} \right)^{-\gamma} \right] + q_t \left(k\left(\ell_t\right) + \ell_t k'\left(\ell_t\right) \right) \tag{10}$$

The first equality reflects the choice of agent A. This agent in equilibrium is the lender, i.e. $b_t^A > 0$. This choice arises even in the case where the discount factors are the same $\beta_A = \beta_B$. The reason is that the pessimism of agent A works like a desire for precautionary savings, which is larger than for agent B and thus drives up the price at which agent A is willing to hold bonds. In turn, as shown by the second equality, the borrower will hold the debt in equilibrium because of the leverage cost. Indeed, if not for this cost, the confident agent B will want to sell as much of the bonds as possible at the seemingly high price q_t obtained from the strong desire of agent A to save. The marginal cost of borrowing, given by the last two terms in equation (10) lower the perceived bond price for the borrower until equilibrium is achieved and both agents are happy to satisfy the bond-market clearing at an interior solution. The marginal cost consists of the higher cost of an additional unit of debt, for a given leverage ℓ_t plus the higher cost arising from the marginal increase in leverage, for a given size of debt.

We are looking for equilibria in which only type B agents hold the tree. This means that for the pricing of the tree we have to characterize just one Euler equation, given by the optimality of agent B:

$$p_t = \beta E_t^B \left[\left(\frac{c_{t+1}^B}{c_t^B} \right)^{-\gamma} \left(p_{t+1} + d_{t+1}\right) \right] + \ell_t^2 k'\left(\ell_t\right) p_t \tag{11}$$

The tree is priced under agent $B's$ belief and the value of the tree is higher due to the collateral benefit, given by the last component of equation (11). We must therefore check that the type A does not want trees. The reason that this may be the case is that the tree looks too expensive from the perspective of agent A, who is more pessimistic about the future outcome of the economy.

3.2 Recursive equilibrium

The market clearing conditions for this economy are as follows. In the goods market

$$c_t^A + c_t^B = \bar{y}_t^A + \bar{y}^B - q_t b_t k\left(\ell_t\right)$$

where we assume that the borrower cost represents wasted resources. The debt market states that bonds are in zero net supply:

$$b_t^A + b_t^B = 0$$

For the market for trees, we use the condition that only agent B owns trees.

The state variables are therefore the type A endowment y_t^A, the stochastic ambiguity a and the distribution of asset holdings. From the debt market condition, this simply boils down to tracking type B debt $b_t = -b_t^B$.

The recursive equilibrium consists of finding the consumption allocations of each agent $c^i\left(y^A, a, b\right)$, the debt allocation $b'\left(y^A, a, b\right)$ and the prices $q\left(y^A, a, b\right)$ and $p\left(y^A, a, b\right)$ .Notice that agents disagree only about income but otherwise we assume that they know the equilibrium mapping given by the functions above. This means we are left with finding find functions from the Euler equations in (10) and (11) together with the budget constraints in (9).

We are looking to solve for the equilibrium using a log-linear approximation. This amounts to finding elasticities for the unknown functions. For example:

$$\hat{c}_t^A = \log c_t^A - \log \bar{c}^A = \varepsilon_{yA}^{cA}\hat{y}_t^A + \varepsilon_a^{cA}\hat{a}_t + \varepsilon_b^{cA}\hat{b}_t$$

where hats denotes log-deviations from the steady state.

3.3 Steady state

We first describe how to characterize the steady state of this economy. We are looking for allocations and prices that are constant.

3.3.1 Type B agents, leverage and collateral value

The confident agents B expect some constant consumption $\bar{c}^B$. To describe the steady state prices, we find it convenient to write the asset pricing conditions with returns.

From the type B Euler equation for bonds we have that the steady state return on bond is given by

$$r^{bond} = \bar{q}^{-1} - 1 = \delta_B - \left(k(\bar{\ell}) + \bar{\ell}k'(\bar{\ell})\right)/\beta_B$$

where $\delta_B = \beta_B^{-1} - 1$, which shows that a higher leverage lowers the interest rate. The borrower is compensated in equilibrium for the higher cost of borrowing by paying a lower interest rate on debt.

From the type B Euler equation for tree holdings we have that the return on holding trees is

$$r^{tree} = \frac{d}{\bar{p}} = \delta_B - \bar{\ell}^2 k'\left(\bar{\ell}\right)/\beta_B$$

which also shows that a higher leverage lowers the tree return. This happens because a higher leverage implies that the tree collateral benefit is larger which raises the value of the tree price, and thus lowers its return.

Using the two conditions for returns, we can compute the excess return as

$$r^{tree} - r^{bond} = \left(k\left(\bar{\ell}\right) + k'\left(\bar{\ell}\right)\ell\left(1-\bar{\ell}\right)\right)/\beta_B$$

which, for low leverage, results in a positive premium on tree.

3.3.2 Type A agents and precautionary savings

Agent A is in turn ambiguous about his income and thus expects his consumption to drop from the steady state $\bar{c}^A$ to a lower value, given by

$$\tilde{c}^A = c^A\left(\bar{y}^A \exp\left(-\bar{a}\right), \bar{a}, \bar{b}\right) \approx \bar{c}^A \exp\left(-\varepsilon_{yA}^{cA}\bar{a}\right) \tag{12}$$

This agent behaves as if he is perpetually surprised by high income. He expects consumption to fall in proportion to ε_{yA}^{cA}, the elasticity response of his equilibrium consumption function to the ambiguous income. The latter is expected to be lower by $\bar{a}$ and thus future consumption is lower, in logs, by $\varepsilon_{yA}^{cA}\bar{a}$.

Agent A holds bonds due to his precautionary motives. The Euler equation results in

$$r^{bond} = \bar{q}^{-1} - 1 \approx \delta_A - \gamma\varepsilon_{yA}^{cA}\bar{a}$$

where $\delta_A = \beta_A^{-1} - 1$. More pessimism, in the form of a larger $\bar{a}$, lowers the interest rate, as the agent is willing to pay a larger bond price to achieve more stability in his consumption profile. This behavior works like a higher discount factor, but it is endogenous, in the sense that the equilibrium response ε_{yA}^{cA} matters for this effect. In addition, the curvature in utility matters. A lower intertemporal elasticity of substitution, given here by γ^{-1}, means that the agent is less willing to accept an intertemporally volatile consumption profile and thus is willing to accept an even lower interest rate to smooth his consumption.

The steady state conditions show clearly that we need to solve *jointly* for the steady states and the elasticities ε_z^y. In particular, here agent A has to have a belief about how his consumption will change next period as he expects a lower income. This will determine his saving decisions, which affects the steady state consumption and pricing decisions. In turn the elasticities are computed around the steady state values of these endogenous decisions.

3.3.3 Tree market participation

It is useful now to revisit the decision of agent A to hold shares in the tree. Type A expects the tree price to drop from its steady state value $\bar{p}$ to a value

$$\tilde{p} = p\left(\bar{y}^A \exp\left(-\bar{a}\right), \bar{a}, \bar{b}\right) \approx \bar{p} \exp\left(-\varepsilon_{yA}^p \bar{a}\right) \tag{13}$$

As long as the elasticity of the tree price with respect to agent A income is positive, i.e. $\varepsilon_{yA}^p > 0$, then agent A is also pessimistic about the (endogenous) price.

For agent A not to hold the tree we mush have his perceived tree return be lower than the bond return

$$\frac{d}{\bar{p}} - \varepsilon_{yA}^p \bar{a} < r^{bond}$$

Notice that this can true, as it turns out to be in the numerical example presented below, even if $r^{bond} < r^{tree} = d/\bar{p}$. For this the price impact of y^A, through the elasticity $\varepsilon_{yA}^p > 0$, is important.

3.4 Implementing an *as if* RE dynamic model

We follow the solution strategy proposed in section (2.3.2). The log-linearized equilibrium consists of the following equations.

Budget constraints: For the lender, type A :

$$c_t^A + q_t b_t = y_t^A + b_{t-1}$$

and for the borrower, type B :

$$c_t^B - q_t b_t \left(1 + k\left(\ell_t\right)\right) = y^B + d - b_{t-1}$$

where we have imposed the market clearing condition in bonds and tree shares.

Euler equations. For the tree holdings:

$$P_t = \ell_t^2 \kappa' (\ell_t) P_t + \beta_B E_t \left[\left(\frac{c_{t+1}^B}{c_t^B} \right)^{-\gamma} (P_{t+1} + D_{t+1}) \right]$$

and for the bond holdings the two Euler equations

$$Q_t = \beta_A E_t \left(\frac{c_{t+1}^{A,New}}{c_t^A} \right)^{-\gamma} = \beta_B E_t \left(\frac{c_{t+1}^B}{c_t^B} \right)^{-\gamma} + Q_t(\kappa (\ell_t) + \ell_t \kappa' (\ell_t))$$

where we introduce the additional forward looking variable for the perspective of agent A about how his future consumption will evolve:

$$\log c_t^{A,New} = \log \overline{c}^A + \varepsilon_b^{cA}(b_{t-1} - b) + \varepsilon_{yA}^{cA}(y_{A,t} - y - a_{t-1}) + \varepsilon_a^{cA}(a_t - a)$$

The solution of this system if the fixed point of jointly solving for constants and elasticities. The steady state of this *as if* rational expectations model gives us the zero risk steady state, where the conjectured policy function for needs to $c_t^{A,New}$ needs to be consistent with the elasticities of the actual $\log c_t^A$ policy function.

4 Numerical example

We now take the model of partial insurance described above and perform a range of simulation exercises. The parametrization for the benchmark model is as follows. Total output is normalized to one, with each agent receiving equal endowments of $\bar{y}^A = \bar{y}^B = 0.45$ and the tree dividends amounting to $d = 0.1$. We set the common CRRA coefficient to $\gamma = 2$ and the two discount factors as $\beta_A = .98$ and $\beta_B = .963$. The borrowing cost is $k(\ell) = .075\ell^2$. We choose these latter three parameters to target the following three moments for the deterministic version of the model: the tree price, the price of bond and the leverage, whose values are reported in the first row of Table (1).

4.1 Steady state effects of uncertainty

We find it useful to compare two versions of how uncertainty about agent $A's$ income is incorporated into our model of partial insurance. One, which we call "risk model", is a standard model of uninsured income risk. There agent A's income is hit by *iid* Gaussian shocks with a standard deviation $\sigma = 0.1$. To capture the effects of risk on decision rules we will solve that model nonlinearly using interpolation methods.

In the second version, which we call the "ambiguity model", agent A's income is ambiguous. Steady state ambiguity is normalized as a multiple of the standard deviation, i.e. $\bar{a} = \eta\sigma$. We solve this model linearly, using the solution strategy that we described above. We choose the parameter η to get a similar zero-risk steady state of debt as the ergodic steady state obtained for the risk model. We find that a value of $\eta = 0.12$ produces this match. In the process, the other variables also are very similar between the risk and the ambiguity steady states, as reported in the second and third rows of Table (1). The interpretation of this parameter is that uncertainty in the form of risk generates similar effects on decision rules through nonlinearities as it does through ambiguity with a set of beliefs on the one-step ahead conditional mean ranging from $[-0.12\sigma, 0.12\sigma]$. This parameter is much smaller that the upper bound of $\eta = 2$ that Ilut and Schneider (2014) argues is theoretically reasonable from the perspective of the statistical fit of the worst-case forecast.

Table 1: Steady states

Steady state	$\bar{c}^A$	$\bar{c}^B$	$\bar{q}$	$\bar{p}$	$\bar{q}\bar{b}$	$\bar{\ell} = \bar{q}\bar{b}/\bar{p}$
Deterministic	0.48	0.518	.98	3	1.2	.40
Risk	0.47	0.517	.987	3.36	1.59	.48
Ambiguity	0.47	0.517	.987	3.33	1.59	.48

Table (1) shows that uncertainty, either in the form of risk or ambiguity, generates in the ergodic steady state gains from trade that come in addition to the difference in patience. This is reflected in the higher equilibrium steady state leverage and debt. The precautionary saving motive of agent A, coming either in the form of risk or worst-case belief, incentivizes agent A to lend more. This not only raises the quantity of debt but also the bond price so that uncertainty aversion leads to a lower interest rate. From the perspective of the borrower, the higher collateral value of the tree means that agent B is willing to pay a higher price for the tree.

4.2 Policy functions

Figure (1) plots the equilibrium decision rules as function of debt in the case of the deterministic, risk and ambiguity model. The green vertical dotted line represents the lower steady state value of debt in the deterministic model compared to the one in the risky and, by construction, the ambiguity model, shown as the vertical red line. There several points of interest.

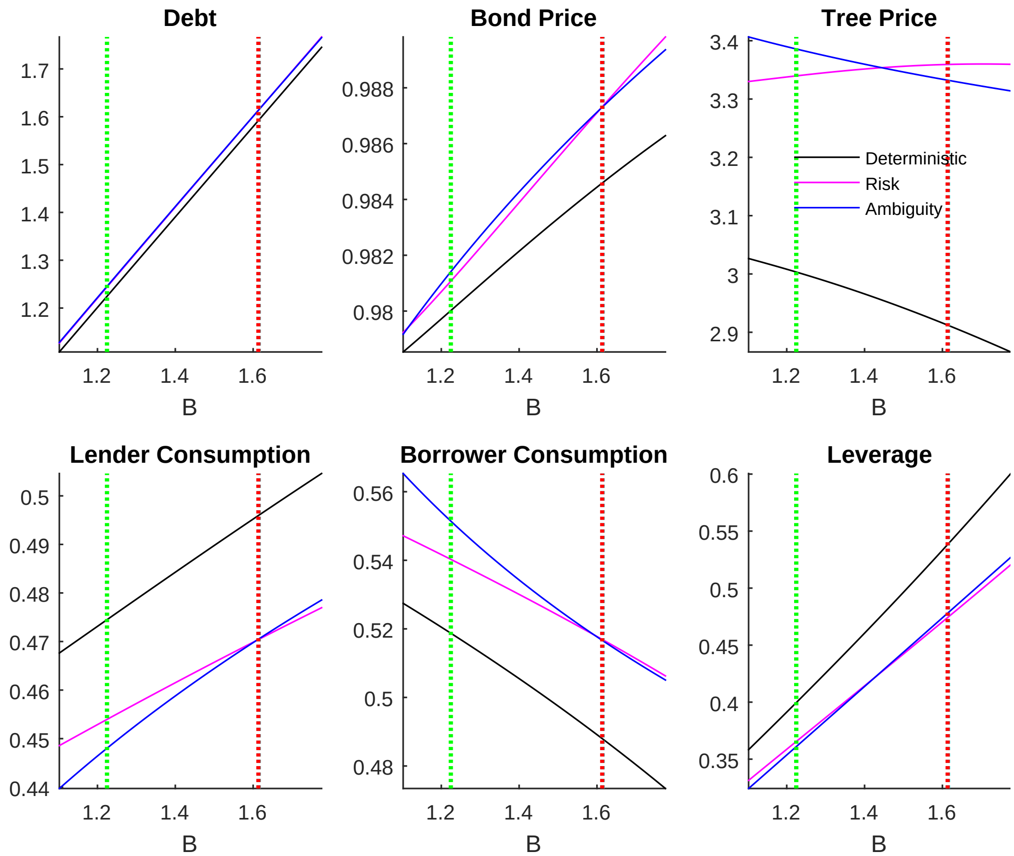

Figure 1: Equilibrium policy rules, as functions of debt, for the deterministic, risky and ambiguous models.

First, policy functions in the risk and ambiguity model are quite similar. While the former was obtained through nonlinear solution methods, the latter captures most of its qualitative properties even if it is solved only up to a first-order approximation.

Second, the shape of the decision rules are as expected. Agent $A's$ consumption and lending is increasing in the amount of bonds that he has entering the period, while agent B's consumption is decreasing since he has to repay that debt. If the lender has lower bonds than in steady state, the transition path he expects is one with a positive consumption growth. This means that the bond price along this path will be lower than the steady state, and will converge to it from below as the state of the economy is one with lower debt. Similar logic holds for the region of debt above its steady state. This explains the upward sloping bond price. Along this path, the borrower expects a negative consumption growth which increases, caeteris paribus, the price he is willing to pay for the tree since this serves as a way to smooth consumption. On the other hand, in this region the collateral value of the tree is lower since debt is lower than the steady state. Thus, there are two competing equilibrium effects. For the linear policy functions, such as the deterministic and ambiguity model, one force is bound to dominate. In this case it is the former and the equilibrium price is higher when debt is lower. In the nonlinear solution, the price is slightly increasing and about flat around the ergodic steady state.

Third, the qualitative effects of uncertainty in shifting the deterministic policy functions are the same, whether it is risk or ambiguity. Sharing uncertainty leads to higher gains from trade, which shifts up debt, for any state value. Even if the shift is small, because debt is very persistent, the fixed point where the debt policy returns a value equal to that of the state value of debt is much larger in the uncertainty case. The uncertainty-averse agent A engages in precautionary saving, so that the consumption function is shifted down with uncertainty. For the same value of debt this results in significantly lower consumption. However, because of the same precautionary reasoning, in steady state agent A also accumulates more debt so the net effect on the steady state consumption is not obvious. For this parametrization it turns out that that the shift down in the function dominates. The desire to save also shifts up the bond price. Together with the higher debt characterizing the new steady state this leads to a significant rise in the bond price. In turn, agent B's consumption is shifted up since he enjoys a lower interest rate on the same debt. However, in the new steady state he will have to pay the now lower interest rate on a larger amount of debt. The overall effect here is such that the second effect dominates and the agent B consumption is also lower in the ergodic steady state than in the deterministic case, as shown in Table (1). The fact that both agents' consumption are lower is possible because of the larger borrowing cost that is a waste in the resource constraint. Because of the larger gains of trade and the higher debt, the collateral value of the tree shifts up with uncertainty and remains high in the steady state. For leverage the price effect dominates the bond effect and the function is shifted down. However, due to its strong upward shape, and intuitively because of the larger gains from trade, the ergodic steady state of leverage is larger.

4.3 Time-varying disagreement

We now focus on the ambiguity model to study time-variation in uncertainty. We have described it as one with heterogeneity in beliefs, where agent A is less confident and thus *as if* more pessimistic about his income. We have developed tools that allows us to study this heterogeneity that remains even in the steady state using linear methods. An advantage of the linearity is that we can expand easily the state space. For example, we can study shocks to the ambiguity, or, in the language of heterogeneous beliefs, shocks to the degree of disagreement between the two agents.

Figure (2) plots the response of the ambiguity model economy to a one-time increase in agent A's ambiguity about his future income. Following a similar logic as with the steady state, this results in a strong desire to save from agent A. This leads to a fall in his consumption, a reduction in the interest rate and an increase in debt. At the same time, the tree price increases as its collateral value is higher. The periods following this shock are characterized by different dynamics, depending on the interest rate effect. For the baseline economy, the fall in the interest rate is so large that the rate becomes negative. This has interesting effects on the dynamics. Even if uncertainty has reverted to its steady state following the first period shock, agent A now sees a negative wealth effect since the debt that he owns comes with a negative interest rate. Thus, starting in period two, we see typical transitional dynamics back to steady state when the agent has suffered a negative wealth shock: consumption is lower than steady state and he reduces debt to smooth this transition. This is the reason why consumption of agent A not only falls in the initial period, which comes from the precautionary savings, but continues to converge back to steady state from below, and at the same time, why debt initially rises and then falls.

To verify this intuition, Figure (3) plots the impulse response to the same ambiguity increase, but for a parametrization where β_A is lower so that the steady state interest rate is larger to the extent that the fall in the interest rate after the initial uncertainty shock still keeps the rate at a positive level. In this case, the dynamics are more standard. Consumption of agent A initially declines strongly and then converges back, slightly from above. This agent has accumulated more debt following the higher perceived uncertainty and, following

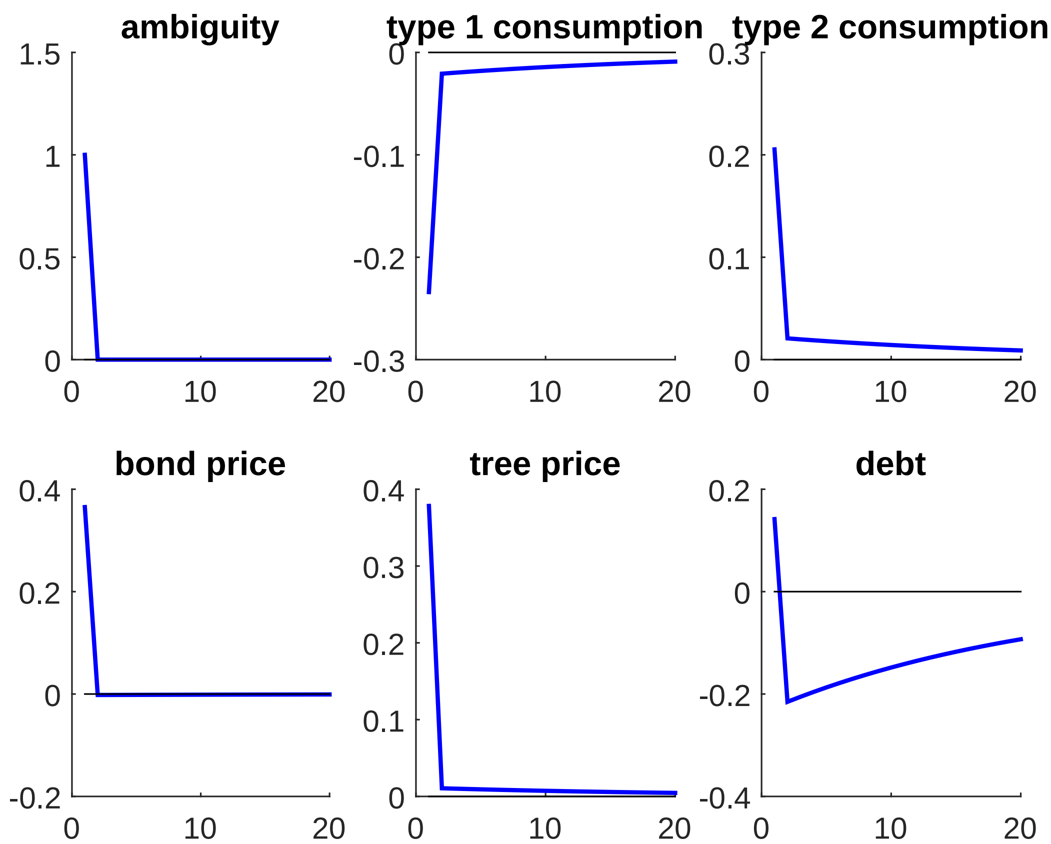

Figure 2: Increase in ambiguity

standard transitional dynamics logic, will now consume out of it to converge smoothly back to steady state. Indeed, debt is initially higher and then it slowly decreases.

4.4 Perturbation without needing deterministic steady state

Our proposed perturbation method involves solving jointly for the zero-risk steady state and the equilibrium elasticities, which has the implication that we do not use the deterministic steady state in the solution method. This is an important technical advantage over standard perturbation methods which usually approximate around the deterministic steady state. That standard approach requires that such a steady state is determinate and thus cannot handle models where the Jacobian of the entire dynamic system is rank deficient. An example for such a situation where one cannot approximate around the deterministic steady state is an asset allocation problem where expected returns are the same, such as in models of international portfolio choice.

In our model of partial insurance we would encounter the same problem is agents have

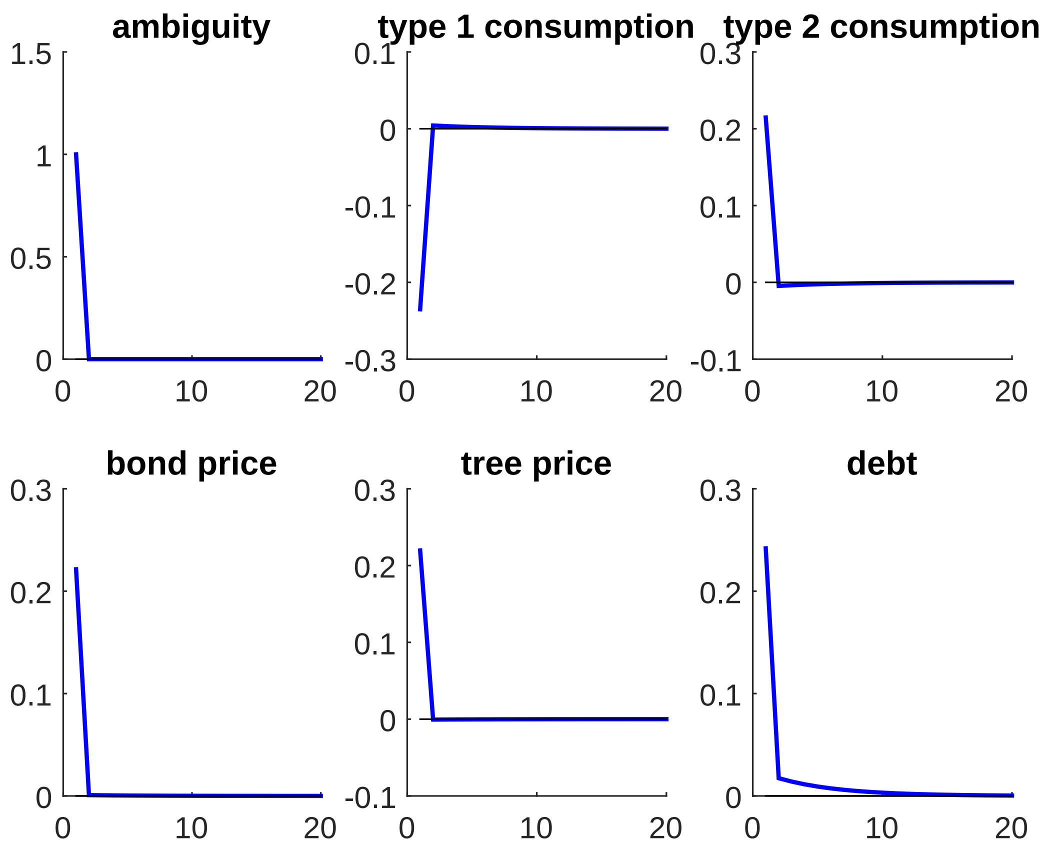

Figure 3: Increase in ambiguity: low discount factor economy

the same discount factor β. In this case, in the deterministic steady state there are no gains from trade. This means that there is zero debt and, importantly, the asset and consumption allocations are indeterminate. Standard approaches cannot handle such a case and need to impose some prior differences between agents to make these allocations determinate. Our strategy does not use the deterministic steady state and instead goes directly after the zero-risk steady state. In the latter there will be differences between agents because of the assumed heterogeneity in confidence.

Consider the following numerical example where the only difference from the baseline model is that both agents have the same discount factor: $\beta_A = \beta_B = .963$. Table (2) indicates that while in the deterministic steady state there is no debt and allocations are indeterminate, in the zero-risk steady state there is positive debt and the allocations are determinate.

Table 2: Steady states

Steady state	$\bar{c}^A$	$\bar{c}^B$	$\bar{q}$	$\bar{p}$	$\bar{q}\bar{b}$	$\bar{\ell} = \bar{q}\bar{b}/\bar{p}$
Deterministic			.963	2.63	0	0
Ambiguity	0.47	0.52	.972	2.76	.81	.29

The key property that emerges from the model with uncertainty is that ambiguity generates gains from trade. The less confident agent, of type A, is concerned about future income and wants to save. This desire makes agent A behave like a saver, which otherwise could be modeled as being the more patient agent. Indeed, heterogeneity in ambiguity works like differences between subjective discount factors. Importantly, the resulting difference is endogenous. Changes in the environment will lead to agent A behaving as if his subjective discount factor has changed.

We can see this analytically. In steady state, optimality conditions for agent A involve the perceived discounted intertemporal marginal rate of substitution between consumption tomorrow and today. Using equation (12) this is given by

$$\beta_A \left(\frac{\tilde{c}^A}{\bar{c}^A} \right)^{-\gamma} = \beta_A \exp\left(\gamma \varepsilon_{yA}^{cA} \bar{a}\right) \tag{14}$$

By defining an adjusted discount factor

$$\widetilde{\beta}_A \equiv \beta_A \exp\left(\gamma \varepsilon_{yA}^{cA} \bar{a}\right)$$

we can reinterpret zero-risk steady state optimality conditions that only involve the pricing kernel in (14) as a deterministic steady state where, if $\bar{a} > 0$, agent A has a higher discount factor than agent B. This adjusted discount factor is however endogenous to the rest of the parameters because the elasticity ε_{yA}^{cA} matters. In addition, even if we fix parameters, in general the model is not perfectly observationally equivalent to the deterministic case of setting exogenously the discount factor of agent A to $\widetilde{\beta}_A$. The reason is that there may be other forward-looking conditions of agent A that involve expectations of other endogenous variables, not just consumption growth. For example, when this agent considers investing in tree shares, it forms expectations over the future tree price, as shown in equation (13). The elasticity of this price with respect to the ambiguous shock appears as an additional effect of uncertainty in this decision, on top of the effect of the pricing kernel.

4.5 Endogenous disagreement and worst-case beliefs

The baseline model features endogenous disagreement to the extent that the structural parameters affect the equilibrium elasticities of allocations and prices with respect to the uncertain shocks. For example, the elasticity of agent A's consumption with respect to his income, ε_{yA}^{cA}, will in general be a function of many of the structural parameters, including the size of his endowment, the subjective discount factor, the borrowing cost and so on. In this sense, the degree of disagreement is on one hand purely exogenous, controlled by the amount of ambiguity $\bar{a}$, and on the other hand determined by the rest of the model parameters.

Part of the a-priori assumed heterogeneity in beliefs is the selection of the identity of the agent whose income is uncertain. However, the approach we propose can be extended to study situations in which agents self-select in equilibrium into types whose beliefs will ex-post differ.

To see this, let us consider a version of the baseline model in which we introduce nominal credit and ambiguous inflation. The payoff of the bond is $(1+\pi)^{-1}$, where π is the uncertain inflation and the price q is now the nominal bond price. Suppose that the true data generating process is that inflation is zero. If there would be no uncertainty about it, or, in the linear model, there would be full confidence in this process, then we would recover the exact same equilibrium as in the real model.

Consider the ambiguity model. Suppose that both agents believe that the one-step ahead conditional mean of inflation belongs to the set $[-\overline{\pi}, \overline{\pi}]$, where $\overline{\pi}$ is a positive parameter. Agent A is in equilibrium the lender. This desire to save follows either from a-priori differences in the subjective discount factor, or, as we have shown above, simply from differences in the ambiguity about income. The lender takes consumption and nominal bond decisions under the worst-case belief about inflation. The future payoff of the bond is decreasing in future inflation, and, in turn, the value function is increasing in the wealth that the agent accumulates. It follows that the lender is concerned that the true DGP for inflation is one in which inflation is high, because this lowers the future continuation utility, by eroding the real value of the nominal bond. Thus, agent A acts as if future inflation is $\overline{\pi}$. At the same time, agent B, who in equilibrium becomes the borrower, is concerned that future inflation is low, and equal to $-\overline{\pi}$. Indeed, a lower future inflation raises the real value of the repayments the agent has to make, which lowers his continuation value.

Let us consider a numerical example. We set the width of the set of beliefs about inflation such that $\overline{\pi} = 0.01$. Table (3) reports the zero-risk steady state of this model, in which the only difference from the baseline version is the addition of the ambiguous inflation. Inflation uncertainty lowers significantly the gains from trade, since both agents now perceive lower real returns to their trading strategies. The uncertainty premium lowers the price of nominal bonds and it leads to less debt. The price of the tree also decreases due to the lower value of collateral.

Table 3: Endogenous worst-case beliefs about inflation

Steady state	$\bar{c}^A$	$\bar{c}^B$	$\bar{q}$	$\bar{p}$	$\bar{q}\bar{b}$	$\bar{\ell} = \bar{q}\bar{b}/\bar{p}$
Baseline	0.47	0.517	.987	3.33	1.59	.48
Ambiguous inflation	0.46	0.53	.981	2.74	0.77	.28

The key property of the ambiguity model is that agents act under the worst-case belief that supports the equilibrium allocation. Agents may ex-ante have the same degree of confidence about the probability distributions of shocks. However, because they endogenously take trading positions of different signs in equilibrium, the ex-ante homogeneity manifests as ex-post heterogeneity of beliefs. In our case, the lender acts as if future inflation is high while the borrower as if future inflation is low.

When the ex-post disagreement in beliefs is sustained by the conjectured equilibrium choices, we have been able to use our proposed strategy of studying heterogeneity. While at the equilibrium allocations, these beliefs will look to an outside observer as dogmatic differences, the model is one in which policy interventions that affect equilibrium choices will also alter this disagreement.